A Nonlinear Continuous Time Optimal Control Model of Dynamic Pricing and Inventory Control with no Backorders

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Abstract

In this paper, we present a continuous time optimal control model for studying a dynamic pricing and inventory control problem for a make-to-stock manufacturing system. We consider a multi-product capacitated, dynamic setting. We introduce a demand-based model where the demand is a linear function of the price, the inventory cost is linear, the production cost is an increasing strictly convex function of the production rate and all coefficients are time-dependent. A key part of the model is that no backorders are allowed. We introduce and study an algorithm that computes the optimal production and pricing policy as a function of the time on a finite time horizon, and discuss some insights. Our results illustrate the role of capacity and the effects of the dynamic nature of demand in the model.

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1 Introduction

1.1 Motivation

Continuous-time optimal control models (sometimes also referred as fluid models) provide a powerful tool for understanding the behavior of systems where the dynamic aspect plays an important role. In recent years, there has been a lot of research in an attempt to provide a deeper understanding of optimal control models from a theoretical as well as an application point of view. In particular, an attractive feature of these models is that they provide good scheduling, production and inventory policies in a variety of settings. Furthermore, they approximate well the underlying stochasticity of problems in a deterministic way. Fluid models arise in applications as diverse as routing and communication systems as well as queueing, supply chain and transportation systems.

A continuous time approach has the advantage of not introducing any approximation to the real setting: it provides the exact solution of the system. When taking a discrete time approach, one has to decide what a reasonable time step should be, and to allow price and production changes only at those times. In reality, in some settings a supplier may need more flexibility. In order to avoid being too restrictive, the time step needs be very small, and if the time horizon is large the size of the problem may become exceedingly large, in terms of number of variables and number of constraints. Therefore, the problem size usually implies significant delays in obtaining good solutions. Examples of supply chain industries where continuous-time optimal control models of the type we discuss in this paper are relevant, include industries with a high volume of throughput and data on costs and demand that change a lot. The hardware as well as the semiconductor industries are such examples. Moreover, we believe that a similar approach can be applied to problems in areas other than dynamic pricing and inventory control, where the evolution of the system evolves dynamically and justify a continuous time approach. We believe that the techniques presented in this paper may be helpful to those areas as well.

The overall goal of this research is to introduce and study a nonlinear continuous time optimal control model and its application to dynamic pricing and inventory control. In our analysis we use ideas from deterministic continuous-time optimal control theory together with nonlinear optimization techniques. We consider non-perishable goods sharing a production capacity that is time-varying and derive an open-loop optimal production and pricing policy over time.

1.2 Some Related Literature

A large part of the literature has focused on the solution of linear continuous-time optimal control models (see for example Anderson [4], [5], Pullan [50]). This part of the literature shows existence of an optimal solution with piecewise constant controls. Pullan in particular showed strong duality and designed a class of algorithms solution convergent. However, when fluid models are nonlinear, the dynamic together with the nonlinear aspect of the problem make them harder to analyze. Nonlinear fluid models are particularly useful for dynamic pricing and inventory management applications, as we explained above. A variety of models have been proposed in the literature for such applications (see references below). These models typically differ due to the production cost, inventory cost, and demand functions considered.

A large volume of literature studies a demand model for the single-product case. For example Pekelman [49] solves the dynamic pricing and production policy problem for a single product optimizing over a finite time horizon. He models the demand as a linear function of the price with time-varying coefficients. The model uses linear inventory cost with a constant coefficient, and a general strictly convex production cost. The model does not allow backorders. Feichtinger and Hartl [25] extend this model by considering a general nonlinear demand function and allowing backorders, with both piecewise linear and strictly convex inventory costs. They obtain phase diagrams for the equilibrium and transient behavior of the optimal solution with a finite or infinite time horizon. Another extension is introduced by Thompson, Sethi and Teng in [52], where the production rate and the level of inventory are bounded, and the production cost is either linear or strictly convex. Gaimon [27] considers additional controls by allowing decisions on the maximal production rate as well as price and production output, where the change in maximal production rate has an effect on the production cost. [21], [33] and [39] consider the case of centralized or decentralized decisions between a distributor and a manufacturer in an industrial channel of distribution. Locke Anderson [41] considers production decisions when the production of a final good requires as input the production of an intermediate good. In the single-product model, Jørgensen [34] uses a continuous time optimal control model to study demand learning effects while Laurent-Varin [40] introduces an interior-point solution algorithm.

In a multi-product setting, Bertsimas and Paschalidis [8], Harrison [31] and Meyn [47] study a make-to-stock problem using fluids. Specifically, Bertsimas and Paschalidis in [8] study an inventory control problem with fixed demand rate and capacity rate shared among all classes. Their model allows backorders and computes a production policy by minimizing either a linear or a quadratic inventory cost over successive small intervals. Luo [42] considers a make-to-stock multi-class queueing scheduling problem that minimizes a convex quadratic backorder and holding cost and finds an optimal production policy over the entire time horizon. Kleywegt [38] uses a cutting plane algorithm to solve a multi-class optimal control problem of dynamic pricing with profit linear in terms of selling rate. Fleisher and Sethuraman [26] provide an approximation algorithm to solve the optimal control of fluid queueing networks. Moreover, Van Ryzin and McGill [54] designed an adaptive approach within the framework of airline revenue management based on historical observed data. They study an algorithm through stochastic approximation theory. Gallego and van Ryzin [28], [29] consider the problem of dynamically pricing over a finite horizon when demand is stochastic and price sensitive. Finally, Kachani and Perakis [35], [36] take a delay-based approach to determine optimal pricing and production policies, where the price and level of inventory affect the delay (time that a product remains in inventory).

A stream of research has focused on a dynamic programming approach to solve pricing and/or inventory problems (see [3], [7]). In these models, the (possibly infinite) time horizon is divided into time periods and allowing decisions at the beginning of each period, as opposed to the research cited above that takes a continuous time approach. Maglaras and Meissner [45] approach the pricing problem under fixed capacity by reducing the problem to determining the aggregate rate at which all products jointly consume resource capacity, and defining an efficient frontier. Federgruen and Heching [24] address pricing and inventory control for a single product under stochastic demand, with backorders, no fixed cost, and in a periodic review model. They characterize the value function and show that a base-stock policy is optimal. Chen and Simchi-Levi ([14], [15]) consider the model where ordering costs include a fixed cost component, both in the finite and infinite time horizon case. They show that if the time horizon is finite, an (s, S, p) policy is optimal for additive demand, but not for multiplicative demand. Using k-convexity, they show that for an infinite horizon and input parameters independent of time, an (s, S, p) policy is optimal for both types of demand models.

The literature on dynamic pricing is growing fast. Elmaghraby and Keskinocak in [22] and the references therein provide a comprehensive literature review of dynamic pricing models while Bitran and Caldentey [11] provide an overview of research on dynamic pricing and its relation to Revenue Management. Furthermore, Zipkin [56] and the references therein provide a thorough review of recent advances in inventory control theory and its relation to supply chain. Chan, Shen, Simchi-Levi and Swann [12] review research on coordination of pricing and inventory decisions. Finally, Yano and Gilbert [55] and the references therein provide a review of pricing and production/procurement decisions.

In a setting where the problem has a dynamic aspect, such as traffic control, queueing networks, supply chain, or transportation, there is a connection with fluid models, which can be viewed (when there is no stochasticity) as continuous time optimal control models. More theoretically, many papers study general continuous time optimal control models. [6], [32], [37] and [51] give formulations of the Maximum Principle under state variable constraints. Clarke ([16], [17], [18], [19] with others) and Devdariani and Ledyaev [20] provide theoretical results on global optimality conditions.
For the solution of linear fluid models, Bertsimas and Luo [43] construct an algorithm solving state constrained separated continuous linear programs under some assumptions. Fluid models also connect with semi-infinite programming problems. Tunçel and Todd [53] study the asymptotic behavior of interior point methods for semi-infinite programming by finding the limits of search directions, potential functions and central paths as the number of variables becomes infinite.

1.3 Contributions

In this paper, we will consider a finite time horizon problem. Although we will focus on a dynamic pricing application, our results apply to more general continuous time optimal control models. The model we study incorporates the following features:

(i) a continuous-time, dynamic environment;
(ii) multiple, non-perishable products;
(iii) a dynamic production capacity shared among all products;
(iv) time-varying coefficients;
(v) two controls for each product at each time: price and production rate;
(vi) no backorders are allowed.

Our goal is twofold: study the structure of the optimal policy, and propose a method for computing it. Furthermore, we note that when we consider this problem in a setting where data are uncertain and we take a robust optimization approach, the robust counterpart problem can be formulated in a form very similar to the deterministic problem. As a result the solution algorithm we present in this paper can be adapted to solving the problem under uncertainty (see [2]).

We assume that multiple products share a single common production capacity. This assumption is a standard one in the literature that considers multiclass systems. For example, Bertsimas and Paschalidis [8] consider a multiclass make-to-stock system and assume that a single facility produces several products, with the production process over time taken as an arbitrary stationary stochastic process. Also in a make-to-stock manufacturing setting with multiple products, Kachani and Perakis [35] suppose that the total production capacity rate across all products is bounded. Gilbert [30] addresses the problem of jointly determining prices and production schedules for a set of items that are produced on the same production equipment and with a limited capacity. Maglaras and Meissner [45] consider a monopolist firm that owns a fixed capacity of a resource that is consumed in the production of multiple products. Finally, Biller et al. [10] extend a single product model of dynamic pricing to cover supply chains with multiple products, each of which is assembled from a set of parts and shares common production capacity. In order to keep the model simple in this paper, we make a similar assumption of a single production capacity constraint, and we leave as a direction of future research the case of multiple capacity constraints which could be applicable to certain production settings.

Inventory problems may allow or deny backorders, i.e. the possibility of having a negative inventory level. In a manufacturing system which does not allow backorders and the demand rate is not external, but determined by a relationship with price, the price can be adjusted so that no demand is actually lost. That is, the price is set so that the accumulated demand never exceeds the inventory level, in such a way that the selling rate equals the demand rate. These models add a constraint that ensures inventory of a given product i is always non negative (i.e., $I_i(t) \geq 0 \quad \forall t \in [0,T]$), where T is the time horizon and $I_i(t)$ the inventory level of product i at time t. There is a holding cost associated with inventory. We assume the holding cost is linear: $h_i(t)I_i(t)$, where $h_i(t)$ is a positive holding cost coefficient for product i at time t.
Nevertheless, the constraint of having no backorders is a difficult one. In this paper, we study this difficulty using ideas from optimal control and nonlinear optimization.

We will consider a solution approach when the objective of the continuous time optimal control model incorporates (a) linear inventory costs, (b) a nonlinear cost structure due to the strictly convex production cost, (c) a nonlinear revenue component. The demand for product i will be modelled as a linear decreasing function of the price of that product : $d_i(t) = \alpha_i(t) - \beta_i(t)p_i(t)$, where $d_i(t)$ and $p_i(t)$ are respectively the demand and the price at time t for product i, and $\alpha_i(t)$ and $\beta_i(t)$ are known positive real valued functions of time.
This will allow us to derive a continuous time optimal policy as a function of a Lagrange multiplier that applies to the entire time horizon determining simultaneously the prices and the production rates of all products (as a function of the Lagrange multiplier). We then propose an heuristic algorithm to compute the Lagrange multiplier, and thus obtain the optimal solution. Our approach does not introduce a time discretization. It illustrates the effect of capacity in the problem as well as the effect of the dynamic nature of the problem.

Previous work by the same authors [1] shows that under some assumptions, we can derive the exact Lagrange multiplier over time as well, rather than using a heuristic algorithm to determine it. However, the procedure may be quite complex and the assumptions difficult to verify. Therefore, in this paper, it is important to propose a heuristic algorithm that exhibits computationally good convergence results and is not too complex to implement.

We would like to point out that it is possible to extend this approach to consider a nonlinear demand function of the price by using a very similar reasoning. Nevertheless, for the sake of simplicity and ease of computation, we will discuss our approach in terms of a linear demand function of the price as is often done in the literature.

We have assumed that the demand for a product depends only on the price for this product and not on the prices of other products. This assumption is standard in multi-product pricing problems when the products are considered distinct so that they target distinct classes of customers. The automotive industry is one example of industry where such an assumption is valid (see [10]). Bertsimas and de Boer [7] study a joint pricing and resource allocation problem in which a finite supply of resource can be used to produce multiple products and the demand for each product depends on its price. They apply this problem to airline revenue management. Paschalidis and Liu [48] consider a communication network with fixed routing that can accommodate multiple service classes and in which the arrival rate of a given class (or demand for that class) depends on the price per call of that class only. In their multi-product case, Biller et al. [10] assume that there are no diversions among products, i.e. that a change in the price for one product does not affect the demand for another product. They motivate this assumption by focusing on items that appeal to various consumer market segments, such as luxury cars, SUV, small pickup, etc. for example of the automotive industry. We position this paper in the same line of research and make the similar assumption of a demand independent of prices of other products. A more general model would allow the demand to depend on all prices with various price elasticities. However, such a model would significantly increase the complexity of the problem. This problem would go beyond the scope of this paper but could be the focus of follow-up research.

1.4 Structure of the paper

The remainder of this paper is structured as follows: in Section 2 we describe the notations we will use throughout the paper, we describe the model and we explain the general solution approach. In Section 3, we detail the computation of the optimal solution assuming the capacity constraint multiplier is given, and we provide some results on the structure of the solution. In Section 4, we provide a heuristic algorithm that allows to obtain that multiplier. In Section 5, we discuss insights from numerical examples.

2 Notations, model and solution approach

2.1 Notations and definitions

Inputs

T	time horizon;
N	number of products;
$K(t)$	shared production capacity rate at time t (non negative);
I_i^0	initial non negative inventory level for product i;
$h_i(t)$	holding cost of one unit of product i at time t;
$f_i(.)$	production cost function for product i with respect to the production rate;
$\alpha_i(t), \beta_i(t)$	coefficients used for product i at time t in the linear relationship between price and demand $d_i(t) = \alpha_i(t) - \beta_i(t)p_i(t)$.

Outputs

$p_i(t)$	price of one unit of product i at time t (control variable);
$u_i(t)$	production flow rate of product i at time t (control variable);
$I_i(t)$	inventory level (number of units) of product i at time t (state variable).

Definitions

We denote by $I(.), p(.), u(.), \alpha(.), \beta(.)$ the vectors with respective components

$$I_i(.), p_i(.), u_i(.), \alpha_i(.), \beta_i(.), \quad i = 1, \dots, N.$$

$I^*(.), p^*(.), u^*(.)$ will denote the optimal solution.
When x and y are vector of the same size n , $x \times y$ will denote the vector with components $x_i y_i$, $i = 1, \dots, n$.
We define:

Constrained interval: Interval of time where the inventory level equals zero (also called boundary interval).

Constrained product: Product that is on a constrained interval.

Unconstrained interval: Interval of time where the inventory level is positive.

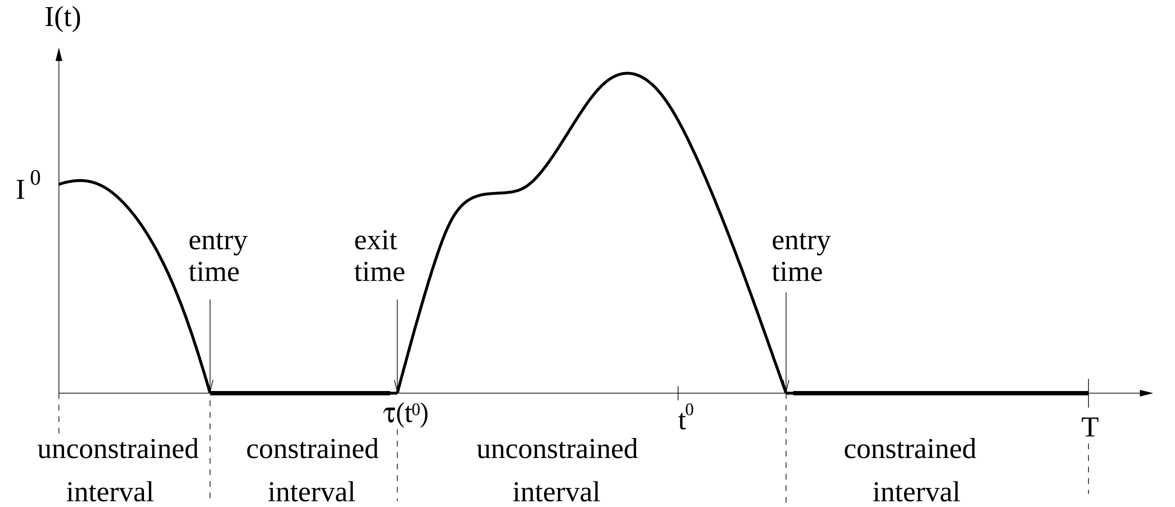
Unconstrained product: Product that is on an unconstrained interval.

Active product: Product with a positive production rate.

Inactive product: Product with a production rate equal to zero.

We notice that for any pricing and production policy we consider (and in particular an optimal policy), the inventory level will be structured via a sequence of intervals, where the inventory level is successively positive and equal to zero. A constrained interval starts at an *entry time* and finishes at an *exit time*, i.e. the time the inventory level becomes again positive.

Assumption 1. *We will assume throughout the paper that for each product, there is a finite number of entry and exit times.*



2.2 Description of the model and assumptions on the data

This problem was solved by Pekelman [49] for the single product case with a holding cost coefficient constant in time and no production capacity constraint. Nevertheless, the presence of multiple products sharing production capacity makes the problem rather complex.

Assumption 2. *For all products i, $\alpha_i(.), \beta_i(.), h_i(.)$ as well as $K(.)$ are assumed to be positive, continuous functions of the time. Moreover, $\alpha_i(.), \beta_i(.)$ and $K(.)$ are assumed to be continuously differentiable.*

Assumption 3. *For all products i, function $f_i(.)$ is assumed to be twice continuously differentiable, strictly convex, non-negative and increasing on $[0, K(t)]$.*

The problem seeks to maximize the revenues minus the inventory and production costs. As a result, it can be written as follows:

$$\begin{aligned} \max \quad & \int_0^T \left[\sum_{i=1}^N \Big(p_i(t) d_i(t) - f_i(u_i(t)) - h_i(t) I_i(t) \Big) \right] dt \qquad (1) \\ \text{s.t.} \quad & \dot{I}_i(t) = u_i(t) - d_i(t), \quad \forall t \in [0,T] \quad i = 1, \ldots, N, \\ & d_i(t) = \alpha_i(t) - \beta_i(t) p_i(t), \quad \forall t \in [0,T] \quad i = 1, \ldots, N, \\ & \sum_{i=1}^N u_i(t) \leq K(t), \quad \forall t \in [0,T], \\ & u_i(t), \quad p_i(t), \quad d_i(t), \quad I_i(t) \geq 0, \quad \forall t \in [0,T] \quad i = 1, \ldots, N, \\ & I_i(0) = I_i^0, \quad i = 1, \ldots, N. \end{aligned}$$

Equivalently:

$$\max \quad \int_0^T \left[\sum_{i=1}^{N} \Big(p_i(t)(\alpha_i(t) - \beta_i(t)p_i(t)) - f_i(u_i(t)) - h_i(t)I_i(t) \Big) \right] dt \tag{2}$$

$$\text{s.t.} \quad \dot{I}_i(t) = u_i(t) - \alpha_i(t) + \beta_i(t)p_i(t), \quad \forall t \in [0,T] \quad i = 1,\dots,N, \tag{3}$$

$$\sum_{i=1}^{N} u_i(t) \leq K(t), \quad \forall t \in [0,T], \tag{4}$$

$$I_i(t) \geq 0, \quad \forall t \in [0,T] \quad i = 1,\dots,N, \tag{5}$$

$$u_i(t) \geq 0, \quad \forall t \in [0,T] \quad i = 1,\dots,N \tag{6}$$

$$0 \leq p_i(t) \leq \frac{\alpha_i(t)}{\beta_i(t)}, \quad \forall t \in [0,T] \quad i = 1,\dots,N, \tag{7}$$

$$I_i(0) = I_i^0, \quad i = 1,\dots,N.$$

We observe that in this continuous time optimal control model, constraint (3) is the dynamic equation that describes the evolution of the level of inventory, modelled as a continuous and differentiable function of time.
Constraint (4) corresponds to the common production capacity that is shared among all the products. This is the only constraint that is coupling the products and prevents us from simply solving N times a single-product problem.
Constraint (5) represents the no backorder constraint. Notice that these are constraints on the state variables. This makes their treatment different from constraints on control variables but also harder. We will apply the Maximum Principle in the case of inequality constraints on the state variables (see [6], [37], [51]).
We introduce constraints (6) and (7) to ensure that prices and production rates are non-negative. Furthermore, the upper bounds on the prices reflect the fact that the demand should remain non-negative. These are constraints on the control variables, which are taken into account by simply restricting the feasible domain of admissible controls.

We assume that the following assumption holds.

Assumption 4.

- $f_i'(0) < \frac{\alpha_i(t)}{\beta_i(t)}, \quad i = 1,\dots,N \quad \forall t \in [0,T]$;
- $h_i(t) < \frac{\alpha_i(t)}{\beta_i(t)}$.

Assumption 4 means that the intercept of the marginal production cost function on the one hand, and the cost of holding one unit of good on the other hand, are smaller than the maximum price that may be charged at any fixed time.

2.3 Existence of an optimal solution

Theorem 1. *Under Assumptions 2, 3 and 4, there exists an optimal solution $u^*(.), p^*(.)$ to Problem (1).*

The proof of this result is given in Appendix C.

2.4 Solution approach

In order to solve problem (1), we will employ ideas from control theory and nonlinear optimization. Since we are dealing with a continuous-time control problem, we will first define the Hamiltonian function using adjoint variables corresponding to the dynamic equations. We will also introduce a Lagrangian function by dualizing the difficult constraints, i.e. the capacity constraint and the no backorder constraints. Dualizing the capacity constraint will enable us to decouple the problem, and reduce it to several single-product problems. Subsequently, we will use the Maximum Principle under constraints on the state variables and the indirect adjoining method to the Lagrangian function (see Hartl, Sethi and Vickson [32] or Sethi and Thompson [51]).

- We will assign adjoint variables $q_i(t)$ in order to dualize the dynamic constraint i at time t;
- We will write the Hamiltonian function;
- We will assign multipliers $\rho_i(t)$ to dualize the constraint on the non negativity of $I_i(t)$;
- We will assign a multiplier $\eta(t)$ to dualize the capacity constraint;
- Through these multipliers, we will construct the Lagrangian function (8).

We will simultaneously compute, as a function of the capacity Lagrange multiplier, the no back-order Lagrange multipliers and adjoint variables that satisfy these optimality conditions, and will allow us to compute an optimal policy.

We use the vector notation $q(t) \equiv (q_1(t), \ldots, q_N(t))$, $\rho(t) \equiv (\rho_1(t), \ldots, \rho_N(t))$. In Section 3, we will present a solution approach for the problem, consisting in:
(i) first writing the optimality conditions and the optimal solution for each product i at a given fixed time t as a function of multipliers $q_i(t) + \rho_i(t)$ and $\eta(t)$. (ii) Then we will augment the approach by also computing the vector $q(t) + \rho(t)$ as a function of multiplier $\eta(t)$ instead of assuming it is given. We will solve the problem for any time t under the assumption that we know multiplier $\eta(t)$. This allows us to "ignore" the state variable and capacity constraints but allows us to take into account the dynamic equations.
(iii) Finally, we consider the problem solution over the whole time horizon and we introduce a heuristic algorithm that computes multiplier $\eta(.)$ iteratively. Thus, using the previous results, the computation of $\eta(.)$ gives rise to $q(.) + \rho(.)$ which in turn enables to obtain the optimal solution.

3 Solution for multiplier $\eta(t)$ given

3.1 Approach overview

In what follows we will assume that the multiplier $\eta(.)$ corresponding to the capacity constraint is given (see also Subsection 2.4 above). Our goal in this section is to illustrate how we can obtain an optimal solution in the case we know the value of this multiplier. As a result, we need to find the multipliers and adjoint variables that satisfy the necessary conditions for optimality. In particular, we take the following approach.
In Section 3.2, we write the necessary condition for optimality. The condition that the Lagrangian should be maximized allows us to obtain the optimal price and production rate at a given time,

as well as the derivative of the optimal inventory level, as a function of the multipliers and adjoint variables, $\eta(.)$ assumed to be known, as shown in Section 3.3. Therefore we next focus on determining the multipliers and adjoint variables.
The system consists of a sequence of constrained and unconstrained intervals. We first consider constrained intervals in Section 3.4. Since the inventory level is kept constant at zero, its derivative also equals zero. This observation enables us to characterize the adjoint variables and multipliers on such intervals.
By complementary slackness, on unconstrained intervals, the Lagrange multiplier for the no stock-out constraint equals zero, and the adjoint equation is used to determine the adjoint variable, as shown in Section 3.5.
We next need to make the connection between intervals (Section 3.6), and determine the entry and exit times, and conditions to enter constrained intervals. To this end, we use continuity conditions to examine transitions from unconstrained to constrained interval. Transversality conditions enable to show that there exists a critical initial inventory level beyond which it is optimal to idle on the entire time horizon, as detailed in Section 3.7. Finally, in Section 3.8, we synthesize the result from the entire section to provide a solution method providing the optimal controls over time for a given multiplier $\eta(t)$.
In the subsections to follow we discuss these steps with further details.

3.2 Necessary conditions for optimality

The reader should refer to Appendix A for more details on the results from optimal control theory we will use.
We note that after dualizing the capacity constraint, the problem separates across products and yields N subproblems with no production capacity constraint. However, the problem remains unchanged if we add the constraint that each production rate does not exceed the total production capacity.

We express the Hamiltonian function as follows:

$$H(I,p,u,q,t) = \sum_{i=1}^{N} \Big(p_i\big(\alpha_i(t) - \beta_i(t)p_i\big) - f_i(u_i) - h_i(t)I_i + q_i\big(u_i - \alpha_i(t) + \beta_i(t)p_i\big)\Big)$$

where the arguments I, p, u, q are vectors with N components and t is the time argument.
The Lagrangian function relaxing the no backorder constraints and the capacity constraint can then be written as:

$$L(I,p,u,q,\rho,\eta,t) \;=\; H(I,p,u,q,t) + \sum_{i=1}^{N} \rho_i\Big(u_i - \alpha_i(t) + \beta_i(t)p_i\Big) + \eta\big(K(t) - \sum_{i=1}^{N} u_i\big) \qquad (8)$$

where the arguments I, p, u, q, ρ are vectors with N components, argument η is a non negative real number, and t is the time argument.
Notice that we dualized only the difficult constraints, i.e. the capacity constraint and the no back-order constraint, and not those that bound the admissible controls.

Using Theorem 2 in Appendix A (note that Lemmas 3, 4 and 5 in Appendix B show that the assumptions of Theorem 2 hold), at the optimal solution,

- The state trajectory satisfies:

$$\begin{aligned} I_i^*(0) &= I_i^0, \quad i = 1,\ldots,N \\ I_i^*(t) &\geq 0, \quad \forall t \in [0,T], \quad i = 1,\ldots,N \\ \dot{I}_i^*(t) &= u_i^*(t) - \alpha_i(t) + \beta_i(t)p_i^*(t), \quad \forall t \in [0,T], \quad i = 1,\ldots,N. \end{aligned}$$

- The optimal control on $[0,T]$ is then given as a function of the adjoint variable and Lagrange multipliers by:

$$(p^*(t), u^*(t)) = \arg \max_{(p,u)\in W(t)} L(I^*(t), p, u, q(t), \rho(t), \eta(t)), \tag{9}$$

where $W(t)$ is the set of admissible controls (p,u) such that:

$$0 \leq u_i \leq K(t), \quad i = 1,\ldots,N,$$
$$0 \leq p_i \leq \frac{\alpha_i(t)}{\beta_i(t)}, \quad i = 1,\ldots,N.$$

- Additional feasibility constraints on $[0,T]$ include constraint (4), i.e.:

$$\sum_{i=1}^{N} u_i^*(t) \leq K(t),$$

as well as

$$\dot{I}_i^*(t) = 0 \ \ \forall i, \ t \text{ such that } I_i^*(t) = 0.$$

- Complementary slackness conditions on $[0,T]$ give rise to:

$$\begin{aligned} \eta(t)\big(K(t) - \sum_{i=1}^{N} u_i^*(t)\big) &= 0 \\ \rho_i(t)I_i^*(t) &= 0, \quad i = 1,\ldots,N \\ \dot{\rho}_i(t) &\leq 0 \text{ on boundary interval of } I_i^*(.), \quad i = 1,\ldots,N \\ \rho_i(t) &\geq 0, \quad i = 1,\ldots,N \\ \eta(t) &\geq 0, \quad i = 1,\ldots,N. \end{aligned}$$

- The vector of adjoint variables $q(.)$ satisfies the adjoint equation almost everywhere (i.e. except at the entry times to the boundary condition $I_i^*(t) = 0$):

$$\begin{aligned} \dot{q}_i(t) &= -\nabla_{I_i} \ L(I^*(t), p^*(t), u^*(t), q(t), \rho(t), \eta(t), t) \\ &= h_i(t) \quad i = 1,\ldots,N, \end{aligned}$$

as well as transversality conditions[1]

$$(q_i + \rho_i)(T) \geq 0, \quad i = 1,\ldots,N$$
$$I_i^*(T)(q_i + \rho_i)(T) = 0, \quad i = 1,\ldots,N.$$

[1] Notice that the transversality conditions are written using the direct adjoining method.

- Finally, for $i = 1, \dots N$,
  - $I_i^*(.)$ is a continuous function of time;
  - $q_i(.)$ is a continuous function of time except at the entry times to the boundary condition[2] $I_i^*(t) = 0$;
  - $(q_i + \rho_i)(.)$ is a continuous function of time everywhere[3].

3.3 The optimal solution as a function of the multipliers and adjoint variables

Proposition 1. *Under Assumptions 2, 3 and 4, given* $q(.), \rho(.)$ *and* $\eta(.)$*, there exist at each time* $t \in [0, T]$*, unique optimal controls given by:*

$$\begin{aligned} p_i^*(t) &= arg \max_{0 \le p_i \le \frac{\alpha_i(t)}{\beta_i(t)}} \Big(\alpha_i(t) - \beta_i(t)p_i + (q_i(t) + \rho_i(t))\beta_i(t)\Big)p_i \\ &= \begin{cases} 0 & \text{if } \; q_i(t) + \rho_i(t) \le -\frac{\alpha_i(t)}{\beta_i(t)}, \\ \frac{1}{2}\Big(\frac{\alpha_i(t)}{\beta_i(t)} + q_i(t) + \rho_i(t)\Big) & \text{if } \; -\frac{\alpha_i(t)}{\beta_i(t)} < q_i(t) + \rho_i(t) \le \frac{\alpha_i(t)}{\beta_i(t)}, \\ \frac{\alpha_i(t)}{\beta_i(t)} & \text{otherwise}, \end{cases} \end{aligned} \tag{10}$$

$$\begin{aligned} u^*(t) &= arg \max_{u \ge 0} \Big(q_i(t) + \rho_i(t) - \eta(t)\Big)u_i - f_i(u_i) \\ &= \begin{cases} 0 & \text{if } \; q_i(t) + \rho_i(t) - \eta(t) \le f_i'(0), \\ f_i^{'-1}(q_i(t) + \rho_i(t) - \eta(t)) & f_i'(0) < q_i(t) + \rho_i(t) - \eta(t) \le f_i'(K(t)) \\ K(t) & \text{otherwise}. \end{cases} \end{aligned} \tag{11}$$

Proof. We solve optimization problem (9) in order to determine the optimal policy as a function of the multipliers and adjoint variables. We notice that the Lagrangian function is separable across products and in p_i and u_i (we have dualized the coupling constraints). Furthermore, it is a strictly concave continuously differentiable function in p_i and u_i (since for every product i, f_i is a strictly convex function, which implies that the function $u_i \mapsto (q_i(t) + \rho_i(t) - \eta(t))u_i - f_i(u_i)$ is strictly concave on $\mathbb{R}^+$). Moreover, the remaining constraints (those not dualized, which are constraints on the control variables only) are linear, i.e. they constrain the control variables within a convex set. Therefore, there are unique optimal controls $(u^*(t),\ p^*(t))$, which are the maximizers of the Lagrangian function over the feasible control variables. To compute them, we consider the partial derivatives of the Lagrangian function:

$$\begin{aligned} \frac{\partial L}{\partial u_i}(I, p, u, q, \rho, \eta, t) &= -f_i'(u_i) + q_i + \rho_i - \eta \\ \frac{\partial L}{\partial p_i}(I, p, u, q, \rho, \eta, t) &= \alpha_i(t) - 2\beta_i(t)p_i + \beta_i(t)q_i + \beta_i(t)\rho_i. \end{aligned}$$

To obtain the optimal solution, we proceed as follows. We first solve the equations setting these partial derivatives to zero. If the solution obtained lies within the set of feasible controls (defined by

[2]The adjoint variable may be discontinuous at the entry or exit times to constrained intervals. However, by convention, we impose continuity at the exit times. This allows to constrain the multiplier ρ to be non negative. See [32] for more details.

[3]This is a consequence from the theory of the direct adjoining method. See [46] or [49] for a justification.

the linear, not dualized constraints), then it is the optimal control. Otherwise, the optimal control lies on a boundary of the set of feasible controls, i.e. zero for production rates, and either zero or $\frac{\alpha_i(t)}{\beta_i(t)}$ for the price, depending on which value corresponds to the higher value of the Lagrangian.
Before ending the proof, we recall that we assumed f_i defined on $\mathbb{R}^+$ to be positive, strictly convex and increasing. Moreover, f_i' is also defined on $\mathbb{R}^+$ and is strictly increasing with range $[f_i'(0), +\infty)$. Therefore, on the one hand it is invertible and on the other hand $f_i'(u) \geq f_i'(0) \geq 0 \quad \forall u \geq 0$. Moreover, it implies that $f_i^{'-1}$ is positive valued and $f_i^{'-1}(u)$ is defined for $u \geq f_i'(0)$. The result then follows.

□

We can also derive the expression for $\dot{I}_i^*(t) = u_i^*(t) - \alpha_i(t) + \beta_i(t)p_i^*(t) = v_{i,t}(q_i(t) + \rho_i(t), \eta(t))$ where function $v_{i,t}$ is defined below:

$$v_{i,t}(x,\eta) = \begin{cases} -\alpha_i(t) & x < -\frac{\alpha_i(t)}{\beta_i(t)} \\ \frac{1}{2}\big(\beta_i(t)x - \alpha_i(t)\big) & -\frac{\alpha_i(t)}{\beta_i(t)} \leq x < \min\{f_i'(0) + \eta,\ \frac{\alpha_i(t)}{\beta_i(t)}\} \\ 0 & \frac{\alpha_i(t)}{\beta_i(t)} \leq x < f_i'(0) + \eta \\ f_i^{'-1}(x - \eta) & \max\{\frac{\alpha_i(t)}{\beta_i(t)},\ f_i'(0) + \eta\} \leq x < f_i'(K(t)) + \eta \\ f_i^{'-1}(x - \eta) + \frac{1}{2}\big(\beta_i(t)x - \alpha_i(t)\big) & f_i'(0) + \eta \leq x < \min\{\frac{\alpha_i(t)}{\beta_i(t)},\ f_i'(K(t)) + \eta\} \\ K(t) + \frac{1}{2}\big(\beta_i(t)x - \alpha_i(t)\big) & f_i'(K(t)) + \eta < x \leq \frac{\alpha_i(t)}{\beta_i(t)} \\ K(t) & x \geq \max\{\frac{\alpha_i(t)}{\beta_i(t)},\ f_i'(K(t)) + \eta\} \end{cases} \tag{12}$$

We notice that function $v_{i,t}$ depends only on the inputs. It is illustrated in Figure 1

We observe that $v_{i,t}(., \eta(t))$ is continuous, piecewise differentiable, non-decreasing, and taking values on $[-\alpha_i(t), K(t)]$.
Moreover, we recall that on a constrained interval, the inventory level is maintained at the constant level zero, thus its derivative is kept at zero. It is therefore of interest to study whether for some given multiplier η, function $v_{i,t}(., \eta)$ may take the value zero:
- if $\frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0) \leq \eta(t)$, then $v_{i,t}(., \eta(t))$ equals zero on interval $[\frac{\alpha_i(t)}{\beta_i(t)},\ f_i'(0) + \eta(t))$ (Figure 1 (a)). We notice that if $q_i(t) + \rho_i(t) \in [\frac{\alpha_i(t)}{\beta_i(t)},\ f_i'(0) + \eta(t)]$, its actual value is irrelevant as $u_i(t) = 0$ and $p_i(t) = \frac{\alpha_i(t)}{\beta_i(t)}$. This case is impossible for $\eta(t)$ sufficiently small however.
- if $\frac{\alpha_i(t) - 2K(t)}{\beta_i(t)} - f_i'(K(t)) \leq \eta(t) \leq \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0)$, $v_{i,t}(., \eta(t))$ has exactly one zero $s_{i,t}$, that is on $[f_i'(0) + \eta(t), \min\{\frac{\alpha_t}{\beta_t},\ f_i'(K(t)) + \eta(t)\})$, verifying

$$f_i^{'-1}(s_{i,t} - \eta(t)) + \frac{1}{2}\big(\beta_i(t)s_{i,t} - \alpha_i(t)\big) = 0$$

(Figure 1 (b) and (c)) - else, $v_{i,t}(., \eta(t))$ has exactly one zero $s_{i,t} = \frac{\alpha_i(t) - 2K(t)}{\beta_i(t)} \in [f_i'(K(t)) + \eta(t),\ \frac{\alpha_i(t)}{\beta_i(t)}]$ (Figure 1 (d)).

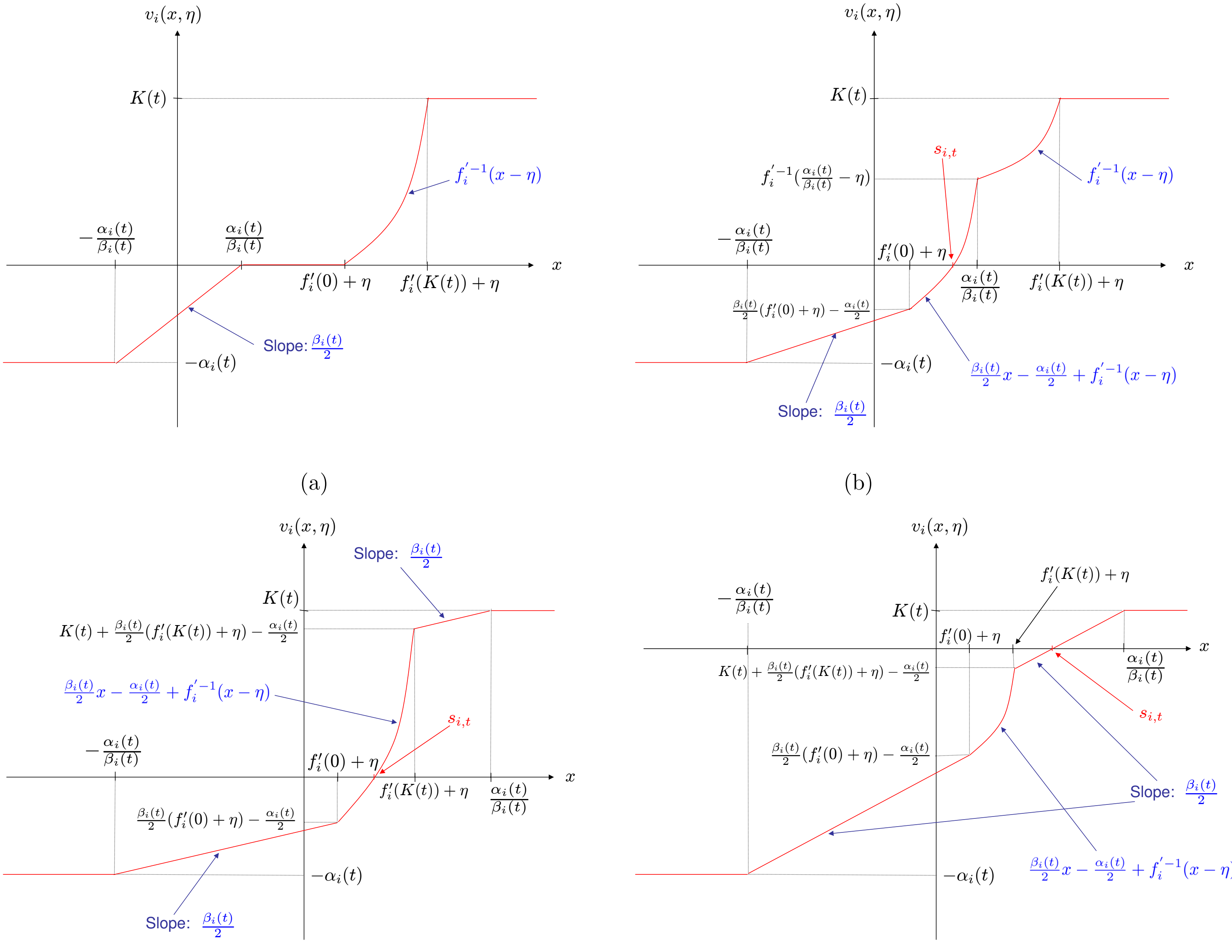


Figure 1: Plot of function $v_{i,t}(.,\eta)$ for (a) $\frac{\alpha_i(t)}{\beta_i(t)} < f_i'(0)+\eta < f_i'(K(t))+\eta$, or (b) $f_i'(0)+\eta < \frac{\alpha_i(t)}{\beta_i(t)} < f_i'(K(t))+\eta$, or (c),(d): $f_i'(0)+\eta < f_i'(K(t))+\eta < \frac{\alpha_i(t)}{\beta_i(t)}$

3.4 Constrained intervals

3.4.1 Preliminary results

Let

$$g_i(t,z) = z - f_i'\Big(\frac{\alpha_i(t) - \beta_i(t)z}{2}\Big)$$

defined for $t \in [0,T]$ and, (for a given value of t) for $z \in \big[\frac{\alpha_i(t)-2K(t)}{\beta_i(t)},\ \frac{\alpha_i(t)}{\beta_i(t)}\big]$.
We define $l_{i,t} : z \mapsto g_i(t,z)$.

Proposition 2. *Under Assumptions 2, 3 and 4, function $l_{i,t}(.)$ is an invertible mapping on $\big[\frac{\alpha_i(t)-2K(t)}{\beta_i(t)},\ \frac{\alpha_i(t)}{\beta_i(t)}\big]$ and its range for a fixed t is $\big(\frac{\alpha_i(t)-2K(t)}{\beta_i(t)} - f_i'(K(t)),\ \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0)\big]$.*

Proof. It is clear that $l_{i,t}(.)$ is continuous, strictly increasing and differentiable.
We compute

$$l_{i,t}'(z) = 1 + \frac{\beta_i(t)}{2} f_i''\big(\frac{\alpha_i(t) - \beta_i(t)z}{2}\big) > 0$$

since f_i is strictly convex, so $l_{i,t}(.)$ is strictly increasing (for a fixed t), and hence invertible. □

Corollary 1. *Under Assumptions 2, 3 and 4, given $0 \le \eta \le \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0)$, equation $v_{i,t}(z,\eta) = 0$ for argument z (and fixed t) has a unique solution $z_0 \equiv \phi_i(t,\eta)$ satisfying $f_i'(0) + \eta < \phi_i(t,\eta) \le \frac{\alpha_i(t)}{\beta_i(t)}$. Moreover, $\phi_i(.,.)$ is continuous in both arguments.*

We see that $\phi_i(t,\eta(t))$ represents $q_i(t) + \rho_i(t)$ expressed as a function of $\eta(t)$ for constrained products.

Proof.

first case: $\eta \ge \frac{\alpha_i(t)-2K(t)}{\beta_i(t)} - f_i'(K(t))$**:**

Since η is in the range of $l_{i,t}(.)$ (for a fixed t), using the previous proposition $l_{i,t}^{-1}(\eta)$ is well defined and unique. As a result, the solution of the equation is uniquely defined by $z_0 = l_{i,t}^{-1}(\eta) \in \big[\frac{\alpha_i(t)-2K(t)}{\beta_i(t)},\ \frac{\alpha_i(t)}{\beta_i(t)}\big]$ and we have $\eta = l_{i,t}(z_0) = g_i(t,z_0)$.
Let $\phi_i(t,\eta) = l_{i,t}^{-1}(\eta)$. In particular, since $l_{i,t}(.)$ is continuous, $\phi_i(.,.)$ is continuous in its second argument. The continuity of $\phi_i(.,.)$ with respect to its first argument follows from the fact that $g_i(.,.)$ is continuous in both arguments and from the relation $g_i(t,\phi_i(t,\eta)) - \eta = 0$.
Since

$$\eta = \phi_i(t,\eta) - f_i'\Big(\frac{\alpha_i(t) - \beta_i(t)\phi_i(t,\eta)}{2}\Big)$$

with the argument $\frac{\alpha_i(t)-\beta_i(t)\phi_i(t,\eta)}{2} \ge 0$ in the expression above, it follows that $\eta \le \phi_i(t,\eta) - f_i'(0)$.
We easily verify that $v_{i,t}(.,\eta)$ is non zero elsewhere.

second case: $\eta < \frac{\alpha_i(t)-2K(t)}{\beta_i(t)} - f_i'(K(t))$

The function $z \mapsto K(t) + \frac{1}{2}\big(\beta_i(t)z - \alpha_i(t)\big)$, defined on $[f_i'(K(t)) + \eta(t), \frac{\alpha_i(t)}{\beta_i(t)}]$ and taking its values on $[K(t) + \frac{1}{2}\big(\beta_i(t)(f_i'(K(t)) + \eta(t)) - \alpha_i(t)\big),\ K(t)]$ has exactly one zero $\phi_i(t,\eta) =$

$\frac{\alpha_i(t)-2K(t)}{\beta_i(t)} \in [f_i'(K(t)) + \eta(t),\ \frac{\alpha_i(t)}{\beta_i(t)}]$. We easily verify that $v_{i,t}(.,\eta)$ is non zero elsewhere. Continuity is clear. Since $f_i'(.)$ is increasing, $\phi_i(t,\eta) \geq f_i'(K(t)) + \eta(t)$ implies $\phi_i(t,\eta) \geq f_i'(0) + \eta(t)$.

$\square$

Corollary 2. ***(i)*** *Function $\phi_i(.,.)$ is piecewise continuously differentiable in its first argument and*

$$\frac{\partial \phi_i}{\partial t}(t,\eta) = \begin{cases} \frac{\frac{1}{2}(\alpha_i'(t)-\beta_i'(t)\phi_i(t,\eta))f_i''\left(\frac{\alpha_i(t)-\beta_i(t)\phi(t,\eta)}{2}\right)}{1+\frac{\beta_i(t)}{2}f_i''\left(\frac{\alpha_i(t)-\beta_i(t)\phi(t,\eta)}{2}\right)}, & \eta \geq \frac{\alpha_i(t)-2K(t)}{\beta_i(t)} - f_i'(K(t)) \\ \frac{(\alpha_i'(t)-2K'(t))\beta_i(t)-(\alpha_i(t)-2K(t))\beta_i'(t)}{\beta_i^2(t)}, & \eta < \frac{\alpha_i(t)-2K(t)}{\beta_i(t)} - f_i'(K(t)) \end{cases}$$

(ii) *Function $\phi_i(.,.)$ is piecewise continuously differentiable in its second argument and*

$$\frac{\partial \phi_i}{\partial \eta}(t,\eta) = \begin{cases} \frac{1}{1+\frac{\beta_i(t)}{2}f_i''\left(\frac{\alpha_i(t)-\beta_i(t)\phi(t,\eta)}{2}\right)}, & \eta \geq \frac{\alpha_i(t)-2K(t)}{\beta_i(t)} - f_i'(K(t)) \\ 0, & \eta < \frac{\alpha_i(t)-2K(t)}{\beta_i(t)} - f_i'(K(t)) \end{cases}$$

Proof. Clear in the second case $\eta < \frac{\alpha_i(t)-2K(t)}{\beta_i(t)} - f_i'(K(t))$. For the first case:
(i) Differentiability follows from the differentiability of $g(.,.)$ with respect to both arguments and from the relation $g_i(t,\phi_i(t,\eta)) - \eta = 0$. The expression is obtained by observing that the relation $g_i(t,\phi_i(t,\eta)) - \eta = 0$ implies by differentiating with respect to t:

$$\frac{\partial g_i}{\partial t}(t,\phi_i(t,\eta)) + \frac{\partial \phi_i}{\partial t}(t,\eta)\frac{\partial g_i}{\partial z}(t,\phi_i(t,\eta)) = 0$$

and since $\frac{\partial g_i}{\partial z}(t,\phi_i(t,\eta)) = l_{i,t}'(\phi_i(t,\eta))$,

$$\frac{\partial \phi_i}{\partial t}(t,\eta) = -\frac{\frac{\partial g_i}{\partial t}(t,\phi_i(t,\eta))}{l_{i,t}'(\phi_i(t,\eta))}$$

(ii) The result follows immediately from $\phi_i(t,\eta) = l_{i,t}^{-1}(\eta)$ and the expression of the derivative of $l_{i,t}(.)$. $\square$

Let $\psi_i(t) = \phi_i(t,0)$. In particular, function $\psi(.)$ is piecewise continuously differentiable.

3.4.2 Structure of the solution on constrained intervals

We observe that the production capacity constraint being non tight at time t implies $\eta(t) = 0$. We now provide some results on the structure of the optimal solution, in both cases of non tight and tight capacity constraint.

Proposition 3. *If $\eta(t) = 0$, then each constrained product i at time t is active and the optimal solution is*

- *if $\frac{\alpha_i(t)-2K(t)}{\beta_i(t)} - f_i'(K(t)) \leq 0$, then $\psi_i(t) = l_{i,t}^{-1}(0)$ and*

$$p_i^*(t) = \frac{1}{2}\Big(\frac{\alpha_i(t)}{\beta_i(t)} + \psi_i(t)\Big), \qquad u_i^*(t) = f_i'^{-1}(\psi_i(t)) = \frac{1}{2}\Big(\alpha_i(t) - \beta_i(t)(\psi_i(t))\Big)$$

- *if* $\frac{\alpha_i(t)-2K(t)}{\beta_i(t)} - f_i'(K(t)) > 0$, *then* $\psi_i(t) = \frac{\alpha_i(t)-2K(t)}{\beta_i(t)}$ *and*

$$p_i^*(t) = \frac{\alpha_i(t) - K(t)}{\beta_i(t)}, \qquad u_i^*(t) = K(t)$$

Proof. Consider product i that has a zero inventory level at time t. The condition $\dot{I}_i(s) = 0$ must hold on the interior of the current constrained interval (which includes time t) since the inventory level remains at the value 0. Therefore, on that interval, $q_i(t) + \rho_i(t) = \phi_i(t, 0) = \psi_i(t) \in [f_i'(0), \frac{\alpha_t}{\beta_t}]$, therefore the product is active. The optimal solution is then obtained using the preliminary results. □

This result means that, when the capacity constraint is not tight, the inventory level of a product is kept at zero by producing with a certain positive rate that balances out the amount being sold. When the capacity constraint cannot be ignored, this result may no longer be true as cumulatively, these necessary production rates may exceed the capacity.

The following results are directly derived from the preliminary results.

Proposition 4. *If* $\eta(t) \geq \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0)$ *and product* i *is constrained at time* t*, then*

$$u_i^*(t) = 0; \quad p_i^*(t) = \frac{\alpha_i(t)}{\beta_i(t)}.$$

In other words, when the capacity constraint is very hard to satisfy, i.e. the system would tend to want to produce a lot beyond the available capacity, then the inventory level of a product is kept at zero by not producing and pricing at the maximum (which yields a zero demand rate).

Lemma 1. *If* $\eta(t) > 0$*, then each constrained product* i *at time* t *is active if and only if*

$$\eta(t) < \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0).$$

Moreover, we then have $q_i(t) + \rho_i(t) = \phi_i(t, \eta(t))$.

Proposition 5. *If* $\frac{\alpha_i(t)-2K(t)}{\beta_i(t)} - f_i'(K(t)) \leq \eta(t) < \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0)$ *and product* i *is constrained at time* t*, then*

$$u_i^*(t) = f_i'^{-1}(\phi_i(t, \eta(t)) - \eta(t)) = \frac{1}{2}\big(\alpha_i(t) - \beta_i(t)\phi_i(t, \eta(t))\big); \qquad p_i^*(t) = \frac{1}{2}\Big(\frac{\alpha_i(t)}{\beta_i(t)} + \phi_i(t, \eta(t))\Big).$$

Proposition 6. *If* $0 < \eta(t) < \frac{\alpha_i(t)-2K(t)}{\beta_i(t)} - f_i'(K(t))$ *and product* i *is constrained at time* t*, then*

$$u_i^*(t) = K(t); \qquad p_i^*(t) = \frac{\alpha_i(t) - K(t)}{\beta_i(t)}.$$

These results illustrate that, when the capacity constraint is not overly constraining (as it is in Proposition 4), it is optimum to produce (rather than idle) and price so that the inventory level remains at zero.

3.5 Unconstrained intervals

We show below some results that hold under Assumptions 2, 3 and 4 and that connect the multipliers with the notions of unconstrained and active products.

Lemma 2. *For each unconstrained product i at time t with last exit time t_i^1 (possibly 0), the following equality holds:*

$$q_i(t) + \rho_i(t) = q_i(t) = q_i(t_i^1) + \int_{t_i^1}^{t} h_i(s)ds.$$

Proof. On the current unconstrained interval (including t, starting at t_i^1), the inventory level for product i is positive, therefore by complementary slackness, $\rho_i(.)$ takes value 0 on that interval. Moreover, the adjoint equation (which holds everywhere except at entry times[4]) gives $\dot{q}_i(s) = h_i(s)$. Since $q_i(.)$ is continuous at exit time t_i^1 implies in particular that this differential equation is valid on $[t_i^1, t]$, which gives rise to the result. □

Corollary 3. *For each unconstrained product i at time t with last exit time t_i^1 (possibly zero),*

$$u_i^*(t) = \begin{cases} 0 & \text{if } \; q_i(t) - \eta(t) \le f_i'(0), \\ f_i'^{-1}(q_i(t) - \eta(t)) & f_i'(0) < q_i(t) - \eta(t) \le f_i'(K(t)) \\ K(t) & \text{otherwise}, \end{cases}$$

$$p_i^*(t) = \begin{cases} 0 & \text{if } \; q_i(t) \le -\frac{\alpha_i(t)}{\beta_i(t)}, \\ \frac{\alpha_i(t)}{2\beta_i(t)} + \frac{1}{2} q_i(t) & \text{if } \; -\frac{\alpha_i(t)}{\beta_i(t)} \le q_i(t) \le \frac{\alpha_i(t)}{\beta_i(t)}, \\ \frac{\alpha_i(t)}{\beta_i(t)} & \text{if } \; q_i(t) \ge \frac{\alpha_i(t)}{\beta_i(t)}, \end{cases}$$

where $q_i(t) \equiv q_i(t_i^1) + \int_{t_i^1}^{t} h_i(s)ds$.

3.6 Transition between constrained and unconstrained intervals

We now attempt to view the problem globally on the time horizon rather than instantaneously as was done so far. We explained earlier in the paper that the horizon is divided in a sequence of constrained and unconstrained intervals (starting with an unconstrained one if $I_i^0 > 0$, and with a constrained one if $I_i^0 = 0$). In this section, we show some properties regarding entering and exiting constrained and unconstrained intervals.

The following result is a direct extension of Lemma 1.

Corollary 4. *If a product i enters (resp. exits) a constrained interval at a time τ such that $\eta(\tau) < \frac{\alpha_i(\tau)}{\beta_i(\tau)} - f_i'(0)$, it does so as an active product.*

This result means that, unless the capacity is overly constraining, it is necessary to be producing in order to enter a phase where the inventory level is kept at zero, even if the product is not particularly profitable. The other alternative would be to increase its price close to the maximum in order to lower the demand close to zero and avoid the inventory level to decrease further and reach zero, but this would yield almost no revenue as the revenue is proportional to the demand.

[4] When writing the necessary conditions for optimality, by convention, the adjoint variables may be discontinuous only at entry times of constrained intervals.

Proposition 7. *If an unconstrained product i enters a constrained interval at time τ such that $\eta(\tau) < \frac{\alpha_i(\tau)}{\beta_i(\tau)} - f_i'(0)$ then it is active at time τ and the entry time τ is determined by*

$$\lim_{t \to \tau^-} q_i(t) = \phi_i(\tau, \eta(\tau)).$$

Proof. Activity follows from Lemma 1. The entry condition results from continuity of $q_i + \rho_i$ at the entry time. □

Proposition 8. *If an unconstrained product i enters a constrained interval at time τ such that $\eta(\tau) \geq \frac{\alpha_i(\tau)}{\beta_i(\tau)} - f_i'(0)$ then it is inactive at time τ and the entry time τ is determined by*

$$\lim_{t \to \tau^-} q_i(t) = \frac{\alpha_i(\tau)}{\beta_i(\tau)}$$

Proof. Inactivity follows from Proposition 4. Suppose i is unconstrained on interval $[\tau - \delta, \tau)$, and is constrained inactive on $[\tau, \tau + \delta']$, where $\delta,\ \delta' > 0$ and $\eta(\tau) \geq \frac{\alpha_i(\tau)}{\beta_i(\tau)} - f_i'(0)$. Since τ is the time when product i becomes constrained, i.e. when the inventory level goes from being positive to being equal to zero, we assume without loss of generality that δ is small enough so that we have $\dot{I}_i(t) < 0$ on $[\tau - \delta, \tau)$. (Otherwise, we decrease δ: since product i is unconstrained on $[\tau - \delta, \tau)$, it is unconstrained on any interval included in $[\tau - \delta, \tau)$.) Since i is inactive constrained on $[\tau, \tau + \delta']$, we have $p_i^*(t) = \frac{\alpha_i(t)}{\beta_i(t)}$ on that interval and thus $q_i(t) + \rho_i(t) \geq \frac{\alpha_i(t)}{\beta_i(t)}$ on $[\tau, \tau + \delta']$.
Suppose that $q_i(\tau^+) + \rho_i(\tau^+) > \frac{\alpha_i(\tau^+)}{\beta_i(\tau^+)}$. Continuity of $q_i + \rho_i$ implies that, for δ small enough, $q_i(t) + \rho_i(t) = q_i(t) \geq \frac{\alpha_i(t)}{\beta_i(t)}$ on $[\tau - \delta, \tau)$, and thus $p_i(t) \geq \frac{\alpha_i(t)}{\beta_i(t)}$. This yields $\dot{I}_i(t) = u_i(t) \geq 0$ which leads to a contradiction. Therefore $q_i(\tau^+) + \rho_i(\tau^+) = \frac{\alpha_i(\tau^+)}{\beta_i(\tau^+)}$. Continuity of $q_i + \rho_i$ implies the result. □

Remark. This result implies that a constrained interval begins and ends as active, unless the capacity is overly constraining. While active, $q_i(t) + \rho_i(t) = \phi_i(t, \eta(t))$, in particular at the entry time. It is possible that during the course of the constrained interval, the product becomes inactive (if $\eta(t) \geq \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0)$), in which case the optimal policy is known - but $q_i(t) + \rho_i(t)$ is undetermined. A product may enter a constrained interval as inactive if the capacity is very constraining. Then the entry time is determined by q_i reaching $\frac{\alpha_i(.)}{\beta_i(.)}$.

Let $\tilde{\phi}_i(t) \equiv \phi_i(t, \eta(t))$. Since $\eta(.)$ is piecewise differentiable, and $\phi_i(,.,)$ is piecewise differentiable with respect to both arguments, then $\tilde{\phi}_i(.)$ is piecewise differentiable. We have

$$\frac{d\tilde{\phi}_i}{dt}(t) = \frac{\partial \phi_i}{\partial t}(t, \eta(t)) + \eta'(t)\frac{\partial \phi_i}{\partial \eta}(t, \eta(t))$$

and we gave the expression of those partial derivatives earlier in the paper.

We will call *transitive time* for product i a time such that

$$\begin{cases} \lim_{s \to t^-} \frac{d\tilde{\phi}_i}{dt}(s) \leq h_i(t) \\ \eta(t) \leq \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0) \end{cases}$$

or

$$\begin{cases} \lim_{s\to t^-} \frac{d\frac{\alpha_i}{\beta_i}}{dt}(s) \le h_i(t) \\ \eta(t) > \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0) \end{cases}$$

Proposition 9. *Product i may enter a constrained interval only at a transitive time.*

This is simply saying that the *transition* from an unconstrained interval to a constrained interval may occur only at a transitive time. Intuitively, this is due to the fact that it may not be possible to optimally maintain the inventory level of a product at zero under any circumstances.

Proof. Upon entering a constrained interval, $q_i(t) + \rho_i(t) = \phi_i(t, \eta(t)) = \tilde{\phi}_i(t)$ if $\eta(t) \le \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0)$ and $q_i(t) + \rho_i(t) = \frac{\alpha_i(t)}{\beta_i(t)}$ otherwise. By taking derivative with respect to t and using the adjoint equation as well as the fact that $\dot{\rho}_i(t) \le 0$, the result follows. □

In the following, we denote

$$\theta_i(t, \eta(t)) \equiv \begin{cases} \phi_i(t, \eta(t)), & \eta(t) \le \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0) \\ \frac{\alpha_i(t)}{\beta_i(t)}, & \eta(t) > \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0) \end{cases}$$

and $\tilde{\theta}_i(t) \equiv \theta_i(t, \eta(t))$, so that on a constrained interval, $q_i(t) + \rho_i(t)$ follows $\theta_i(t, \eta(t))$, and the condition to enter a constrained interval is (i) $q_i(t)$ intersects $\theta_i(t, \eta(t))$, and (ii) the total derivative with respect to time of $\theta_i(t, \eta(t))$ does not exceed $h_i(t)$ (transitive time).

3.7 A view of the entire time horizon

Proposition 10. *If a product i has a positive level of inventory at time T, under Assumptions 3, and 4, then the inventory level is positive throughout the entire time horizon.*

Proof. Consider a product i that has a positive inventory level at time T. This means by complementary slackness that $\rho_i(T) = 0$. Moreover, using the transversality conditions, it follows that $q_i(T) + \rho_i(T) = q_i(T) = 0$. Suppose that the inventory level of that product has reached zero at some point within the time horizon. Let $\tau < T$ be the last exit time from a constrained interval. We have $I_i(\tau) = 0$ and $I_i(t) > 0 \ \ \forall t \in (\tau, T]$, therefore $\rho_i(t) = 0 \ \ \forall t \in (\tau, T]$. In particular, $\rho_i(\tau^+) = 0$. Then, since $q_i + \rho_i$ is continuous everywhere and since product i is constrained at time τ^-,

$$0 \le f_i'(0) < \theta_i(\tau^-, \eta(\tau^-)) = q_i(\tau^-) + \rho_i(\tau^-) = q_i(\tau^+) + \rho_i(\tau^+) = q_i(\tau^+)$$

and therefore, using the adjoint equation valid on $[\tau, T]$,

$$q_i(T) = q_i(\tau^+) + \int_\tau^T h_i(s)ds > q_i(\tau^+) > 0.$$

This is a contradiction. □

We notice that this result makes sense at an intuitive level. There is no reward at the end of the time horizon for any remaining inventory. Moreover, incurring inventory that is not sold incurs cost but not revenue. Therefore, if the retailer follows an optimal pricing and production policy, she will not incur any inventory that will not be sold by time T. As a result, if there is some remaining inventory at time T, it means that this inventory was not incurred by some additional production, but was incurred from the initial inventory. In other words, no production took place throughout the entire time horizon and therefore, since there is some inventory at time T, the inventory level was positive all along.

Corollary 5. *There exists $\bar{I}_i > 0$ defined as $\bar{I}_i \equiv -\int_0^T w_i(t)dt > 0$ such that*

$$I_i^0 > \bar{I}_i \iff \textit{product } i \textit{ is unconstrained on the entire time horizon including time } T,$$

where

$$0 \geq w_i(t) = \begin{cases} -\alpha_i(t) & \textit{if } G_i(t) < -\frac{\alpha_i(t)}{\beta_i(t)} \\ \frac{1}{2}\Big(-\alpha_i(t) + \beta_i(t)G_i(t)\Big) & \textit{if } G_i(t) \geq -\frac{\alpha_i(t)}{\beta_i(t)} \end{cases}$$

$$G_i(t) = -\int_t^T h_i(s)\ ds \leq 0.$$

Moreover, in that case product i is inactive on the entire time horizon.

Proof. If a product i is such that $I_i^*(t) > 0, \quad \forall t \in [0,T]$, then $\rho_i(t) = 0, \quad \forall t \in [0,T]$ and for this product i there is a unique unconstrained interval, on which the adjoint equation is valid.
Since $q_i(t) = q_i(0) + \int_0^t h_i(s)ds$ and $q_i(T) = 0$, it follows that $q_i(0) = -\int_0^T h_i(s)ds$ and therefore

$$q_i(t) + \rho_i(t) = q_i(t) = -\int_t^T h_i(s)ds \equiv G_i(t) \leq 0 \leq f_i'(0), \quad \forall t \in [0,T].$$

Therefore, $u_i^*(t) = 0, \quad \forall t \in [0,T]$ and

$$p_i^*(t) = \begin{cases} 0 & \text{if } G_i(t) < -\frac{\alpha_i(t)}{\beta_i(t)} \\ \frac{1}{2}\Big(G_i(t) + \frac{\alpha_i(t)}{\beta_i(t)}\Big) & \text{if } G_i(t) \geq -\frac{\alpha_i(t)}{\beta_i(t)}. \end{cases} \tag{13}$$

We will denote $\mathcal{P}$ this pricing and production policy on $[0,T]$.
Therefore, since $\dot{I}^*{}_i(t) = u_i^*(t) - \alpha_i(t) + \beta_i(t)p_i^*(t)$, it follows that

$$\dot{I}_i^*(t) = \begin{cases} -\alpha_i(t) & \text{if } G_i(t) < -\frac{\alpha_i(t)}{\beta_i(t)} \\ \frac{1}{2}\Big(-\alpha_i(t) + \beta_i(t)G_i(t)\Big) & \text{if } G_i(t) \geq -\frac{\alpha_i(t)}{\beta_i(t)} \end{cases}$$

i.e. $\dot{I}_i^*(t) = w_i(t)$. Moreover,

$$0 < I_i^*(T) = I_i^0 + \int_0^T \dot{I}_i^*(t)dt \equiv I_i^0 - \bar{I}_i.$$

For the converse, suppose that the inventory level reaches zero within the time horizon. Let $\tau \leq T$ the first time the inventory level becomes equal to zero. On the first unconstrained interval $[0,\tau)$, since $I_i^*(t) > 0$, by complementary slackness $\rho_i(t) = 0, \quad \forall t \in [0,\tau)$. The adjoint equation then implies

$$q_i(t) = q_i(\tau^-) - \int_t^\tau h_i(s)ds \quad \forall t \in [0,\tau).$$

The adjoint variable $q_i(.)$ may be discontinuous at entry time τ, but $(q_i + \rho_i)(.)$ is continuous and in particular at the entry to the constrained interval, $(q_i + \rho_i)(\tau) = \phi_i(\tau, \eta(\tau)) > 0$.
Continuity of $(q_i + \rho_i)(.)$ along with the fact that $\rho_i(\tau^-) = 0$ then imply

$$q_i(\tau^-) = (q_i + \rho_i)(\tau^-) = \theta_i(\tau, \eta(\tau)) > 0.$$

Therefore,

$$q_i(t) = q_i(\tau^-) - \int_t^\tau h_i(s)ds > -\int_t^\tau h_i(s)ds > -\int_t^T h_i(s)ds = G_i(t) \quad \forall t \in [0, \tau).$$

Furthermore, we observe that policy $\mathcal{P}$ yields a multiplier $q_i(t)$ equal to $G_i(t)$ and a derivative of the inventory level equal to $w_i(t)$. We notice in expression (12) that the derivative of the inventory level on an unconstrained interval is non decreasing with $q_i(t)$. As a result, $\dot{I}_i^*(t) > w_i(t) \;\; \forall t \in [0, \tau)$. In other words, in this case the inventory does not decrease as fast on $[0, \tau]$ as with policy $\mathcal{P}$, but the entire initial inventory level is consumed by time τ. However, $\bar{I}_i$ represents the total inventory consumed on $[0, T]$ with policy $\mathcal{P}$. More rigorously, $\dot{I}_i^*(t) > w_i(t) \;\; \forall t \in [0, \tau)$ implies

$$-\bar{I}_i = \int_0^T w_i(t)dt \leq \int_0^\tau w_i(t)dt < \int_0^\tau \dot{I}_i^*(t)dt = -I_i^0$$

i.e. $I_i^0 > \bar{I}_i$. □

Remark: If $I_i^0 = \bar{I}_i$, then the same policy holds and we obtain $I_i(T) = 0$. (The inventory level reaches zero for the first time at time T, and the optimal strategy is given by policy $\mathcal{P}$.)

This result suggests that there exists for each product a critical value of the initial inventory level above which it is optimal to never produce on the entire time horizon. This critical value depends only on the demand parameters and the holding cost of that product.

This result will also be used in its negative form, i.e. if $I_i^0 < \bar{I}_i$ then the inventory level of product i reaches zero on $[0, T]$, and is at zero level at the end of the time horizon T. We will distinguish two possible cases then:

case a: the inventory level of product i reaches zero for the first time before the end of the time horizon T, i.e. enters a constrained interval of non zero length within the time horizon, and as we proved earlier it is on a constrained interval at the end of the time horizon T

case b: the inventory level of product i reaches zero for the first time at the end of the time horizon T (without entering a constrained interval). Then the product is unconstrained on $[0, T)$, and the initial inventory level is totally consumed by the end of the time horizon T.

We will refer to these two cases in the remaining of the paper and the description of the algorithm. Note that if $I_i^0 = \bar{I}_i$, the inventory level also reaches zero for the first time at time T (like in case **b**) but the optimal strategy is to idle while in case **b** the optimal strategy will not be to idle in general.

3.8 Solving the model

In what follows, we describe how to derive the optimal solution for a given multiplier $\eta(.)$

First we test whether each product has an initial inventory level high enough (i.e. higher that $\bar{I}_i$) so that it is optimal to never produce them. The pricing policy is as shown in the proof of Corollary 5.

Otherwise, we know that the inventory level reaches zero by time T.

As we discussed earlier in the paper, once for all i the value of $q_i(.) + \rho_i(.),\ i = 1, \ldots, N$, is known, assuming that $\eta(.)$ is given, the optimal pricing and production policies p_i^* and u_i^* are easy to compute.

To determine $q_i(.) + \rho_i(.),\ i = 1, \ldots, N$, we have to solve N single product problems. We proceed as follows for each product i:

Step 1: (first unconstrained interval)

If there is a non zero initial inventory level I_i^0, we start on an unconstrained interval. (If there is no initial inventory level, we start on a constrained interval: set $t_i^0 = 0$ and go to Step 2.)

On that unconstrained interval, $\rho_i(t) = 0$ and $q_i(t) + \rho_i(t) = q_i(t)$. Using the adjoint equation, the value of $q_i(t) + \rho_i(t)$ on that interval can be determined as a function of time t and the initial value of the adjoint variable $q_i^0 \equiv q_i(0)$. Precisely, we have

$$q_i(t) = q_i^0 + \int_0^t h_i(s)ds.$$

Supposing we are in case **a**, this interval ends at the first entry time t_i^0, the time when the product becomes constrained. By continuity of $(q_i + \rho_i)(.)$, we have

$$q_i(t_i^{0-}) = (q_i + \rho_i)(t_i^{0-}) = (q_i + \rho_i)(t_i^{0+}) = \theta_i(t_i^0, \eta(t_i^0)).$$

To determine simultaneously q_i^0 and t_i^0, we solve the nonlinear system of equations and an inequality that ensures that the change of inventory on $[0, t_i^0]$ equals $-I_i^0$ and that the adjoint variable intersects for the first time at a transitive time the function $\tilde{\theta}_i(.)$ at time t_i^0.

More specifically, we attempt to solve the following system of two equations for t_i^0 and q_i^0 such that t_i^0 is the smallest positive number satisfying the equations and the inequality:

$$\begin{aligned} \int_0^{t_i^0} \dot{I}_i(t)dt &= -I_i^0 \\ q_i^0 + \int_0^{t_i^0} h_i(s)ds &= \theta_i(t_i^0, \eta(t_i^0)) \\ \tilde{\theta}_i'(t_i^0, \eta(t_i^0)) &\leq h_i(t_i^0) \end{aligned}$$

where $\dot{I}_i(t)$ is given by expression (12) in which we use $q_i(t) = q_i^0 + \int_0^t h_i(s)ds$.

If this system has a solution, once we have solved this system, we know the bound of the first unconstrained interval and the expression of $q_i(t) + \rho_i(t) = q_i(t)$ on that interval, and we can calculate the optimal solution.

If this system has no solution, we are in case **b** and we must only determine q_i^0 such that

$$\int_0^T \dot{I}_i(t)dt = -I_i^0.$$

In particular, $q_i(t)$ does not reach $\theta_i(t,\eta(t))$ on $[0,T]$ in that case. Then $[0,T)$ is unconstrained and we have determined $q_i(t)$ on that interval, so we can calculate the optimal strategy on the entire time horizon.

Step 2: (constrained interval and following unconstrained interval)
On a constrained interval, the trajectory of $q_i+\rho_i$ follows that of θ_i. In order to determine whether this constrained interval is followed by another unconstrained interval, we will attempt to compute the exit time t_i^1 ($> t_i^0$) of this constrained interval, and the next entry time t_i^2 ($> t_i^1$) (if there is another unconstrained interval, it must be followed by a constrained interval since all products are constrained at time T). If we find no solution we will conclude that product i remains constrained until the end of the time horizon.
We first suppose that there is an unconstrained interval (t_i^1, t_i^2). We have $\rho_i(t) = 0$ and $q_i(t) + \rho_i(t) = q_i(t) \;\; \forall t \in (t_i^1, t_i^2)$. Using the adjoint equation, the value of $q_i(t) + \rho_i(t)$ on that interval can be determined as a function of time t and the initial value of the adjoint variable $q_i(t_i^1)$. Using the necessary conditions $\rho_i(t) = 0 \;\; \forall t \in (t_i^1, t_i^2)$, $\quad (q_i+\rho_i)(t) = \theta_i(t,\eta(t))$ on the constrained interval (t_i^0, t_i^1), and the continuity of $(q_i+\rho_i)(.)$, we obtain

$$q_i(t_i^{1+}) = (q_i+\rho_i)(t_i^{1+}) = (q_i+\rho_i)(t_i^{1-}) = \theta_i(t_i^1, \eta(t_i^1)).$$

$$q_i(t_i^{2-}) = (q_i+\rho_i)(t_i^{2-}) = (q_i+\rho_i)(t_i^{2+}) = \theta_i(t_i^2, \eta(t_i^2)).$$

We then attempt to solve the nonlinear system of equations that ensures that the change of inventory on $[t_i^1, t_i^2]$ equals zero and that the adjoint variable intersects for the first time on a transitive interval the function $\tilde{\theta}_i(.)$ at time t_i^2.
More specifically, we want to solve the following system of two equations for t_i^1 and t_i^2 such that $t_i^1 \in [t_i^0, T]$ and t_i^2 is the smallest number in $[t_i^1, T]$ satisfying the equations and the inequality:

$$\begin{aligned}
\int_{t_i^1}^{t_i^2} \dot{I}_i(t)dt &= 0 \\
\theta_i(t_i^1, \eta(t_i^1)) + \int_{t_i^1}^{t_i^2} h_i(s)ds &= \theta_i(t_i^2, \eta(t_i^2)) \\
\tilde{\theta}_i'(t_i^2, \eta(t_i^2)) &\leq h_i(t_i^2)
\end{aligned}$$

where $\dot{I}_i(t)$ is given by expression (12) in which we use $q_i(t) = \theta_i(t_i^1, \eta(t_i^1)) + \int_{t_i^1}^t h_i(s)ds$.
If we can solve this system, we know the bounds on the constrained interval and the following unconstrained interval. Moreover, we have

$$q_i(t) + \rho_i(t) = \theta_i(t, \eta(t)) \;\; \forall t \in (t_i^0, t_i^1)$$

(unless $\eta(t)$ exceeds $\frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0)$, in which case we use Proposition 4)

$$q_i(t) + \rho_i(t) = \theta_i(t_i^1, \eta(t_i^1)) + \int_{t_i^1}^{t} h_i(s)ds \quad \forall t \in (t_i^1, t_i^2)$$

which we can calculate and as a result we can obtain the optimal policy on these intervals. We set $t_i^0 \leftarrow t_i^2$ and we repeat Step 2.
If we cannot solve this system we conclude that product i is constrained on $(t_i^0, T]$. We set $t_i^1 = T$ and stop.

We assumed that the number of junction times is finite (see Assumption 1), so that this process iterates only a finite number of times (i.e. there is a finite number of constrained and unconstrained intervals).

4 Heuristic algorithm for determining Lagrange multiplier $\eta(.)$

The value that multiplier $\eta(.)$ takes is determined by how difficult it is to satisfy the capacity constraint. If the constraint is not tight, the multiplier takes value zero according to the complementary slackness condition. Conversely, if satisfying it is very constraining (i.e. if, in an uncapacitated setting, the product would cumulatively optimally require a production rate much greater than the capacity rate), then the multiplier is high. We provide in [1] a method for computing exactly $\eta(.)$. The method is valid when inputs are steady enough so that (i) the number of times a product goes from being inactive to active (and vice-versa), and (ii) the number of times the capacity constraint goes from being tight to being non tight (and vice-versa), are rather small. the method may be quite complex, especially when the number of products increases.
In this paper, we introduce an approach that can be simply applied to any inputs and any number of products. We use a heuristics based on an iterative trial and error approach of Everett [23] that was used for instance by Bertsimas and Patterson [9]. The idea is to determine multiplier $\eta(.)$ via an iterative algorithm based on the following idea. For a given non negative $\eta(.)$, the previous section describes how to derive the optimal solution. If for all times t, the capacity constraint

$$\sum_{i=1}^{N} u_i(t) \leq K(t)$$

and complementary slackness condition

$$\eta(t)\Big(\sum_{i=1}^{N} u_i(t) - K(t)\Big) = 0$$

are satisfied, the procedure ends. Otherwise, we need to update the value of multiplier $\eta(.)$, and iterate the process. More precisely, based on the intuition behind the meaning of that multiplier, at times when the capacity constraint is violated, the multiplier will be increased. At times when the capacity constraint is satisfied and non tight with a positive multiplier, the multiplier will be decreased (while remaining non negative).

To this end, we update the multiplier at a finite number of time instants, according to some time discretization, and at the next iteration we consider the corresponding piecewise constant function for the new multiplier. At convergence of the algorithm, we consider the final piecewise constant multiplier function as an approximation to the optimal multiplier. Alternatively, we can use some smoothing technique to find a function that goes through all points where the multiplier has been updated.

Specifically, let's consider S a discretization of $[0,T]$, i.e. $S = \{\frac{j}{M}T,\ j = 1,\dots,M\}$ where M is an integer.
At iteration k of the algorithm, multiplier $\eta^{k-1}(t),\ t \in [0,T]$ is given. We use the procedure outlined in the previous section to obtain the corresponding optimal solution $u_i^{k-1}(t),\ p_i^{k-1}(t),\ t \in [0,T],\ i = 1,\dots,N$. We compute $\Delta^{k-1}(t) = K(t) - \sum_{i=1}^{N} u_i^{k-1}(t),\ t \in [0,T]$.
We update the multiplier as follows:

- for all times $t \in S$ such that $\Delta^{k-1}(t) \geq 0$, we define $\eta^k(t) = \max\{\epsilon', (1-\delta_t^{k-1})\eta^{k-1}(t)\}$
- for all times $t \in S$ such that $\Delta^{k-1}(t) < 0$, we define $\eta^k(t) = (1+\delta_t^{k-1})\eta^{k-1}(t)$

(ϵ' being a positive real number close to zero. It allows to ensure $\eta^k(t)$ remains non negative. We do not use zero to avoid $\eta^k(t)$ to be kept at zero for all following iterations if δ_t^{k-1} was ever greater than 1.)
The $\eta^k(t)$ is defined on $[0,T]$ as the piecewise constant function: for $t_0 \equiv \frac{j}{M}T < t < \frac{j}{M}T$, $\eta^k(t) = \eta^k(t_0)$.
The parameters $\delta_t^k, t \in S$ are updated using the following rule:

- if $\Delta^k(t)\Delta^{k-1}(t) > 0$, we define $\delta^{k+1}(t) = \epsilon_1\delta^k(t)$
- if $\Delta^k(t)\Delta^{k-1}(t) < 0$, we define $\delta^{k+1}(t) = \epsilon_2\delta^k(t)$
- if $\Delta^k(t)\Delta^{k-1}(t) = 0$, we define $\delta^{k+1}(t) = \delta^k(t)$

The values of ϵ_1 and ϵ_2 are fixed parameters where $\epsilon_1 > 1$ and $\epsilon_2 < 1$. The intuition behind this method is as follows. If the capacity constraint is violated at some time t ($\Delta^{k-1}(t) < 0$), the cumulative production rate across products is too high at that time given the available production capacity. Therefore, we increase the Lagrange multiplier to further penalize the violation. Likewise, if the constraint is not violated, we decrease the Lagrange multiplier. The amount of increase or decrease of the multiplier (step size) is updated at each iteration.
If the capacity constraint at a given time is repeatedly unsatisfied then the step size is gradually increased as the value of $\eta^k(t)$ may still be quite far from its optimal value. If the constraint fluctuates between feasibility and infeasibility, then the step size is reduced substantially as the value of $\eta^k(t)$ has come close to its optimal value. It is interesting to note that updating the Lagrange multipliers depends only upon whether or not the constraint was satisfied, not on the magnitude of the difference.

Stopping criterion: we stop when the complementary slackness condition at each time of the discretized time horizon is satisfied within some ϵ:

$$\forall t \in S,\ |\eta^k(t)\Delta^k(t)| < \epsilon.$$

In the next section, we gain insight by investigating how the fixed term of the demand, the price sensitivities, and the capacity constraint affect the optimal solution. Moreover, we discuss in detail the performance of the heuristics determining multiplier $\eta(t)$ one a particular scenario of input parameters.

5 Computational Results and Insights

5.1 Example 1: impact of a demand peak and of the capacity constraint

5.1.1 Input parameters

In order to illustrate our results, we consider an example with 2 products and 3 different maximum demand scenarios (coefficient $\alpha_i(.)$), on a time horizon $[0, 10]$. In each scenario, we let the capacity take 3 different values chosen as we illustrate below. For each demand scenario we keep the capacity constant throughout the time horizon. For simplicity, and in a similar fashion as in numerical results from the literature (see [13], [27], [30], [44], [45], [52]), we consider coefficients $\beta_i(t)$ (describing the elasticity of the demand with respect to the price) and holding cost coefficients that are constant. We also assume that the production cost is quadratic, that is,

$$f_i(u_i) = \frac{\gamma_i}{2} u_i^2,$$

with coefficients $\gamma_i,\ i = 1,\ 2$, constant.
The inputs chosen are summarized in the following table:

	β	h	γ	I^0
product 1	1	1	10	10
product 2	1	2	20	10

Product 1 has smaller holding and production costs, but both products start with the same initial inventory and their demands have the same sensitivity to price. This is to ease the comparison of results.

In a similar fashion as in the literature, we model the maximal demand (coefficient α) increasing on the first half of the time horizon and decreasing on the second half to study the effect of a demand peak in the middle of the time horizon. We will consider 3 scenarios. In all scenarios, the average demand for both products is the same (equal to 46.67). However, the amplitude differs: in scenario 1, $\alpha_1(t)$ and $\alpha_2(t)$ both have an amplitude of 25; in scenario 2, we double the amplitude of $\alpha_2(t)$ only, while in scenario 3, we double the amplitude of $\alpha_1(t)$ only, as shown in the following table:

	Scenario 1	Scenario 2	Scenario 3
$\alpha_1(t)$	$30 + 10t - t^2$	$13.33 + 20t - 2t^2$	$30 + 10t - t^2$
$\alpha_2(t)$	$30 + 10t - t^2$	$30 + 10t - t^2$	$13.33 + 20t - 2t^2$

The corresponding plots are shown in Figure 5.1.1.

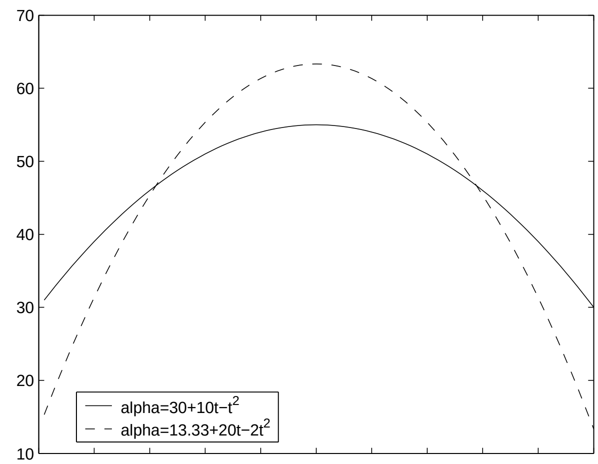


Figure 2: choices of parameters α

In each scenario, we first compute the optimal pricing and production policy for both products separately based on the assumption that there is no capacity constraint. We then determine the maximum value of the total production $\max_{t\in[0,T]} u_1(t) + u_2(t) \equiv u^*$. Clearly, if the capacity remains greater than or equal to u^*, the policies obtained are optimal.
Then in each demand scenario, we compute the solution for lower values of the capacity, that is, for a capacity equal to $0.75u^*$ and to $0.5u^*$. In these two cases, the capacity will be binding at least at some point within the time horizon.

5.1.2 Interpretation of the results

The results can be seen in Figures 3, 4, and 5. These figures show the evolution of inventory levels, production rates, and prices for both products in the optimal solution for each scenario, and in each scenario for three values of the capacity as explained above.
We also report the objective value (profit) under each scenario and for each value of the capacity available, as well as the proportion of the total profit generated by product 1.

We observe that in all cases, the system builds up some inventory at the beginning of the time horizon because of the upcoming demand peak, and then maintains a level of inventory at zero for the remaining time. Of course, the lower the capacity, the least the system has the ability to build up inventory.
We observe that the prices increase and production rates decrease when capacity decreases.
We also observe that in Scenario 1, the capacity is tight from the beginning of the time horizon both for capacity levels of $0.75u^*$ and $0.5u^*$, and only in the latter case it is tight over the whole time horizon.
In scenario 2 and 3, in both cases the capacity is tight from the beginning, but is not tight near the end of the time horizon.
Moreover, by comparing the scenarios, we notice that the amplitude of variation for prices increases when the amplitude of the coefficient $\alpha(t)$ increases.
Finally, it is worth noticing that in all scenarios, under no capacity constraint, the production rate for both products increases in the first part of the time horizon (while the inventory level is non

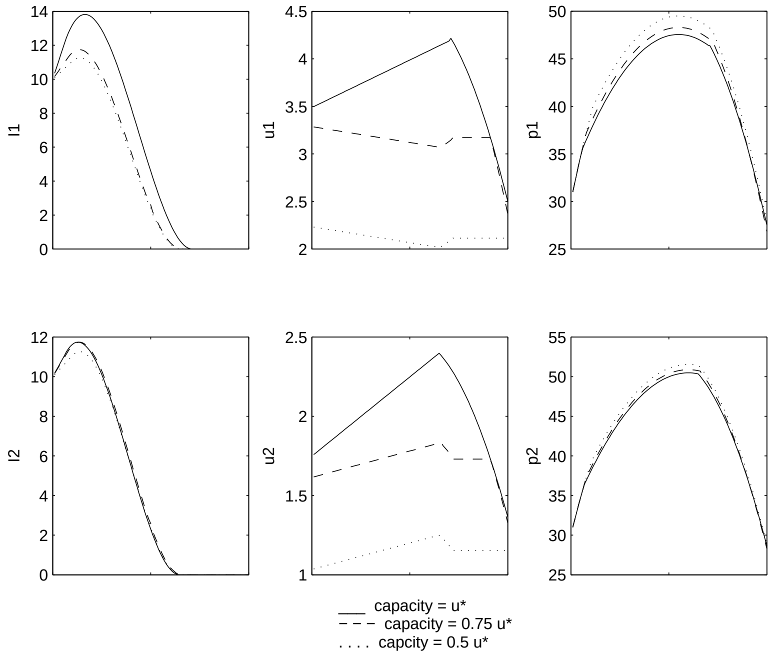


Figure 3: Solution for demand scenario 1

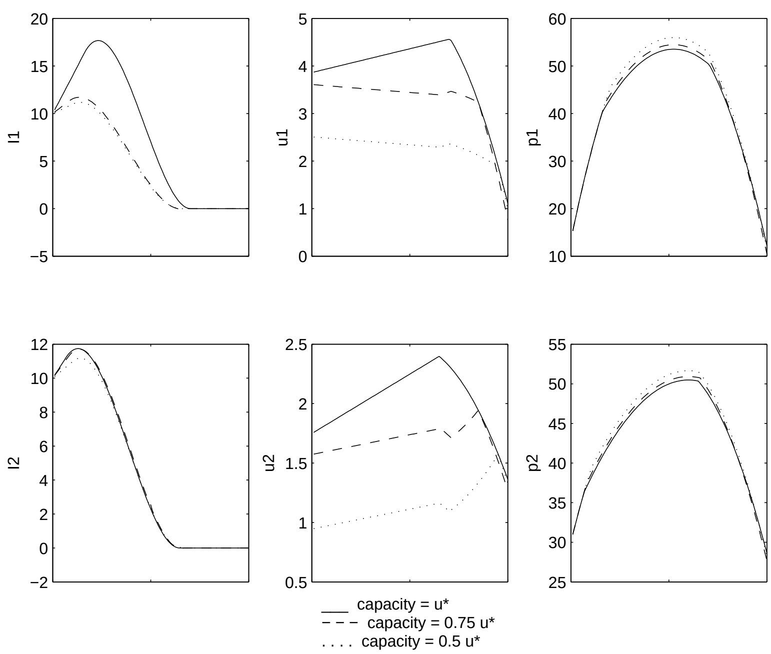


Figure 4: Solution for demand scenario 2

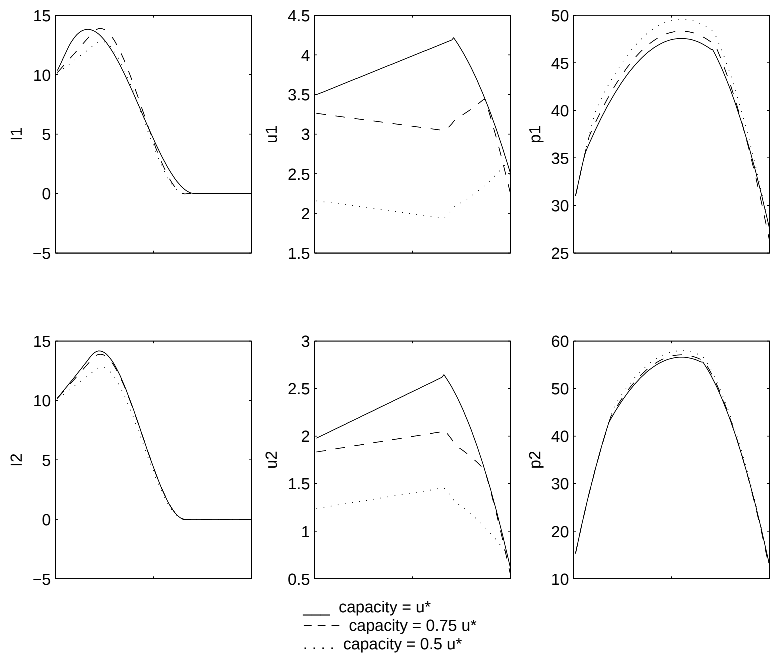


Figure 5: Solution for demand scenario 3

zero) since the system attempts to build up some inventory due to the upcoming demand peak. However, under lower capacity, the production rate for product 1 decreases in that phase, while production rate for product 2 keeps increasing (but is lower in all cases). The fact that the system tends to produce more of the less expensive product is quite natural. Therefore, introducing a capacity constraint has more effect on the production for that product. It can also be seen that the level of inventory changes much more for product 1 than for product 2 in the presence of a capacity constraint, with maybe the exception of scenario 3 where a noticeable peak of demand for the expensive product justifies to stock some inventory in the beginning of the time horizon, despite the higher holding and production costs.

		Scenario 1	Scenario 2	Scenario 3
	u^*	6.5352	6.9099	6.7935
capacity = u^*	Profit from product 1	1327.5 (54.6%)	1523.4 (58.01%)	1327.5 (51.24%)
	Profit from product 2	1102.8	1102.8	1263.1
	Total profit	2430.3	2626.2	2590.5
capacity = 0.75 u^*	Profit from product 1	1311.6 (59.79%)	1520.4 (62.78%)	1328.5 (56.46%)
	Profit from product 2	881.9	901.5	1024.4
	Total profit	2193.6	2421.9	2352.9
capacity = 0.5 u^*	Profit from product 1	1160.2 (59.19%)	1435.0 (67.11%)	1156.3 (55.30%)
	Profit from product 2	799.9	793.2	934.9
	Total profit	1960.1	2138.3	2091.1

We observe that profits decrease as the capacity decreases (when the capacity drops by 25% and 50% respectively, there is a 9.74% and a 19.35% decrease in scenario 1, 7.78% and 18.58% in scenario 2 and 9.17% and 19.28% in scenario 3.

We also notice that the capacity constraint increases the proportion of total profit due to product 1, which is the least expensive product (to hold in inventory and to produce).
When a product has a demand that is more time varying (but with the same average), the proportion of profit that product generates is also greater. Also the total profit increases if the demand for one of the products is more varying, compared with demands that are both less varying.
Finally, the maximum demand satisfied under no capacity constraint increases when the demand for one of the products is more varying. This effect is more marked when the demand for product 1 (the cheapest product) is more varying.

To conclude this section, the major insights from the numerical tests we performed are the following:

1. The optimal solution tends to build up some inventory prior to the demand peak (and more so for the cheapest product), and subsequently lets the inventory level remain at zero.

2. As the capacity decreases (i.e. the capacity constraints more the system), inventory levels and production rates tend to decrease, prices tend to increase, and profits decrease.

3. As the capacity decreases, the proportion of profits due to the cheapest product increases.

4. As the capacity decreases, the production rate of the most expensive product decreases less than the other product, while remaining smaller.

5. The shape of the evolution of prices over time is similar to the shape of the evolution of coefficient $\alpha(t)$.

6. As the amplitude of the coefficient α for a product increases, the amplitude of prices increases as well and the proportion of profits this product generates increases. Moreover, the maximal demand satisfied over the time horizon increases.

5.2 Example 2: Impact of constant price sensitivities (coefficients $\beta_i(.)$) with a demand peak

We consider the same inputs as above in scenario 1 of coefficients $\alpha_i(.)$ and a capacity level constant and equal to 1, but with 3 different cases of coefficients $\beta_i(.)$, defined by

- $\beta_1(t) = \beta_2(t) = 1, \quad t \in [0, T]$
- $\beta_1(t) = \beta_2(t) = 2, \quad t \in [0, T]$
- $\beta_1(t) = \beta_2(t) = 3, \quad t \in [0, T]$.

The results are shown in Figure 6.

In all three cases the capacity was tight all along the time horizon. We observe that, as suggested by intuition, the prices decrease when the price sensitivities increase, and the inventory levels reach zero earlier. Moreover, the amplitude of prices decrease as well.

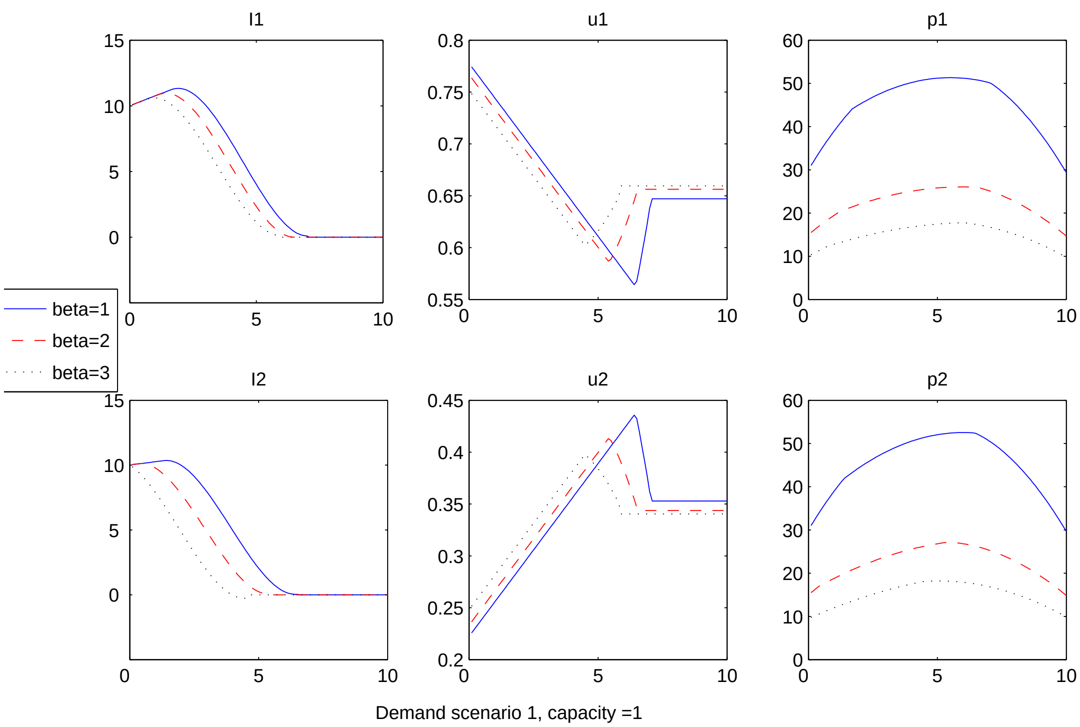


Figure 6: Solution for price sensitivities equal to 1

5.3 Example 3: Impact of time-varying price sensitivities (coefficients $\beta_i(.)$) with a constant maximum demand

We now consider that coefficients $\alpha_i(.)$ are fixed at 15, $t \in [0, T]$ and that the capacity level is constant and equal to 1. We want to study the effect of time varying price sensitivities, both increasing and decreasing. We will run the solution method for

- $\beta_1(t) = \beta_2(t) = 0.5 + 0.1t$
- $\beta_1(t) = \beta_2(t) = 1.5 - 0.1t$.

Price sensitivities that increase with time correspond to products that become less attractive to the customer towards the end of the time horizon, for example products subject to a seasonality effect, or such that there have appeared on the market newer products that can serve as a substitute. Price sensitivities that increase with time correspond to products that become more attractive to the customer towards the end of the time horizon, for example because of a marketing campaign or an appearing trend.

The results are shown in Figure 7. The capacity level was tight all along the time horizon

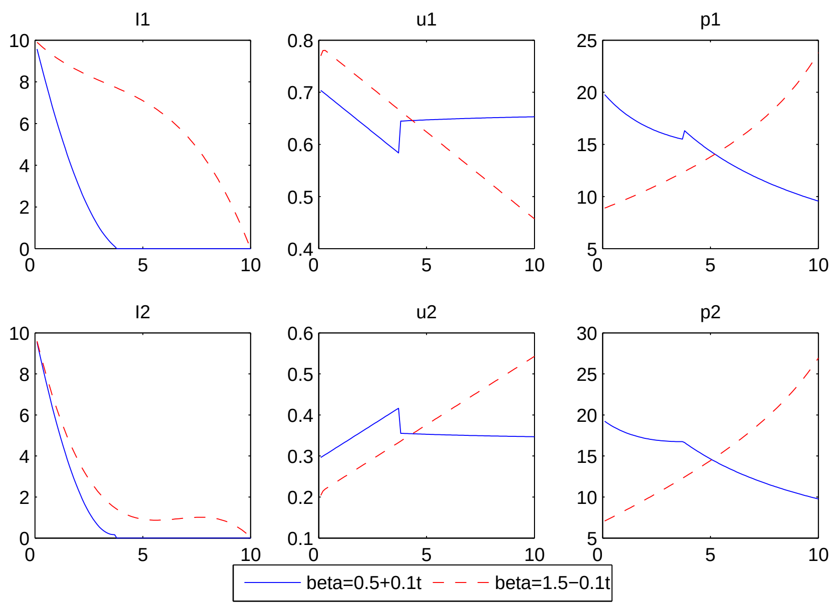


Figure 7: Solution for price sensitivities equal to 1

(except in one case at the very beginning). Similarly as above, the trend of prices is intuitive: the prices evolve with time in an way opposite to the way the price sensitivities evolve with time. Notice that for decreasing price sensitivities, the products are in case **b**, i.e. the inventory level is positive on $[0, T)$ and reach zero at time T. Indeed, it is optimal to save inventory to be sold at the end of the time horizon when the price sensitivity is lower and the products can be sold at a higher price. This effect is stronger for product 1 which has a lower holding cost. When price sensitivities increase with time, the inventory levels reach zero faster that when they were constant since it is optimal to sell all inventory before price sensitivities become too high and the prices are low.

5.4 Heuristics: computational results

The method introduced in Section 4 yields good convergence results in the computations below. We provide as an example the results for Example 1, Scenario 1 described in Section 5.1, and a capacity rate constant equal to 3.5.
We use the following parameters: $M = 100$ steps in the discretization, $\epsilon_1 = 1.1$, $\epsilon_2 = 0.9$, $\delta_0 = 0.1$, $\eta^0 = 15$, and $\epsilon = 0.2$. For this example, 95 iterations were required (taking less than 2 minutes overall). Note that each iteration implements the procedure detailed in Section 3: the optimal solution at all times is calculated for a given vector $\eta(t)$, and then updates the multiplier $\eta(t)$ as described in Section 4. Each iteration is very fast, so the overall time is primarily due to the number of iterations that are necessary to satisfy the stopping criterion. The graphs below show the multiplier η obtained (Figure 8a), the amplitude of capacity constraint violation (Figure 8b), the amplitude of complementary slackness condition violation (Figure 8c), as well as the production rates, prices, and inventory levels (Figure 8d). Notice that the capacity constraint is basically tight at all times (the maximum unused capacity is approximately 0.006), as multiplier $\eta(t)$ is positive throughout the time horizon. Discontinuities of $\eta(t)$ occur when the inventory level of a product reaches zero and enters a constrained phase. At convergence, the complementary slackness is also approximately satisfied. Decreasing parameter ϵ would decrease the maximum violation of the complementary slackness solution, but would increase the number of iterations and thus the overall running time.

Conclusions
In this paper, we studied a continuous time optimal control model for a dynamic pricing and inventory management problem with no backorders. In particular, we studied a demand based model in a make-to-stock system and in a multi-product capacitated dynamic setting. We considered a particular cost structure, allowing time flexibility in the coefficients and in the production cost. A particular feature of the model we considered is that it does not allow backorders. We introduced a continuous time solution approach utilizing the KKT conditions and an extension of Pontryagin's Principle for state-constrained problems. Through numerical examples, we illustrate the role of capacity and of the dynamic nature of demand in the model.

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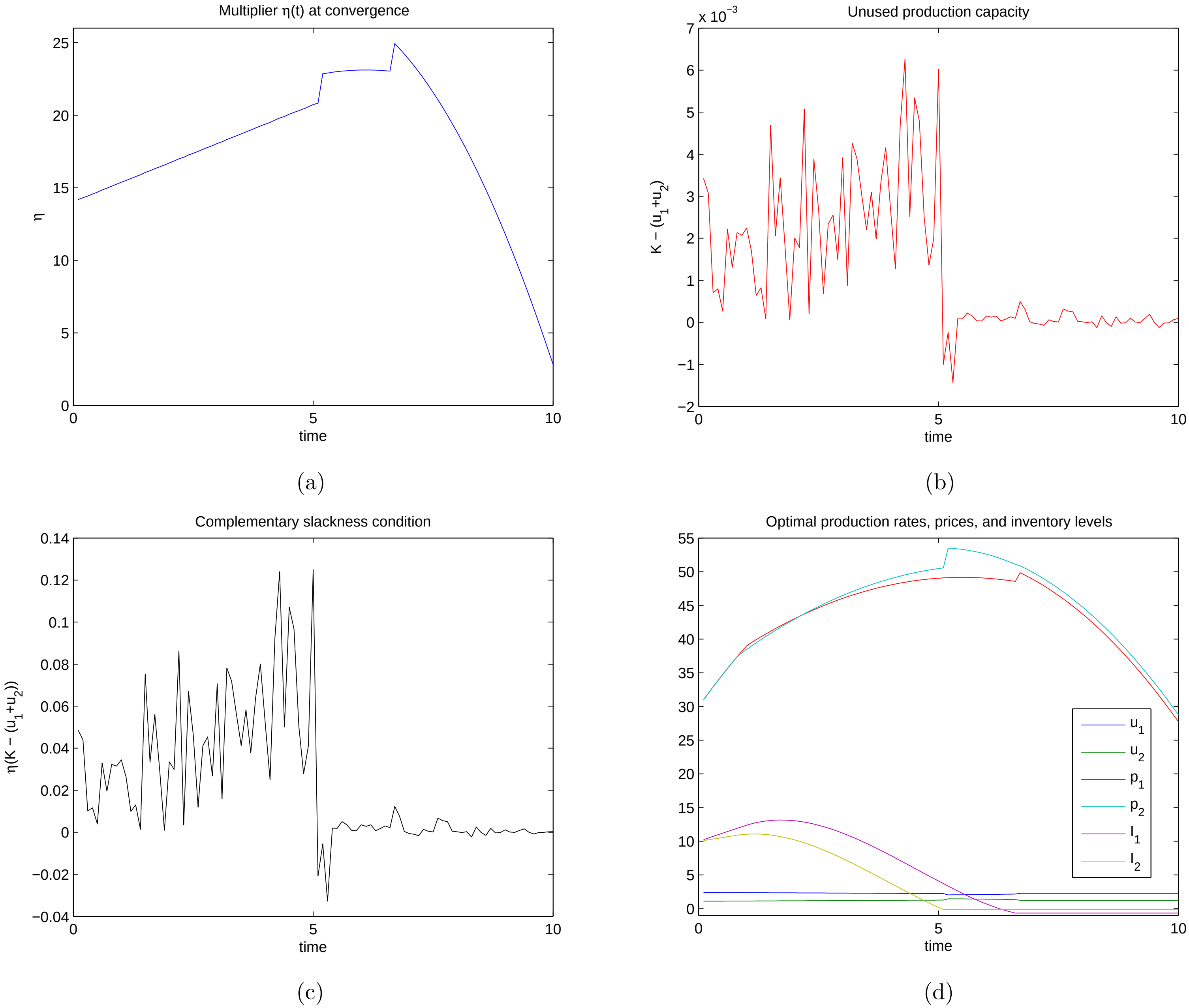

Figure 8: Results for Example 1, Scenario 1 using the heuristic algorithm to determine multiplier $\eta(.)$. Figure (a) shows multiplier $\eta(t)$ at convergence, Figure (a) shows the capacity constraint $K(t)-(u_1(t)+u_2(t))$, Figure (c) shows the complementary slackness condition $\eta(t)\Big(K(t)-(u_1(t)+u_2(t))\Big)$, and Figure (d) shows the optimal prices $p_1(t)$, $p_2(t)$, production rates $u_1(t)$, $u_2(t)$, and inventory levels $I_1(t)$, $I_2(t)$ corresponding to multiplier $\eta(t)$ at convergence.

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A Maximum Principle with mixed inequality constraints and pure state variable inequality constraints: Theoretical results. Sufficiency Conditions.

In this section, we state the maximum principle for optimal control problems with mixed inequality constraints and pure state variable inequality constraints. These results are described in more detail in Sethi and Thompson [51], Hartl, Sethi and Vickson [32], Arrow and Kurz [6].
Consider the following control problem

$$\max \quad \int_0^T F(I(t), w(t), t)dt \tag{14}$$

$$\text{subject to} \quad \dot{I}(t) = \vartheta(I(t), w(t), t) \tag{15}$$

$$I(0) = I^0 \tag{16}$$

$$\xi(I(t), w(t), t) \geq 0 \tag{17}$$

$$\varpi(I(t), t) \geq 0 \tag{18}$$

where:

T	is the time horizon,
$I(t) \in E^n$	is the vector of state variables at time t,
I^0	is the vector of initial conditions,
$w(t) \in E^m$	is the vector of control variables (prices and production rates) at time t,
$F : E^n \times E^m \times E \to E$	is a function assumed to be continuously differentiable,
$\vartheta : E^n \times E^m \times E \to E^n$	is a function assumed to be continuously differentiable,
$\xi : E^n \times E^m \times E \to E^a$	is a function assumed to be continuously differentiable in all its arguments and depends explicitly on $w(t)$,
$\varpi : E^n \times E \to E^b$	is a function assumed to be continuously differentiable.

We notice that constraint (17) involves control variables and possibly state variables as well (we refer to this as a mixed inequality constraint) while constraint (18) involves the state variable only (we refer to this as a pure state variable inequality constraint).
We define a control $w(.)$ to be admissible if it is piecewise continuous and if, together with the state trajectory $I(.)$ it generates through (15) and (16), it satisfies (17) and (18).
Inequality (18) represents by definition a set of constraints $\varpi_i(I(t), t) \geq 0,\ i = 1, \ldots, b$. The constraint $\varpi_i(I(t), t) \geq 0$ is called a constraint of r^{th} order if the r^{th} time derivative of $\varpi_i(I(t), t)$ is the first in which a term in control $w(.)$ appears. To make this dependency in the control variable clear, we will add $w(t)$ as an argument of the r^{th} time derivative of $\varpi_i(I(t), t)$, even though $w(t)$ is not an argument of $\varpi_i(I(t), t)$. For the sake of simplicity and because it is satisfied in the application we are interested in, we will assume that each constraint $\varpi_i(I(t), t) \geq 0$ is of the first order.
We define

$$\varpi^1(I(t), w(t), t) = \frac{d\varpi}{dt}(I(t), t) = \frac{\partial \varpi}{\partial I}\vartheta(I(t), w(t)t) + \frac{\partial \varpi}{\partial t}(I(t), t).$$

With respect to the i^{th} constraint $\varpi_i(I(t), t) \geq 0$, an interval $(\theta_i^1, \theta_i^2) \subset [0, T]$ is called an *interior* or *unconstrained* interval if $\varpi_i(x(t), t) > 0, \quad \forall t \in (\theta_i^1, \theta_i^2)$. If the optimal trajectory "hits the boundary," i.e., satisfies $\varpi_i(x(t), t) = 0,\ \forall t \in (\tau_i^1, \tau_i^2)$, for some i and some interval $(\tau_i^1, \tau_i^2) \subset [0, T]$, then $[\tau_i^1, \tau_i^2]$ is called a *boundary* or *constrained interval*. An instant τ_i^1 is called an *entry time*

if there is an interior interval ending at time τ_i^1 and a boundary interval starting at time τ_i^1. Correspondingly, τ_i^2 is called an *exit time* if a boundary ends and an interior interval starts at time τ_i^2. If the trajectory touches the boundary at time τ_i, i.e., $\varpi_i(I(\tau_i), \tau_i) = 0$ for some i and if the trajectory is in the interior just before and just after τ_i, then τ_i is called a *contact time*. Taken together, entry, exit and contact times are called *junction times*.
We assume that the following constraint qualification holds:

$$\text{rank}\left[\frac{\partial \xi}{\partial w}, \text{diag}(\xi)\right] = a$$

as well as the full-rank condition on any boundary interval $[\tau_j^1, \tau_j^2]$:

$$\text{rank}\begin{bmatrix} \partial \varpi_1^1/\partial w \\ \vdots \\ \partial \varpi_{\hat{b}}^1/\partial w \end{bmatrix} = \hat{b},$$

where for $t \in [\tau_j^1, \tau_j^2]$, $\varpi_i(I^*(t), t) = 0 \;\; i = 1, \ldots, \hat{b} \leq b$ and $\varpi_i(I^*(t), t) > 0 \;\; i = \hat{b}+1, \ldots, b$.

We define the Hamiltonian function $H : E^n \times E^m \times E^n \times E \to E$ as

$$H(I, w, q, t) \equiv F(I, w, t) + q\vartheta(I, w, t),$$

where $q \in E^n$ (a row vector). We also define the Lagrangian function $L : E^n \times E^m \times E^n \times E^q \times E \to E$ as[5]

$$L(I, w, q, \eta, \rho, t) = H(I, w, q, t) + \eta\xi(I, w, t) + \rho\varpi^1(I, w, t),$$

where $\eta \in E^a$ and $\rho \in E^b$ are row vectors, whose components are called Lagrange multipliers. These Lagrange multipliers satisfy the complementary slackness conditions

$$\eta(t) \geq 0, \quad \eta(t)\xi(I(t), w(t), t) = 0,$$

$$\rho(t) \geq 0, \quad \dot{\rho}(t) \leq 0, \quad \rho(t)\varpi(I(t), t) = 0.$$

We now state the maximum principle for the problem under consideration.

Theorem 2. *(Maximum Principle) We suppose that $I^*(.)$ has only finitely many junction times, that each pure state constraint $\varpi_i(I(t), t)) \geq 0$ is of the first order, that constraint qualification holds, and that the full rank condition holds. The necessary conditions for w^* (with state trajectory I^*) to be an optimal control policy for the problem we defined above are the following:*
There exist piecewise continuous[6] and piecewise continuously differentiable adjoint variable $q(.)$, piecewise continuous multipliers $\eta(.)$, $\rho(.)$, parameter ν, and jump parameter $\zeta(.)$, such that the following conditions hold almost everywhere:

[5]We form the Lagrangian function by adjoining indirectly (i.e. via their first time derivative) the constraints on the state variable. This method is called indirect adjoining method. In the direct adjoining method, the Lagrangian function is formed by adjoining directly the constraints as follows: $L^d(I, w, q, \eta^d, \rho^d, t) = H(I, w, q, t) + \eta^d\xi(I, w, t) + \rho^d\varpi(I, t)$, with $H = F(I, w, t) + q^d\vartheta(I, w, t)$. It is shown in [32] that $\eta^d(t) = \eta(t)$, $\; q^d(t) = q(t) + \rho(t)\frac{\partial \varpi}{\partial I}(I^*(t), t)$.

[6]In the direct adjoining method, $q^d(.)$ is continuous.

- $\dot{I}^*(t) = \vartheta(I^*(t), w^*(t), t)$, $I^*(0) = I_0$, *satisfying constraints* $\xi(I^*(t), w^*(t), t) \geq 0$, $\varpi(I^*, t) \geq 0$;
- $\dot{q}(t) = -\frac{\partial L}{\partial I}(I^*(t), w^*(t), q(t), \eta(t), \rho(t), t)$ *except at entry/contact times, with transversality conditions*[7]

$$q(T^-) = \nu \frac{\partial \varpi}{\partial I}(I^*(T), T), \quad \nu \geq 0, \quad \nu \varpi(I^*(T), T) = 0;$$

- *the Hamiltonian maximizing condition*

$$H(I^*(t), w^*(t), q(t), t) \geq H(I^*(t), w(t), q(t), t),$$

at each $t \in [0, T]$, *for all* w *satisfying*

$$\xi(I^*(t), w, t) \geq 0, \text{ and } \varpi_i^1(I^*(t), w, t) \geq 0, \text{ whenever } \varpi_i(I^*(t), t) = 0, \; i = 1, \ldots, b;$$

- *at any entry/contact time*[8] τ, *the adjoint variable* q *may have a discontinuity of the form*

$$q(\tau^-) = q(\tau^+) + \zeta(\tau) \frac{\partial \varpi}{\partial I}(I^*(\tau), \tau) \text{ and}$$

$$H(I^*(\tau), w^*(\tau^-), q(\tau^-), \tau) = H(I^*(\tau), w^*(\tau^+), q(\tau^+), \tau) - \zeta(\tau) \frac{\partial \varpi}{\partial t}(I^*(\tau), \tau);$$

- *the Lagrange multipliers* $\eta(t)$ *are such that*

$$\frac{\partial L}{\partial w}(I^*(t), w^*(t), q(t), \eta(t), \rho(t), t) = 0$$

and the complementary slackness conditions

$$\eta(t) \geq 0, \quad \eta \xi(I^*(t), w^*(t), t) = 0,$$

$$\rho \geq 0, \quad \dot{\rho} \leq 0 \text{ on boundary intervals of } \varpi, \quad \rho \varpi(I^*(t), t) = 0, \text{ and}$$

$$\zeta(\tau) \geq 0, \quad \zeta(\tau) \varpi(I^*(\tau), \tau) = 0.$$

Theorem 3. *Let* $(I^*(.), w^*(.), q(.), \eta(.), \rho(.), \nu, \zeta(.))$ *satisfy the necessary conditions above. Suppose that constraint qualification and full-rank condition hold. Let* $q^d(t) = q(t) + \rho(t) \frac{\partial \varpi}{\partial I}(I^*(t), t)$. *If* $H(I, w, q^d, t)$ *is a concave function in* (I, w), *at each* $t \in [0, T]$, $\xi(I, W, t)$ *is a quasi-concave function in* (I, w), $\varpi(I, t)$ *is a quasi-concave function in* I, *then policy* $(I^*(.), w^*(.))$ *is optimal.*

[7] In the direct adjoining method, the transversality conditions are:

$$q^d(T) = \nu^d \frac{\partial \varpi}{\partial I}(I^*(T), T), \quad \nu^d \geq 0, \quad \nu^d \varpi(I^*(T), T) = 0.$$

[8] We are using the convention specifying that the adjoint variable is continuous at exit times.

B Proof that assumptions made in Appendix A hold for Problem (1)

In this section, we illustrate why the assumptions in Appendix A apply to the pricing problem (1) we are studying under Assumptions 1, 2, 3, and 4. In particular, we show that the assumption of constraints of the first order, constraint qualification, full-rank condition, and sufficiency conditions defined in Appendix A hold.
In Problem (1),

- the control variables are $(u_i(.),\ p_i(.),\ i = 1, \ldots, N)$ which are functions defined on $[0, T]$,
- the state variables are $(I_i(.),\ i = 1, \ldots, N)$ which are functions defined on $[0, T]$,
- the dynamic evolution of the system is given by

$$\dot{I}(t) = \vartheta(I(t), u(t), p(t), t) = u(t) - \alpha(t) + \beta(t) \times p(t),$$

- the mixed inequality constraints are $\xi(u(t), p(t), t) =$

$$\Big(p_1(t), \ldots, p_N(t), \frac{\alpha_1(t)}{\beta_1(t)} - p_1(t), \ldots, \frac{\alpha_N(t)}{\beta_N(t)} - p_N(t), u_1(t), \ldots, u_N(t), K(t) - \sum_{i=1}^{N} u_i(t)\Big) \geq 0$$

  (note that $\xi(u, p, t) \in \mathbb{R}^{3N+1}$),

- the pure state variable inequality constraint is given by $\varpi(I(t)) = (I_1(t), \ldots, I_N(t)) \geq 0$.

We have $q_i^d(t) = q_i(t) + \rho_i(t)$ the adjoint variables within the framework of the direct adjoining method.

Lemma 3. *The pure state variable inequality constraints are of order 1.*

Proof. The i^{th} pure state variable inequality constraint is $\varpi_i(I(t)) = I_i(t) \geq 0$. By taking the derivative with respect to time once, we obtain

$$\frac{d\varpi_i}{dt}(I(t)) = \frac{d\varpi_i}{dI_i}(I(t))\frac{dI_i}{dt}(t) = \frac{dI_i}{dt}(t) = u_i(t) - \alpha_i(t) + \beta_i(t)p_i(t).$$

We thus observe that the first time derivative depends explicitly on the controls, therefore the constraint is of the first order. □

We define

$$\varpi^1(I(t), u(t), p(t), t) \equiv \frac{d\varpi}{dt}(I(t)) = \frac{d\varpi}{dI}(I(t))\dot{I}(t) = \dot{I}(t) = u(t) - \alpha(t) + \beta(t) \times p(t)$$

Lemma 4. *Constraint qualification holds.*

Proof. This condition guarantees that the gradients with respect to (u, p) of active constraints on controls are linearly independent.
We need to show that $\text{rank}\Big[\frac{\partial \xi}{\partial u}, \frac{\partial \xi}{\partial p}, \text{diag}(\xi)\Big] = 3N + 1$.

Let $M \equiv \left[\frac{\partial \xi}{\partial u}, \frac{\partial \xi}{\partial p}, \mathrm{diag}(\xi)\right]$. Then we can write matrix $M \in \mathbb{R}^{3N+1} \times \mathbb{R}^{5N+1}$ as follows (omitting the time argument for a lack of space):

$$
\left[\begin{array}{cccccccccccccccc}
 & & & 1 & & & p_1 & & & & & & & & & \\
 & & & & \ddots & & & \ddots & & & & & & & & \\
 & & & & & 1 & & & p_N & & & & & & & \\
 & & & -1 & & & & & & \frac{\alpha_1(t)}{\beta_1(t)} - p_1 & & & & & & \\
 & & & & \ddots & & & & & & \ddots & & & & & \\
 & & & & & -1 & & & & & & \frac{\alpha_N(t)}{\beta_N(t)} - p_N & & & & \\
1 & & & & & & & & & & & & u_1 & & & \\
 & \ddots & & & & & & & & & & & & \ddots & & \\
 & & 1 & & & & & & & & & & & & u_N & \\
-1 & \dots & -1 & & & & & & & & & & & & & K(t) - \sum_{i=1}^{N} u_i
\end{array}\right]
$$

To show that $\mathrm{rank}(M) = 3N+1$, we observe that:
(i) there are $3N+1$ rows in M, so the rank is at most $3N+1$;
(ii) if there is no binding constraint on the control variables, the last $3N+1$ columns are non zero and linearly independent, implying that the rank is equal to $3N+1$;
(iii) for each i such that the price $p_i = 0$, column $2N+i$ is the zero vector; however we can replace it with column $N+i$ to obtain a set of linearly independent columns.
Similarly, for each i such that $p_i = \alpha_i(t)/\beta_i(t)$, we replace column $3N+i$ with column $N+i$. We observe that the constraints $p_i = 0$ and $p_i = \alpha_i(t)/\beta_i(t)$ cannot be binding simultaneously.
Using the same reasoning, for each i such that $u_i = 0$, we replace column $4N+i$ with column i.
Finally, if $\sum_{i=1}^{N} u_i = K(t)$, we replace the last column with any column i, $1 \le i \le N$ such that $u_i > 0$. We observe that this is possible because when the capacity constraint is tight, at least one of the production rates must be positive (i.e. the last $N+1$ columns cannot be simultaneously equal to the zero vector).
This implies that there are $3N+1$ linearly independent columns. □

Lemma 5. *Let $[\tau_1, \tau_2]$ be a boundary interval. The full-rank condition holds on $[\tau_1, \tau_2]$.*

Proof. Suppose that $I_i(t) = 0,\ i = 1, \dots, n_0$, and $I_i(t) > 0,\ i = n_0+1, \dots, N$, on $[\tau_1, \tau_2]$. Let

$$
Q \equiv \begin{bmatrix}
\frac{\partial \varpi_1^1}{\partial u} & \frac{\partial \varpi_1^1}{\partial p} \\
\frac{\partial \varpi_2^1}{\partial u} & \frac{\partial \varpi_2^1}{\partial p} \\
\vdots & \vdots \\
\frac{\partial \varpi_{n_0}^1}{\partial u} & \frac{\partial \varpi_{n_0}^1}{\partial p}
\end{bmatrix}
$$

We need to show that $\mathrm{rank}(Q) = n_0$. We can express matrix $Q \in \mathbb{R}^{n_0} \times \mathbb{R}^{2n_0}$ as follows:

$$
Q = \begin{bmatrix}
1 & & & & \beta_1(t) & & & \\
 & 1 & & & & \beta_2(t) & & \\
 & & \ddots & & & & \ddots & \\
 & & & 1 & & & & \beta_{n_0}(t)
\end{bmatrix}
$$

which clearly has rank n_0. □

Remark. In the direct adjoining method, $q_i^d(t) = q_i(t) + \rho_i(t)$ and the transversality conditions are given by

$$q^d(T) = \nu^d \frac{\partial \varpi}{\partial I}(I^*(T), T) = \nu^d, \quad \nu^d \geq 0, \quad \nu^d \varpi(I^*(T), T) = \nu^d I^*(T) = 0,$$

which can be rewritten

$$q_i(T) + \rho_i(T) \geq 0, \quad (q_i(T) + \rho_i(T))I^*(T) = 0.$$

Lemma 6. *Under Assumptions 1, 2, 3, and 4, the assumptions in Theorem 3 (see Appendix A) hold for Problem (1).*

Proof. Notice that

- $H(I, u, p, q^d, t) = \sum_{i=1}^N \Big(p_i(\alpha_i(t) - \beta_i(t)p_i) - f_i(u_i) - h_i(t)I_i + q_i^d(u_i - \alpha_i(t) + \beta_i(t)p_i) \Big)$ is a concave function in (I, u, p);
- ξ is a linear function in (p, u), and thus quasi-concave in (I, p, u);
- ϖ is a linear (thus quasi-concave) function of I.

□

C Proof of existence of a solution for Problem (1)

The following existence result follows similarly to [32]. We will provide the theorem by using the notations we defined in Appendix A.
Define the (state-dependent) control region

$$\Omega(I, t) = \{w \in E^m | \xi(I, w, t) \geq 0\} \subset E^m$$

and the set

$$Q(I, t) = \{(F(I, w, t) + c, \ \vartheta(I, w, t)) | c \leq 0, w \in \Omega(I, t)\} \subset E^{n+1}.$$

Theorem 4 (Filippov-Cesari Theorem)**.** *Consider problem (14). Assume that F, ϑ, ξ and ϖ are continuous in all their arguments at all points $(I, w) \in E^n \times E^m$. Suppose that there exists an admissible pair and that the following Roxin's condition holds:*

$$\textit{set } Q(I, T) \textit{ is convex, for all } I \in E^n.$$

Suppose further that

$$\textit{there exists } \delta > 0 \textit{ such that } \|I(t)\| < \delta, \textit{ for all admissible } \{I(t), w(t)\} \textit{ and } t,$$

and that

$$\textit{there exists } \delta_1 > 0 \textit{ such that } \|w\| < \delta_1, \textit{ for all } w \in \Omega(I, t) \textit{ with } \|I\| < \delta.$$

Then there exists an optimal couple $\{I^*, w^*\}$ with $w^*(.)$ measurable.

Theorem 1. Under Assumptions 2, 3, and 4, an optimal control policy exists for Problem (1).

Proof. We will show that conditions of Theorem 4 hold for Problem (1).
For Problem (1), the control region is independent of the state and may be expressed as:

$$\begin{aligned}\Omega(t) &= \{(u,p) \in \mathbb{R}^{2N} | \xi(u,p,t) \geq 0\} \\ &= \{(u,p) \in \mathbb{R}^{2N} | u \geq 0,\ p \geq 0,\ p_i \leq \alpha_i(t)/\beta_i(t),\ i = 1,\ldots,N,\ \sum_{i=1}^{N} u_i \leq K(t).\}\end{aligned}$$

The set $Q(I,t)$ is given by

$$Q(I,t) = \Big\{ \Big(\sum_{i=1}^{N} \big(p_i(\alpha_i(t) - \beta_i(t)p_i) - f_i(u_i) - h_i(t)I_i\big) + c,\ \ u - \alpha(t) + \beta(t) \times p \Big) \Big| c \leq 0, (u,p) \in \Omega(t) \Big\}.$$

- Continuity:
  It is clear that
  (i) the integrand F of the objective function

  $$(I,u,p,t) \mapsto \sum_{i=1}^{N} \Big(p_i(\alpha_i(t) - \beta_i(t)p_i) - f_i(u_i) - h_i(t)I_i\Big),$$

  (ii) the function ϑ describing the dynamic evolution $(u,p,t) \mapsto u - \alpha(t) + \beta(t) \times p$,
  (iii) the function ξ giving rise to control inequality constraints

  $$(u,p) \mapsto \Big(p,\ \alpha(t)/\beta(t) - p,\ u,\ K(t) - \sum_{i=1}^{N} u_i\Big),$$

  and
  (iv) the function ϖ giving rise to pure state variables inequality constraints $I \to I$ are continuous functions in all their arguments.

- There exists an admissible pair:
  Consider the policy

  $$p_i(t) = \alpha_i(t)/\beta_i(t), \quad u_i(t) = 0, \quad i = 1,\ldots,N, \quad \forall t \in [0,T].$$

  Under this policy, $\dot{I}_i(t) = u_i(t) - \alpha_i(t) + \beta_i(t)p_i(t) = 0$, so the generated state trajectory satisfies $I_i(t) = I_i^0 \geq 0 \ \ \forall t \in [0,T]$. As a result, this policy satisfies all constraints so it is an admissible pair.

- Roxin's condition holds:

Let $x_1,\ x_2 \in \mathbb{R},\ y_1,\ y_2 \in \mathbb{R}^N$ such that $(x_1, y_1),\ (x_2, y_2) \in Q(I, T)$ with

$$\begin{aligned}
x_1 &= \sum_{i=1}^{N} \big(p_i^1(\alpha_i(T) - \beta_i(T)p_i^1) - f_i(u_i^1) - h_i(T)I_i\big) + c^1 \\
x_2 &= \sum_{i=1}^{N} \big(p_i^2(\alpha_i(T) - \beta_i(T)p_i^2) - f_i(u_i^2) - h_i(T)I_i\big) + c^2 \\
y_1 &= u^1 - \alpha(T) + \beta(T) \times p^1 \\
y_2 &= u^2 - \alpha(T) + \beta(T) \times p^2 \\
c^1 &\leq 0 \\
c^2 &\leq 0 \\
(u^1, p^1) &\in \Omega(T) \\
(u^2, p^2) &\in \Omega(T)
\end{aligned}$$

Let $\lambda \in [0, 1]$. We want to show that $(\bar{x}, \bar{y}) \equiv \lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2) \in Q(I, T)$.
Let $(\bar{u}, \bar{p}) = \lambda(u^1, p^1) + (1 - \lambda)(u^2, p^2)$.
It is easy to verify that $(\bar{u}, \bar{p}) \in \Omega(T)$.
It is also clear that $\bar{y} = \bar{u} - \alpha(T) + \beta(T) \times \bar{p}$.
Since the function $(u, p) \mapsto \sum_{i=1}^{N} \big(p_i(\alpha_i(t) - \beta_i(t)p_i) - f_i(u_i)$ is concave in (u, p), it follows that

$$\sum_{i=1}^{N} \Big(\bar{p}_i(\alpha_i(T) - \beta_i(T)\bar{p}_i) - f_i(\bar{u}_i) - h_i(T)I_i\Big) \geq$$

$$\lambda \sum_{i=1}^{N} \Big(p_i^1(\alpha_i(T) - \beta_i(T)p_i^1) - f_i(u_i^1) - h_i(T)I_i\Big) + (1-\lambda) \sum_{i=1}^{N} \Big(p_i^2(\alpha_i(T) - \beta_i(T)p_i^2) - f_i(u_i^2) - h_i(T)I_i\Big).$$

By observing that the right hand side of this inequality may be rewritten as $\lambda(x_1 - c^1) + (1 - \lambda)(x_2 - c^2)$, we obtain that there exists $c^3 \leq 0$ such that

$$\sum_{i=1}^{N} \Big(\bar{p}_i(\alpha_i(T) - \beta_i(T)\bar{p}_i) - f_i(\bar{u}_i) - h_i(T)I_i\Big) + c^3 = \bar{x} - \lambda c^1 - (1 - \lambda)c^2$$

Letting $\bar{c} \equiv \lambda c^1 + (1 - \lambda)c^2 + c^3 \leq 0$ implies that

$$\bar{x} = \sum_{i=1}^{N} \Big(\bar{p}_i(\alpha_i(T) - \beta_i(T)\bar{p}_i) - f_i(\bar{u}_i) - h_i(T)I_i\Big) + \bar{c}.$$

Therefore, $(\bar{x}, \bar{y}) \in Q(I, T)$ and $Q(I, T)$ is a convex set.

- The admissible controls are bounded:
  The constraints defining the admissible controls provide bounds to the prices $0 \leq p_i(t) \leq \alpha_i(t)/\beta_i(t)$ and to the production rates $0 \leq u_i(t) \leq K(t)$, where $\alpha_i(.), \beta_i(.)$ and $K(.)$ are positive- and finite-valued functions of time. Since the time horizon is finite, there exists bounds on the control variables at each time.

- The state variable is bounded:
  The inventory level is bounded below by zero. Moreover, the control variables $p_i(t)$ and $u_i(t)$ are bounded (as we discussed above). As a consequence, $\dot{I}_i(t) = u_i(t) - \alpha_i(t) + \beta_i(t)p_i(t)$ is bounded too. Since the time horizon is finite, it follows that there exists also an upper bound on the state variable $I_i(t)$ for all t.

This proves that all assumptions in Theorem 4 hold for Problem (1). □