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A Nonlinear Derivative Schroedinger-Equation: Its Bi-Hamilton Structures, Their Inverses, Nonlocal Symmetries and Mastersymmetries

W. STRAMPP and W. OEVEL*
Fachbereich 17, Mathematik, Wilhelmshoeher Allee 73
Gesamthochschule Kassel, 3500 Kassel
**Fachbereich 17, Mathematik, Warburger Str. 100*
Universitaet Paderborn, 4790 Paderborn.

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We investigate the bi-Hamiltonian formulation for the nonlinear derivative Schroedinger-equation. The underlying symplectic and implectic operators can be inverted explicitly yielding a Laxhierarchy of both local and nonlocal symmetry transformations and conservation laws. An infinite sequence of mastersymmetries for this hierarchy is constructed.

In Ref. 1) the following derivative nonlinear Schroedinger-equation (DNLS):

$$q_t = iq_{xx} + (\bar{q}q^2)_x \tag{1}$$

was solved by using the inverse scattering method. Kaup and Newell introduced the following spectral problem:

$$\varphi_{1x} + i\lambda^2\varphi_1 = \lambda u\varphi_2\,,$$
$$\varphi_{2x} - i\lambda^2\varphi_2 = \lambda v\varphi_1\,, \tag{2}$$

being appropriate for the system of equations

$$u_t = iu_{xx} + (u^2v)_x\,,$$
$$v_t = -iv_{xx} + (v^2u)_x\,. \tag{3}$$

Obviously, solutions to (3) are related to solutions of (1) by setting $q = u = \bar{v}$.

By considering gauge transformations from the spectral problem (2) into the AKNS spectral problem, Wadati and Sogo[2] found a gauge transformation mapping solutions of (3) into AKNS field functions.

Furthermore in the more general context of Wadati-Konno-Ichikawa spectral problems,[3] a hierarchy of conservation laws for (3) has been given.

The Hamiltonian approach to the DNLS has been studied first by Kulish[4] who obtained a recursion operator yielding a hierarchy of symmetries and conserved quantities being in involution.

In this paper we study the bi-Hamilton formulation[5)~8)] of the DNLS (3). The recursion operator splits up into the two operators constituting these Hamiltonian structures. Since the inverse of both operators can be obtained easily, we also get an explicit expression for the inverse of the recursion operator. We thus can extend the known hierarchy of symmetries and conserved covariants to a bi-infinite hierarchy, being in involution with respect to the Poisson-brackets given by both Hamiltonian structures. The new symmetries and conservation laws turn out to be nonlocal in the space-variable.

Furthermore mastersymmetries[9] (of degree 1) for this hierarchy are constructed, which give rise to time dependent symmetries for every member of this hierarchy.

Bi-Hamiltonian Systems

In this section we give a brief description of notations being basic for the Hamiltonian machinery.[5)~8)] Let $M := S \oplus S$, S being the Schwartz space of C^∞-functions on the real line vanishing rapidly as $x \to \pm\infty$. Let M^* be the dual of M (which we identify with M). We introduce the following pairing between M and its dual:

$$\langle f, s\rangle = \left\langle \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \right\rangle = \int_{-\infty}^{\infty} (f_1(x)s_1(x) + f_2(x)s_2(x))\,dx\,;\ f, s \in M\,.$$

With each C^∞-vectorfield K: $M \to M$ we associate a dynamical system

$$\begin{bmatrix} u \\ v \end{bmatrix}_t = K(u, v)\,. \tag{4}$$

Following quantities will be discussed:

A C^∞-vectorfield $\sigma(u, v) \in M$ is called a symmetry of (4) if $[\sigma, K] = K'[\sigma] - \sigma'[K] = 0$. If σ depends explicitly on the time-variable t then it generates a symmetry of (4) if $\partial/\partial t\ \sigma + [K, \sigma] = 0$.

A C^∞-covectorfield $\gamma(u, v) \in M^*$ is called a conserved covariant of (4) if $\langle\gamma(u(t), v(t)), m(t)\rangle$ is time independent whenever u, v solve (4) and m is a solution of its linearization, i.e., $m_t = K'(u, v)[m]$. We are especially interested in those conserved covariants which possess potentials, i.e., which are gradients of conservation laws.

A field $J(u, v)$: $M \to M^*$ is called symplectic if it is anti-symmetric w.r.t. to $\langle,\rangle$ and if the Jacobi-identity holds for the bracket $\ll X_1, X_2, X_3 \gg := \langle J'[X_1]X_2, X_3\rangle$ for all $X_1, X_2, X_3 \in M$.

A field $\Theta(u, v)$:$M^* \to M$ is called implectic if Θ is anti-symmetric w.r.t. $\langle,\rangle$ and if the Jacobi-identity holds for the bracket $\ll X_1^*, X_2^*, X_3^* \gg := \langle X_1^*, \Theta'[\Theta X_2^*]X_3^*\rangle$ for all $X_1^*, X_2^*, X_3^* \in M^*$. If a symplectic J (implectic Θ, respectively) is invertible then its inverse J^{-1} (Θ^{-1}) is implectic (symplectic).

The dynamical system (4) is called bi-Hamiltonian if there are functions f_0, f_1:$S \to R$, such that $K = \Theta \mathrm{grad} f_0$ and $JK = \mathrm{grad} f_1$ with implectic Θ and symplectic J. Note that in this situation J maps symmetries of (4) to symmetries and Θ maps conserved covariants to conserved covariants.

A field $\Phi(u, v)$: $M \to M$ is called a recursion operator for (4) if $\Phi'[K] - K'\Phi + \Phi K' = 0$. A recursion operator maps symmetries to symmetries and its adjoint maps conserved covariants to conserved covariants.

Such an operator $\Phi(u, v)$: $M \to M$ is called hereditary if $\Phi'[\Phi X_1]X_2 - \Phi\Phi'[X_1]X_2$ is symmetric in X_1 and X_2 for all $X_1, X_2 \in M$.

In Ref. 9) the notion of mastersymmetries was introduced. Let X be a space of commuting vectorfields, then a vectorfield τ is called a mastersymmetry (of degree 1) for this X if $[K, \tau] \in X$ for every $K \in X$. As X is assumed to be abelian we find for every $K \in X$ a time dependent symmetry $\tau + t[\tau, K]$.

The following theorem summarizes the results about bi-Hamiltonian systems with hereditary recursion operators:[5)~8)]

THEOREM 1

Let K_0 be a bi-Hamiltonian vectorfield with symplectic Θ and implectic J, i.e., $K_0 = \Theta$ grad f_0 and $JK_0 = \mathrm{grad} f_1$ for some functions f_0 and f_1. Let $\Phi := \Theta J$ be hereditary and invertible. Define $K_n := \Phi^n K_0$, $\gamma_{n+1} := JK_n$, $n \in Z$. Then every K_n generates a symmetry for the dynamical system given by any K_m, i.e., $[K_n, K_m] = 0$ for all $n, m \in Z$. All the γ_n's are closed, i.e., $\gamma_n = \mathrm{grad} f_n$, for any K_m every f_n is a conservation law. We have $\{f_n, f_m\} := \langle \mathrm{grad} f_n, \Theta \mathrm{grad} f_m\rangle = 0$ for all $n, m \in Z$, where $\{,\}$ defines a Poisson-bracket. All the dynamical systems associated with the K_n's are bi-Hamiltonian: $K_n = \Theta \mathrm{grad} f_n$, $JK_n = \mathrm{grad} f_{n+1}$.

As to mastersymmetries and time-dependent symmetries we will apply:

THEOREM 2

Let K_0 be a vectorfield with an invertible hereditary recursion operator Φ. Let there be a vectorfield τ_0 "scaling" both K_0 and Φ, i.e.,

$$[\tau_0, K_0] = \alpha K_0;\ \alpha \in R \tag{5}$$

and

$$\Phi'[\tau_0] - \tau_0'\Phi + \Phi\tau_0' = \beta\Phi;\ \beta \in R\,. \tag{6}$$

Let $K_n := \Phi^n K_0$, $\tau_n := \Phi^n \tau_0$, $n \in Z$, then we find

$$\begin{aligned}&[K_n, K_m] = 0\,,\\ &[\tau_n, K_m] = (\alpha + m\beta)K_{n+m}\,,\\ &[\tau_n, \tau_m] = (m-n)\beta\tau_{n+m}\end{aligned} \tag{7}$$

for all $n, m \in Z$. Hence every τ_n is a mastersymmetry of the hierarchy given by the K_m's, every $\tau_n + t(\alpha + m\beta)K_{n+m}$ is a time-dependent symmetry for the dynamical system given by K_m.

Proof (of theorem 2):

From the hereditary property of Φ (and hence Φ^{-1}) one concludes inductively that i) every K_n admits Φ as recursion operator and ii) a relation similar to (6) holds for Φ and every τ_n, $n \in Z$:

$$\Phi'[\tau_n] - \tau_n'\Phi + \Phi\tau_n' = \beta\Phi^{n+1};\ \beta \in R\,. \tag{8}$$

From this (7) is obtained easily by induction. A more elaborate proof of such a situation is given in Ref. 10), where these structures are interpreted in a purely geometrical framework.

The results for the DNLS

The operator

$$\Theta(u, v) := \begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix} \tag{9}$$

is implectic ($D := \partial/\partial x$), whereas

$$J(u, v) := \begin{bmatrix} vD^{-1}v & -i + vD^{-1}u \\ i + uD^{-1}v & uD^{-1}u \end{bmatrix} \tag{10}$$

is symplectic (on a suitable submanifold on which $D^{-1} := \int_{-\infty}^{x} \cdots$ is antisymmetric). The operator $\Phi := \Theta J$ turns out to be hereditary. Note that both operators can be inverted easily, one finds

$$\Theta^{-1}(u, v) := \begin{bmatrix} 0 & D^{-1} \\ D^{-1} & 0 \end{bmatrix} \tag{11}$$

and

$$J^{-1}(u,v):=\begin{bmatrix}-uD^{-1}u & -i+uD^{-1}v\\ i+vD^{-1}u & -vD^{-1}v\end{bmatrix} \quad (12)$$

leading to an explicit inverse of Φ. Choosing

$$K_0(u,v):=\begin{bmatrix}u_x\\ v_x\end{bmatrix}, \quad (13)$$

one checks $K_0=\Theta$ grad f_0 and $JK_0=$grad f_1 with

$$f_0(u,v)=\int_{-\infty}^{\infty} uv\, dx\,, \quad (14\cdot1)$$

$$f_1(u,v)=1/2\int_{-\infty}^{\infty}(u^2v^2-i(uv_x-vu_x))\,dx\,. \quad (14\cdot2)$$

Therefore according to theorem 1, Φ generates a hierarchy of commuting bi-Hamiltonian equations $K_n:=\Phi^n K_0$, $n\in Z$, the Hamiltonians of which form an integrable system. Note that the positive part of this hierarchy is the Lax-hierarchy of symmetries found in Ref. 4), all these quantities turn out to be local in the space-variable. The DNLS (3) is found among these vectorfields as the element K_1. But apart from this known results we also get the negative hierarchy K_{-1}, K_{-2},$\cdots$ of nonlocal commuting symmetries, e.g., we find

$$K_{-1}(u,v)=\begin{bmatrix}-iu\\ iv\end{bmatrix}=\Theta \text{ grad}\int_{-\infty}^{\infty} iuD^{-1}v dx\,, \quad (15)$$

(generating a gauge transformation, still local) and

$$K_{-2}(u,v)$$
$$=\begin{bmatrix}-D^{-1}u-iuD^{-1}vD^{-1}u-iuD^{-1}uD^{-1}v\\ -D^{-1}v+ivD^{-1}uD^{-1}v+ivD^{-1}vD^{-1}u\end{bmatrix}$$
$$=\Theta \text{ grad}\int_{-\infty}^{\infty}(-uD^{-2}v+i/4(uD^{-1}v-vD^{-1}u)\times D^{-1}(uD^{-1}v+vD^{-1}u))\,dx. \quad (16)$$

Note that we assume D^{-1} to be anti-symmetric.

Finally, let us consider time-dependent symmetries for the above hierarchy. Note that the DNLS admits a scaling transform which turns out to be an admissible scaling for the symmetries as well:

Choosing

$$\tau_0(u,v)=\begin{bmatrix}xu_x+1/2u\\ xv_x+1/2v\end{bmatrix}, \quad (17)$$

we obtain

$$[\tau_0, K_0]=K_0\,. \quad (18)$$

Checking that this τ_0 scales the recursion operator as well

$$\Phi'[\tau_0]-\tau_0'\Phi+\Phi\tau_0'=\Phi\,, \quad (19)$$

we can apply theorem 2 to obtain an infinite sequence of mastersymmetries for the DNLS hierarchy:

$$[\tau_n, K_m]=(1+m)K_{n+m}\,,$$
$$[\tau_n, \tau_m]=(n-m)\tau_{n+m} \quad (20)$$

for all n, $m\in Z$.

Hence $\tau_n+t(1+m)K_{n+m}$ is a time-dependent symmetry for the dynamical system induced by K_m. From the above structure constants it is clear that the sequences of the K_n's and τ_n's can also be obtained by iterative application of the commutator with

$$\tau_1(u,v)=xK_1+\begin{bmatrix}u^2v+3/2iu_x\\ v^2u-3/2iv_x\end{bmatrix} \quad (21)$$

and

$$\tau_{-1}(u,v)=xK_{-1}+1/2\times\begin{bmatrix}iD^{-1}u+uD^{-1}(uD^{-1}v-vD^{-1}u)\\ -iD^{-1}v+vD^{-1}(vD^{-1}u-uD^{-1}v)\end{bmatrix} \quad (22)$$

starting with the simple vectorfields K_0, K_{-2}, τ_0, τ_2 and τ_{-2}.

We want to remark that the conservation laws mentioned in theorem 1 can also be obtained recursively using the mastersymmetries τ_m:

$$\langle\text{grad } f_n, \tau_m\rangle=(m+n)f_{n+m}\,. \quad (23)$$

A detailed analysis of these recursion schemes is found in Ref. 10).

Conclusions

One interesting aspect of the DNLS is the fact that its crucial structures (the recursion operator and its Hamiltonian formulations) can be inverted explicitly, thus yielding a hierarchy of nonlocal symmetries and conservation laws. Similar results have been obtained for other systems as well.[11] As a Baecklund transformation to the AKNS system is known[2] one now can transform these new quantities to nonlocal symmetries and conservation laws for the AKNS system. This shall be done in a subsequent paper.

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