A Nonlinear Programming Approach to Optimal Design Centering, Tolerancing, and Tuning

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***Abstract*—A theory of optimal worst-case design embodying centering, tolerancing, and tuning is presented. Some simplified problems and special cases are discussed. Projections and slack variables are used to explain some of the concepts. The worst-case tolerance assignment and design centering problem falls out as a special case. Practical implementation requires a reasonable and relevant number of parameters and constraints to be identified to make the problem tractable. Two circuits, a simple *LC* low-pass filter and a realistic high-pass filter, are studied under a variety of different problem situations to illustrate both the benefits to be derived from our approach and the difficulties encountered in its implementation.**

I. Introduction

COMPONENT TOLERANCE ASSIGNMENT is now considered to be an integral part of the design process [1]–[7]. The optimal worst-case tolerance problem with variable nominal point has benefitted in terms of increased tolerances [5]–[7]. Tuning [7], [8], on the other hand, does not seem to have been given its proper place in the design process. This work, therefore, brings in tuning of one or more components basically to further increase tolerances to reduce cost or to make unrealistically toleranced solutions more attractive. The mathematical formulation of an approach which embodies centering, tolerancing, and tuning in a unified manner is presented. Simplified problems and appropriate geometric interpretations are discussed. The worst-case purely toleranced problem and purely tuned problem fall out as special cases, as is to be expected. Numerical examples involving simple functions and a realistic as well as a simple circuit, illustrate the concepts.

II. Fundamental Concepts and Definitions

A *design* consists of design data of the *nominal point* $\boldsymbol{\phi}^0$, the *tolerance vector* $\boldsymbol{\varepsilon}$ and the *tuning vector* $\boldsymbol{t}$ where, for k parameters,

$$\boldsymbol{\phi}^0 \triangleq \begin{bmatrix} \phi_1^{\,0} \\ \phi_2^{\,0} \\ \vdots \\ \phi_k^{\,0} \end{bmatrix}, \ \boldsymbol{\varepsilon} \triangleq \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_k \end{bmatrix}, \text{ and } \boldsymbol{t} \triangleq \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_k \end{bmatrix}. \tag{1}$$

We assume that the parameters can be varied continuously and chosen independently. Extra conditions such as discretization and imposed parameter bounds may be treated as constraints [6]. Some of the parameters can be set to zero or held constant.

An *outcome* $\{\boldsymbol{\phi}^0,\boldsymbol{\varepsilon},\boldsymbol{\mu}\}$ of a *design* $\{\boldsymbol{\phi}^0,\boldsymbol{\varepsilon},\boldsymbol{t}\}$ implies a point

$$\boldsymbol{\phi} = \boldsymbol{\phi}^0 + \boldsymbol{E}\boldsymbol{\mu} \tag{2}$$

where

$$\boldsymbol{E} \triangleq \begin{bmatrix} \varepsilon_1 & & & \\ & \varepsilon_2 & & \\ & & \ddots & \\ & & & \varepsilon_k \end{bmatrix} \tag{3}$$

and $\boldsymbol{\mu} \in R_\mu$. R_μ is a set of multipliers determined from realistic situations of the tolerance spread. For example,

$$R_\mu \triangleq \{\boldsymbol{\mu} \mid -1 \le \mu_i \le -a_i \text{ or } a_i \le \mu_i \le 1,\ i \in I_\phi\} \tag{4}$$

where

$$0 \le a_i \le 1 \tag{5}$$

and

$$I_\phi \triangleq \{1,2,\cdots,k\}. \tag{6}$$

The most commonly used continuous range is obtained by setting a_i to zero. A commercial stock may have the better toleranced components taken out, thus $0 < a_i \le 1$. Unless otherwise stated, we consider

$$R_\mu \triangleq \{\boldsymbol{\mu} \mid -1 \le \mu_i \le 1,\ i \in I_\phi\}. \tag{7}$$

The *tolerance region* R_ε is a set of points described by (2) for all $\boldsymbol{\mu} \in R_\mu$. In the case of $-1 \le \mu_i \le 1$, $i \in I_\phi$,

$$R_\varepsilon \triangleq \{\boldsymbol{\phi} \mid \phi_i = \phi_i^{\,0} + \varepsilon_i\mu_i,\ -1 \le \mu_i \le 1,\ i \in I_\phi\} \tag{8}$$

which is a convex *regular polytope* of k dimensions with sides of length $2\varepsilon_i$, $i \in I_\phi$, and centered at $\boldsymbol{\phi}^0$. The extreme points of R_ε are obtained by setting $\mu_i = \pm 1$. Thus, the set of *vertices* may be defined as

$$R_v \triangleq \{\boldsymbol{\phi} \mid \phi_i = \phi_i^{\,0} + \varepsilon_i\mu_i,\ \mu_i \in \{-1,1\},\ i \in I_\phi\}. \tag{9}$$

The number of points in R_v is 2^k. Let each of these points be indexed by $\boldsymbol{\phi}^i$, $i \in I_v$, where

$$I_v \triangleq \{1,2,\cdots,2^k\}. \tag{10}$$

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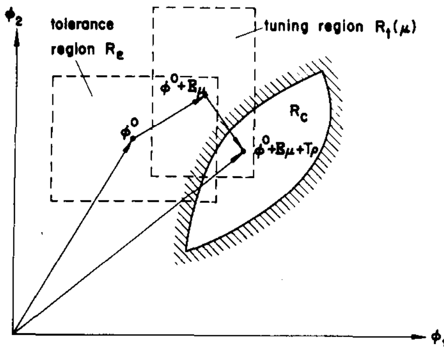


Fig. 1. Illustration of regions R_ε,R_t, and R_c. If $\mu = 0$ then R_t is centered at ϕ^0.

Thus

$$R_v = \{\phi^1,\phi^2,\cdots,\phi 2^k\}.$$

The *tuning region* $R_t(\mu)$ is defined as the set of points

$$\phi = \phi^0 + E\mu + T\rho \tag{11}$$

for all $\rho \in R_\rho$, where

$$T \triangleq \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_k \end{bmatrix}. \tag{12}$$

The components of ρ will be called slack variables since they do not directly contribute to the objective function. Some of the common examples of R_ρ are

$$R_\rho \triangleq \{\rho \mid -1 \le \rho_i \le 1, i \in I_\phi\} \tag{13}$$

or in the case of *one-way tuning* or *irreversible trimming*,

$$R_\rho = \{\rho \mid 0 \le \rho_i \le 1, i \in I_\phi\} \tag{14}$$

or

$$R_\rho = \{\rho \mid -1 \le \rho_i \le 0, i \in I_\phi\}. \tag{15}$$

Unless otherwise indicated, the case given by (13) is considered.

The *constraint region* R_c is given by

$$R_c \triangleq \{\phi \mid g_i(\phi) \ge 0, \quad \text{for all } i \in I_c\} \tag{16}$$

where

$$I_c \triangleq \{1,2,\cdots,m_c\} \tag{17}$$

is the index set for the performance specifications and parameter constraints. R_c is assumed to be not empty. Other conditions and assumptions will be imposed on R_c as the theory is developed further.

The definitions are illustrated in Fig. 1 by a two-dimensional example.

A *tunable constraint region* is denoted by $R_c(\psi)$, where ψ represents other independent variables. Fig. 2 depicts three different regions of an example of $R_c(\psi)$. Overlapping of these regions is allowable. The value of ψ may be continuous or discrete. $R_c(\psi) = R_c$ in the ordinary sense if ψ is a constant.

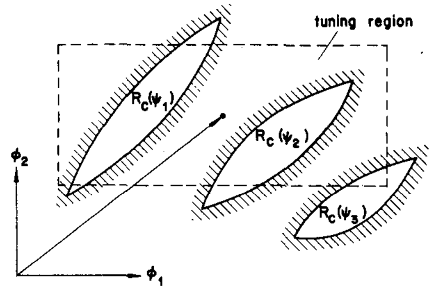


Fig. 2. Example of three different settings of tunable constraint regions.

III. The Original Problem P_0

The problem may be stated as follows: obtain a set of optimal design values $\{\phi^0,\varepsilon,t\}$ such that any outcome $\{\phi^0,\varepsilon,\mu\}$, $\mu \in R_\mu$, may be tuned into R_c for some $\rho \in R_\rho$.

It is formulated as the nonlinear programming problem:

$$P_0\text{: minimize } C(\phi^0,\varepsilon,t)$$

$$\text{subject to } \phi \in R_c$$

where

$$\phi = \phi^0 + E\mu + T\rho \tag{18}$$

and constraints $\phi^0,\varepsilon,t \ge 0$, for all $\mu \in R_\mu$ and some $\rho \in R_\rho$. C is an appropriate function chosen to represent a reasonable approximation to known component cost data.

Stated in an abstract sense, the *worst-case solution* of the problem must satisfy

$$R_t(\mu) \cap R_c \neq \varnothing \tag{19}$$

for all $\mu \in R_\mu$, where $\varnothing$ denotes a null set.

IV. The Reduced Problem P_1

The original problem P_0 of the preceding section can be reduced by separating the components into *effectively tuned* and *effectively toleranced* parameters. Let

$$I_\varepsilon \triangleq \{i \mid \varepsilon_i > t_i, i \in I_\phi\} \tag{20}$$

$$I_t \triangleq \{i \mid t_i \ge \varepsilon_i, i \in I_\phi\} \tag{21}$$

$$\varepsilon_i' \triangleq \varepsilon_i - t_i, i \in I_\varepsilon \tag{22}$$

and

$$t_i' \triangleq t_i - \varepsilon_i, i \in I_t. \tag{23}$$

It is obvious that I_t and I_ε are disjoint and $I_t \cup I_\varepsilon = I_\phi$.

Now, consider the problem

$$P_1\text{: minimize } C(\phi^0,\varepsilon,t)$$

$$\text{subject to } \phi \in R_c$$

where

$$\phi_i = \phi_i^0 + \begin{cases} \varepsilon_i'\mu_i, & \text{for } i \in I_\varepsilon \\ t_i'\rho_i', & \text{for } i \in I_t \end{cases} \tag{24}$$

for all $-1 \leq \mu_i \leq 1$, $i \in I_\varepsilon$, and for some $-1 \leq \rho_i' \leq 1$, $i \in I_t$.

Theorem 1

A feasible solution to the *reduced problem* P_1 is a feasible solution to the original problem P_0.

Proof: Given $\boldsymbol{\phi}^0, \boldsymbol{\varepsilon}, \boldsymbol{t}$ we will show that

1) $\varepsilon_i\mu_i + t_i\rho_i = \varepsilon_i'\mu_i, \; i \in I_\varepsilon$ (25)

2) $\varepsilon_i\mu_i + t_i\rho_i = t_i'\rho_i', \; i \in I_t$ (26)

under the restrictions on μ_i, ρ_i, and ρ_i'.

1) Since ρ_i can be freely chosen from $-1 \leq \rho_i \leq 1$, we can let $\rho_i = -\mu_i$ giving

$$(\varepsilon_i - t_i)\mu_i = \varepsilon_i'\mu_i. \quad (27)$$

2) For any $-1 \leq \rho_i' \leq 1$ and all $-1 \leq \mu_i \leq 1$, we can choose

$$-1 \leq \rho_i = \frac{(t_i - \varepsilon_i)\rho_i' - \varepsilon_i\mu_i}{t_i} \leq 1, \qquad t_i \neq 0. \quad (28)$$

Thus any point with components represented by (24) of the reduced problem can be represented by (18) of the original problem.

Intuitively, this theorem states the fact that a feasible solution to a restrictive problem is also a feasible solution to an appropriate less restrictive problem. The variable transformation (22) and (23) may be considered as extraneous constraints to be satisfied.

Theorem 2

A feasible solution to the original problem P_0 implies a feasible solution to the reduced problem P_1 if R_c is one-dimensionally convex [3].

Proof: 1) We note, for $i \in I_\varepsilon$, that

$$\phi_i^0 - \varepsilon_i + t_i\rho_i(-1) \leq \phi_i^0 - \varepsilon_i + t_i \leq \phi_i^0 + (\varepsilon_i - t_i)\mu_i \leq \phi_i^0 + \varepsilon_i - t_i \leq \phi_i^0 + \varepsilon_i + t_i\rho_i(1) \quad (29)$$

where $\rho_i(-1)$ corresponds to $\mu_i = -1$ and $\rho_i(1)$ corresponds to $\mu_i = 1$. If R_c is one-dimensionally convex, the following assumption

$$\begin{bmatrix} \vdots \\ \phi_i^0 - \varepsilon_i + t_i\rho_i(-1) \\ \vdots \end{bmatrix}, \quad \begin{bmatrix} \vdots \\ \phi_i^0 + \varepsilon_i + t_i\rho_i(1) \\ \vdots \end{bmatrix} \in R_c \quad (30)$$

implies that

$$\begin{bmatrix} \vdots \\ \phi_i^0 + (\varepsilon_i - t_i)\mu_i \\ \vdots \end{bmatrix} \in R_c \quad (31)$$

where we consider changes in the ith component only and impose the required restrictions on μ_i and ρ_i.

2) On the other hand, for $i \in I_t$, given feasible $\rho_i(-1)$ and $\rho_i(1)$ such that

$$\phi_i^0 - \varepsilon_i + t_i\rho_i(-1) \leq \phi_i^0 + \varepsilon_i + t_i\rho_i(1) \quad (32)$$

there exists a feasible ρ_i' such that

$$\phi_i^0 - \varepsilon_i + t_i\rho_i(-1) \leq \phi_i^0 + (t_i - \varepsilon_i)\rho_i' \leq \phi_i^0 + \varepsilon_i + t_i\rho_i(1). \quad (33)$$

This is true for $t_i = \varepsilon_i$ and can be seen for $t_i > \varepsilon_i$ by rewriting this inequality as

$$\frac{-\varepsilon_i + t_i\rho_i(-1)}{t_i - \varepsilon_i} \leq \rho_i' \leq \frac{\varepsilon_i + t_i\rho_i(1)}{t_i - \varepsilon_i}. \quad (34)$$

Hence, if R_c is one-dimensionally convex, the assumption implies that

$$\begin{bmatrix} \vdots \\ \phi_i^0 + (t_i - \varepsilon_i)\rho_i' \\ \vdots \end{bmatrix} \in R_c. \quad (35)$$

Thus, a feasible solution to the original problem can be transformed to a feasible solution of the reduced problem P_1.

A Geometric Interpretation

Let us define a *projection matrix* $\boldsymbol{P}$ as a diagonal matrix such that

$$\boldsymbol{P} \triangleq \begin{bmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_k \end{bmatrix} \quad (36)$$

where

$$p_i = \begin{cases} 0, & \text{for } i \in I_t \\ 1, & \text{for } i \in I_\varepsilon \end{cases}. \quad (37)$$

The projection of a point $\boldsymbol{\phi}$ may be denoted as $\boldsymbol{\phi}_p = \boldsymbol{P\phi}$. It may be noted that the projections of two points $\boldsymbol{\phi}^a, \boldsymbol{\phi}^{b(j)} = \boldsymbol{\phi}^a + \alpha\boldsymbol{e}_j$, where $\boldsymbol{e}_j$ is the jth unit vector, for $j \in I_t$, and some constant α, coincide. The projection concept and the introduction of slack variables provide a key to understanding the tuning concept.

Let

$$R_{\varepsilon t} \triangleq \{\boldsymbol{\phi} \mid \phi_i^0 - \varepsilon_i' \leq \phi_i \leq \phi_i^0 + \varepsilon_i', \; i \in I_\varepsilon\} \quad (38)$$

and

$$R_{t\varepsilon} \triangleq \{\boldsymbol{\phi} \mid \phi_i^0 - t_i' \leq \phi_i \leq \phi_i^0 + t_i', \; i \in I_t\} \quad (39)$$

denote the regions defined by the effectively toleranced and effectively tuned parameters. Then consider the following regions

$$R_{\varepsilon tp} \triangleq \{\boldsymbol{\phi}_p \mid \boldsymbol{\phi}_p = \boldsymbol{P\phi}, \; \boldsymbol{\phi} \in R_{\varepsilon t}\} \quad (40)$$

$$R_{ct\varepsilon} \triangleq R_c \cap R_{t\varepsilon}, \quad (41)$$

and

$$R_{ct\varepsilon p} \triangleq \{\boldsymbol{\phi}_p \mid \boldsymbol{\phi}_p = \boldsymbol{P\phi}, \; \boldsymbol{\phi} \in R_{ct\varepsilon}\}. \quad (42)$$

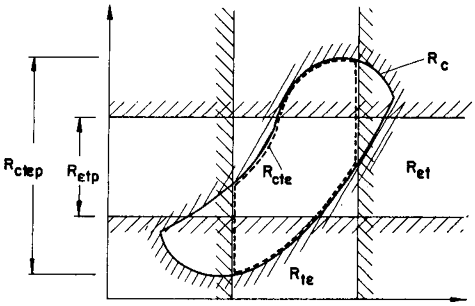


Fig. 3. Geometric interpretation of reduced problem P_1.

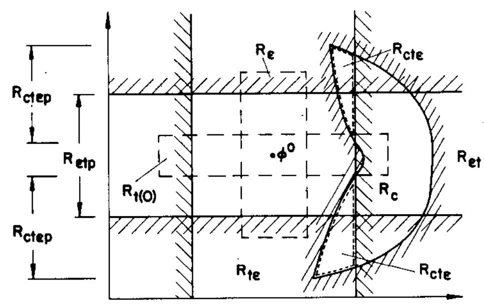


Fig. 4. Example of $R_{\varepsilon tp} \nsubseteq R_{ct\varepsilon p}$. $R_t(\mu)$ for $\mu = 0$ is indicated, for convenience.

Fig. 3 illustrates the definition of the regions. Any point whose components are given by (24) lies in the intersection of $R_{\varepsilon t}$ and $R_{t\varepsilon}$. Suppose the projection of $R_{ct\varepsilon}$ onto the subspace spanned by the effectively toleranced parameters includes the projection of that point. Then it may be tuned into $R_{ct\varepsilon}$ by adjusting the value of ρ_i', $i \in I_t$.

The reduced problem P_1 may be stated as: solve a pure tolerance problem (i.e., no tuning) in the subspace spanned by the toleranced variables with $R_{\varepsilon tp}$ as the tolerance region and $R_{ct\varepsilon p}$ as the constraint region. In other words, the regions defined by a feasible solution must satisfy the condition that

$$R_{\varepsilon tp} \subseteq R_{ct\varepsilon p}. \tag{43}$$

Fig. 4 illustrates a case where $R_{\varepsilon tp} \nsubseteq R_{ct\varepsilon p}$. An outcome, for example, at ϕ^0 cannot be tuned to R_c within the effective tuning range. However, there exists a solution to the original formulation by tuning both components. R_c is not one-dimensionally convex in this case.

Special Cases

Case 1: $I_\varepsilon = \varnothing$, the Pure Tuning Problem: In this case, $R_{\varepsilon t}$ is the entire space and $\boldsymbol{P}$ is a zero matrix. $R_{\varepsilon tp}$ is a single point at the origin. The problem has a solution if

$$R_{ct\varepsilon} \neq \varnothing. \tag{44}$$

Case 2: $I_t = \varnothing$, the Pure Tolerance Problem: In this case, $R_{t\varepsilon}$ is the entire space and $\boldsymbol{P}$ is a unit matrix:

$$R_{\varepsilon tp} = R_{\varepsilon t} \quad \text{and} \quad R_{ct\varepsilon p} = R_{ct\varepsilon} = R_c.$$

The problem has a solution if

$$R_{\varepsilon t} \subseteq R_c. \tag{45}$$

From a tolerance–tuning point of view, the first case is trivial theoretically. Except when there is only one single point R_c, the pure tuning problem is equivalent to an optimization of the nominal parameter values. On the other hand, the pure tolerance problem is very important from a practical point of view.

Extension of P_1 for Tunable Constraint Region

Three types of components can be identified when the constraint region is considered to be tunable. They are a) toleranced components, b) components tuned by the manufacturer, and c) components tunable by the customer. In this case,

$$\boldsymbol{\phi} \in R_c(\boldsymbol{\psi})$$

where

$$\phi_i = \phi_i^0 + \begin{cases} \varepsilon_i'\mu_i, & \text{for } i \in I_\varepsilon \\ t_i'\rho_i', & \text{for } i \in I_{tm} \\ t_i'\rho_i'(\boldsymbol{\psi}), & \text{for } i \in I_{tc} \end{cases} \tag{46}$$

where I_{tm} identifies components b) and I_{tc} identifies components c).

Setting the $\boldsymbol{\psi}$ to a particular value will control the setting of ρ_i', $i \in I_{tc}$, such that $\boldsymbol{\phi}$ will be in that particular constraint region $R_c(\boldsymbol{\psi})$.

V. The Reduced Problem P_2

It is impossible to test all the points in $R_{\varepsilon tp}$ to be in $R_{ct\varepsilon p}$. In order to make the problem tractable a number of simplifying assumptions could be made to obtain an acceptable solution to the problem with reasonable effort. To this end we replace the continuous range $-1 \leq \mu_i \leq 1$ by a discrete set $\mu_i \in \{-1,1\}$, $i \in I_\varepsilon$. Now, consider the problem

$$P_2\text{: minimize } C(\boldsymbol{\phi}^0, \boldsymbol{\varepsilon}, \boldsymbol{t})$$

$$\text{subject to } \boldsymbol{\phi} \in R_c$$

where

$$\phi_i = \phi_i^0 + \begin{cases} \varepsilon_i'\mu_i, & \text{for } i \in I_\varepsilon \\ t_i'\rho_i', & \text{for } i \in I_t \end{cases} \tag{47}$$

for all $\mu_i \in \{-1,1\}$, $i \in I_\varepsilon$, and some $-1 \leq \rho_i' \leq 1$, $i \in I_t$.

Let us define the set of *projected vertices* (or the vertices of the projected region) by

$$R_{vp} \triangleq \{\boldsymbol{\phi}_p \mid \boldsymbol{\phi}_p = \boldsymbol{P}\boldsymbol{\phi},\ \boldsymbol{\phi} \in R_v\}. \tag{48}$$

The condition may be now stated as

$$R_{vp} \subseteq R_{ct\varepsilon p}.$$

Theorem 3

A feasible solution to *reduced problem P_2* implies a feasible solution to reduced problem P_1 if $R_{ct\varepsilon p}$ is one-dimensionally convex.

This is a pure tolerance problem in the subspace spanned by the effectively toleranced parameters. For a proof in the tolerance parameter space, see Bandler [3].

VI. The Objective Functions

Several *objective functions* (or *cost functions*) have been proposed [1]–[5]. In practice, a suitable modeling problem would have to be solved to determine the cost-tolerance relationship. Here, it is assumed that the tolerances and tuning ranges (either absolute or relative) are the main variables and that the total cost of the design is the sum of the cost of the individual components.

The objective function should have the following properties

$$\begin{aligned} &C(\phi^0,\varepsilon,t) \to c, && \text{as } \varepsilon \to \infty \\ &C(\phi^0,\varepsilon,t) \to \infty, && \text{for any } \varepsilon_i \to 0 \\ &C(\phi^0,\varepsilon,t) \to C(\phi^0,\varepsilon), && \text{as } t \to 0 \\ &C(\phi^0,\varepsilon,t) \to \infty, && \text{for any } t_i \to \infty. \end{aligned} \tag{49}$$

Suitable objective functions will be, for example, of the form

$$C = \sum_{i=1}^{k} \frac{c_i}{x_i} + \sum_{i=1}^{k} c_i' y_i \tag{50}$$

where x_i and y_i denote the tolerances and tuning ranges, respectively. In the case of relative tolerances or relative tuning ranges $x_i = \varepsilon_i/\phi_i^0 \times 100$, $y_i = t_i/\phi_i^0 \times 100$. We may set the appropriate c_i' to zero if tuning is considered either free, or fixed or is not required. c_i may be set to zero if the corresponding tolerance is fixed.

VII. Mathematical Example

Consider the constraints

$$\phi_2 - \phi_1 - 2 \geq 0 \tag{51}$$

$$-\phi_2^2 + 16\phi_1 \geq 0. \tag{52}$$

A convex region R_c is defined by these constraints.

We will take R_μ as an infinite set of discrete points $\mu(i)$, $i = 1,2,\cdots$, where $-1 \leq \mu_1(i) \leq 1$ and $-1 \leq \mu_2(i) \leq 1$. Thus a relevant problem may be formulated as follows. Minimize

$$C = \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} \tag{53}$$

with respect to ε_1, ε_2, ϕ_1^0, and ϕ_2^0, and subject to

$$\begin{aligned} g_1 &= \varepsilon_1 \geq 0 \\ g_2 &= \varepsilon_2 \geq 0 \\ g_3 &= \phi_1^0 \geq 0 \\ g_4 &= \phi_2^0 \geq 0 \end{aligned} \tag{54}$$

$$g_5(i) = (\phi_2^0 + \varepsilon_2\mu_2(i)) - (\phi_1^0 + \varepsilon_1\mu_1(i)) - 2 \geq 0, \quad i = 1,2,\cdots \tag{55}$$

$$g_6(i) = -(\phi_2^0 + \varepsilon_2\mu_2(i))^2 + 16(\phi_1^0 + \varepsilon_1\mu_1(i)) \geq 0, \quad i = 1,2,\cdots \tag{56}$$

where $-1 \leq \mu_1(i) \leq 1$ and $-1 \leq \mu_2(i) \leq 1$.

Optimality requires that

$$\begin{bmatrix} -\dfrac{1}{\varepsilon_1^2} \\ -\dfrac{1}{\varepsilon_2^2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \sum_i u_5(i) \begin{bmatrix} -\mu_1(i) \\ \mu_2(i) \\ -1 \\ 1 \end{bmatrix} + \sum_i u_6(i) \begin{bmatrix} 16\mu_1(i) \\ -2\mu_2(i)(\phi_2^0 + \varepsilon_2\mu_2(i)) \\ 16 \\ -2(\phi_2^0 + \varepsilon_2\mu_2(i)) \end{bmatrix} \tag{57}$$

$$u_1 g_1 = \cdots = u_4 g_4 = u_5(i) g_5(i) = u_6(i) g_6(i) = 0, \quad i = 1,2,\cdots \tag{58}$$

$$u_1,\cdots,u_4,u_5(i),u_6(i) \geq 0, \qquad i = 1,2,\cdots \tag{59}$$

where u denotes a multiplier. To solve the above equations, assume that ε_1, ε_2, ϕ_1^0, and ϕ_2^0 are not zero, therefore, set u_1, u_2, u_3, and u_4 to zero. Minimize $g_5(i)$ of (55) and $g_6(i)$ of (56) with respect to $\mu(i)$. This leads, respectively, to

$$(\phi_2^0 - \varepsilon_2) - (\phi_1^0 + \varepsilon_1) - 2 \geq 0 \tag{60}$$

using $\mu(i) = [1\ -1]^T$ and

$$-(\phi_2^0 + \varepsilon_2)^2 + 16(\phi_1^0 - \varepsilon_1) \geq 0 \tag{61}$$

using $\mu(i) = [-1\ 1]^T$. The optimality conditions (57)–(59) are correspondingly reduced yielding the solution

$$\begin{aligned} \varepsilon_1 &= 0.5 \\ \varepsilon_2 &= 0.5 \\ \phi_1^0 &= 4.5 \\ \phi_2^0 &= 7.5. \end{aligned}$$

Consider next the problem of minimizing

$$C = \frac{1}{\varepsilon_2} \tag{62}$$

with respect to t_1', ε_2, ϕ_1^0, ϕ_2^0, and $\rho_1(i)$, and subject to

$$\begin{aligned} g_1 &= t_1' \geq 0 \\ g_2 &= \varepsilon_2 \geq 0 \\ g_3 &= \phi_1^0 \geq 0 \\ g_4 &= \phi_2^0 \geq 0 \end{aligned} \tag{63}$$

$$g_5 = 0.1 - \frac{t_1'}{\phi_1^0} \geq 0 \tag{64}$$

$$g_6(i) = (\phi_2^0 + \varepsilon_2\mu_2(i)) - (\phi_1^0 + t_1'\rho_1'(i)) - 2 \geq 0, \quad i = 1,2,\cdots \tag{65}$$

$$g_7(i) = -(\phi_2^0 + \varepsilon_2\mu_2(i))^2 + 16(\phi_1^0 + t_1'\rho_1'(i)) \geq 0, \quad i = 1,2,\cdots \tag{66}$$

$$g_8(i) = 1 - \rho_1'(i) \geq 0, \qquad i = 1,2,\cdots \tag{67}$$

$$g_9(i) = 1 + \rho_1'(i) \geq 0, \qquad i = 1,2,\cdots. \tag{68}$$

Here, ε_1 is considered fixed at 0.5 and there is a maximum effective tuning range of 10 percent. Hence, the first component does not contribute to the cost. The effective tuning range $t_1' = t_1 - 0.5$ is used as a variable.

The optimality conditions require that

$$\begin{bmatrix} 0 \\ -\dfrac{1}{\varepsilon_2^2} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ 0 \end{bmatrix} + u_5 \begin{bmatrix} -\dfrac{1}{\phi_1^0} \\ 0 \\ \dfrac{t_1'}{\phi_{10}^{02}} \\ 0 \\ 0 \end{bmatrix} + \sum_i u_6(i) \begin{bmatrix} -\rho_1'(i) \\ \mu_2(i) \\ -1 \\ 1 \\ -t_1'e_i \end{bmatrix}$$

$$+ \sum_i u_7(i) \begin{bmatrix} 16\rho_1'(i) \\ -2(\phi_2^0 + \varepsilon_2\mu_2(i))\mu_2(i) \\ 16 \\ -2(\phi_2^0 + \varepsilon_2\mu_2(i)) \\ 16t_1'e_i \end{bmatrix}$$

$$+ \sum_i u_8(i) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -e_i \end{bmatrix} + \sum_i u_9(i) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ e_i \end{bmatrix} \tag{69}$$

$$u_1g_1 = \cdots = u_5g_5 = u_6(i)g_6(i) = \cdots = u_9(i)g_9(i) = 0, \quad i = 1,2,\cdots \tag{70}$$

$$u_1,\cdots,u_5,u_6(i),\cdots,u_9(i) \geq 0, \qquad i = 1,2,\cdots. \tag{71}$$

Minimize $g_6(i)$ of (65) and $g_7(i)$ of (66) with respect to $\mu_2(i)$. We use $\mu_2(i) = -1$ in (65) and $\mu_2(i) = 1$ in (66) for this purpose. The corresponding $\rho_1'(i) = -1$ and $\rho_1'(i) = 1$, respectively, are obtained by maximizing $g_6(i)$ and $g_7(i)$ with respect to $\rho_1'(i)$. This yields the solution

$$\begin{aligned} t_1' &= 0.5432 \\ \varepsilon_2 &= 1.444 \\ \phi_1^0 &= 5.4321 \\ \phi_2^0 &= 8.3333. \end{aligned}$$

As expected, the inclusion of tunable elements can increase the tolerance on the components. The tolerance of the second parameter increases from $\varepsilon_2 = 0.5$ to $\varepsilon_2 = 1.444$ when the first component is allowed to have a maximum effective tuning range of 10 percent. This means that an actual absolute tuning of 1.0432 and a tolerance of 0.5 are designed for ϕ_1. The result can only be accomplished by allowing the nominal point to move. For example, the first component moved from 4.5 to 5.4321, a shift of 20 percent.

VIII. Frequency Domain Implementation

Data for a specific problem is contained in a data vector $\boldsymbol{a}^i$ which has the form

$$\boldsymbol{a}^i \triangleq \begin{bmatrix} r \\ \boldsymbol{\mu} \\ \psi \\ S \\ w \end{bmatrix}, \qquad i = 1,2,\cdots,m_a \tag{72}$$

where ψ is an independent parameter denoting frequency or any number to identify a particular function for which the vertex $\boldsymbol{\phi}^r$ is chosen. $\boldsymbol{\mu}$ is the vector associated with $\boldsymbol{\phi}^r$, in particular,

$$r = 1 + \sum_{j=1}^{k} \left[\frac{\mu_j^r + 1}{2}\right] 2^{j-1}, \qquad \mu_j^r \in \{-1,1\}. \tag{73}$$

m_a is the total number of distinct vectors $\boldsymbol{a}^i$. S is a specification and w a weighting factor associated with each ψ. In our present work,

$$w = \begin{cases} +1, & \text{if } S \text{ is an upper specification} \\ -1, & \text{if } S \text{ is a lower specification.} \end{cases}$$

The performance constraints may now be formulated in the form of

$$g = w(S - F) \geq 0 \tag{74}$$

with appropriate subscripts. F is the circuit response function evaluated at sample point ψ and point $\boldsymbol{\phi}$ which is given by

$$\boldsymbol{\phi} = \boldsymbol{P}\boldsymbol{\phi}^r + \sum_{j \in I_t} (\phi_j^0 + t_j'\rho_j'(r))\boldsymbol{e}_j. \tag{75}$$

The projection matrix $\boldsymbol{P}$ and the index sets I_t and I_ε are fixed for a particular problem. They are determined before optimization takes place.

Let the n optimization variables be denoted by $\boldsymbol{x}$ including the variable nominal values, tolerances, tuning variables and all the appropriate slack variables $\rho_j'(r)$, $j \in I_t$, $r \in I_v$. Let m be the total number of constraints which include the performance specifications, slack variable bounds, parameter bounds, and any other extra constraints not considered above. In general, for linear network design in the frequency domain

$$n = k_0 + k_\varepsilon + k_t(1 + n_v) \tag{76}$$

and

$$m = \left[\sum_{i=1}^{n^\psi} n_v(i)\right] + 2k_tn_v + \cdots \tag{77}$$

where k_0, k_ε, and k_t are the number of variable nominal parameters, toleranced and tuned parameters, respectively; $n_v \leq 2^{k\varepsilon}$ is the number of distinct vertices chosen; n^ψ is the number of frequency points considered; $n_v(i)$ is the number of vertices chosen at the ith frequency point and $2k_tn_v$ is the number of slack variable bounds.

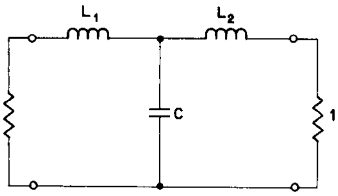


Fig. 5. Circuit for LC low-pass filter example.

TABLE I
SPECIFICATIONS FOR LC LOW-PASS FILTER

Frequency Range (rad/s)	Sample Points (rad/s)	Insertion Loss Specification (dB)	Type	Weight w
0 - 1	0.45, 0.50, 0.55, 1.0	1.5	upper (passband)	+1
2.5	2.5	25	lower (stopband)	-1

TABLE II
DATA FOR LOW-PASS FILTER

ℓ \ i	1	2	3	4	5
r	6	6	6	8	1
	+1	+1	+1	+1	-1
μ	-1	-1	-1	+1	-1
	+1	+1	+1	+1	-1
ψ	0.45	0.50	0.55	1.0	2.5
S	1.5	1.5	1.5	1.5	25
w	1	1	1	1	-1

Low-Pass Filter

The LC low-pass filter shown in Fig. 5 is considered [5], [6]. Table I summarizes the specifications. The critical vertices used in the data vector a^i can be obtained from published vertex selection schemes [6]. These schemes utilize first partial derivative information at some local points or local regions to predict the worst vertices. Very often updating of a^i is required at suitable intervals. In this case, the numerical experience we have gained previously from the tolerance problems [5], [6] allows us to chose the minimal set of vertices. These are: ϕ^6 at $\psi = 0.45, 0.50, 0.55$ rad/s; ϕ^8 at $\psi = 1.0$ rad/s and ϕ^1 at $\psi = 2.5$ rad/s, where $\phi = [L_1 C L_2]^T$. Updating was not required in this example except when all the three components are toleranced and tuned simultaneously. Table II summarizes the data for the filter.

Several cases have been studied [9] but the results of the case L_1 tuned with C and L_2 toleranced will be presented. The objective function used is based on the relative tolerances of C and L_2 in the form

$$C = \frac{x_2}{x_5^2} + \frac{x_3}{x_6^2} \tag{78}$$

where, assuming $t_C = t_{L_2} = 0$, and some fixed value of ε_{L_1}, we take

$$\begin{aligned} x_1 &= \phi_1{}^0 = L_1{}^0 \\ x_2 &= \phi_2{}^0 = C^0 \\ x_3 &= \phi_3{}^0 = L_2{}^0 \\ x_4{}^2 &= t_1{}' = t_{L_1} - \varepsilon_{L_1} \\ x_5{}^2 &= \varepsilon_2 = \varepsilon_C \\ x_6{}^2 &= \varepsilon_3 = \varepsilon_{L_2}. \end{aligned}$$

The cost of element L_1 is assumed fixed. It, therefore, is not included in (78). The last three transformations are chosen to avoid changes of sign. There are three distinct projected vertices: $\phi_p{}^6$, $\phi_p{}^8$, and $\phi_p{}^1$. The projection matrix in this case is

$$P = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}. \tag{79}$$

Therefore, the other variables may be identified as

$$\begin{aligned} x_7 &= \rho_1{}'(6) \\ x_8 &= \rho_1{}'(8) \\ x_9 &= \rho_1{}'(1). \end{aligned} \tag{80}$$

Substituting the numerical values from Table II into (75) we have the following:

$$a^1, a^2, a^3 \Rightarrow \phi = P\phi^6 + (\phi_1{}^0 + t_1{}'\rho_1{}'(6))e_1 = \begin{bmatrix} x_1 + x_4{}^2 x_7 \\ x_2 - x_5{}^2 \\ x_3 + x_6{}^2 \end{bmatrix} \tag{81}$$

$$a^4 \Rightarrow \phi = P\phi^8 + (\phi_1{}^0 + t_1{}'\rho_1{}'(8))e_1 = \begin{bmatrix} x_1 + x_4{}^2 x_8 \\ x_2 + x_5{}^2 \\ x_3 + x_6{}^2 \end{bmatrix} \tag{82}$$

$$a^5 \Rightarrow \phi = P\phi^1 + (\phi_1{}^0 + t_1{}'\rho_1{}'(1))e_1 = \begin{bmatrix} x_1 + x_4{}^2 x_9 \\ x_2 - x_5{}^2 \\ x_3 - x_6{}^2 \end{bmatrix}. \tag{83}$$

The performance specifications g_i, $i = 1,2,\cdots,5$, may now be formed. Additional constraints are given by

$$\left.\begin{aligned} g_{5+2i-1} &= 1 + x_{6+i} \\ g_{5+2i} &= 1 - x_{6+i} \end{aligned}\right\} \quad i = 1,2,3 \tag{84}$$
$$g_{12} = t_r - x_4{}^2/x_1.$$

The last constraint g_{12} is designed to limit the effective tuning range to t_r.

The resulting nonlinear programming problem (9 variables, 12 constraints) is solved by a least pth optimization algorithm due Charalambous [10] and the quasi-Newton method developed by Fletcher [11] and Gill and Murray [12]. The starting point corresponds to the optimally toleranced nominal point and arbitrary small tolerance and tuning values. Typically, a few hundred function evaluations with less than 30 s of CDC 6400 computing time is

TABLE III
L_1 TUNED, C AND L_2 TOLERANCED

Parameters	$t_r = 0.2$	$t_r = 0.1$	$t_r = 0.05$
L_1^0	2.0932	2.2442	2.1953
C^0	0.9360	0.9059	0.9062
L_2^0	1.7718	1.7569	1.7920
$100\, t_1'/L_1^0$	20.00 %	10.00 %	5.00 %
$100\, \varepsilon_2/C^0$	15.99 %	14.23 %	12.60 %
$100\, \varepsilon_3/L_2^0$	21.62 %	18.41 %	16.23 %
$\rho_1'(6)$		-1.0000	
$\rho_1'(8)$		-1.0000	
$\rho_1''(1)$		1.0000	
	n = 9	m = 12	

† For the optimally toleranced solution [5] $L_1{}^0 = L_2{}^0 = 1.9990$, $C^0 = 0.9056$, $100\varepsilon_1/L_1{}^0 = 100\varepsilon_3/L_2{}^0 = 9.89\%$, $100\varepsilon_2/C^0 = 7.60\%$.

required. Table III summarizes the results. Three different tuning ranges are used. The 5-percent tuning of L_1 increases the tolerances of the other two components by as much as 65 percent. A 10-percent positive and negative shift is obtained for $L_1{}^0$ and $L_2{}^0$, respectively. C^0 is shifted slightly. The slack variables assume values of -1, -1, and 1, indicating that the worst cases do occur at the vertices and, correspondingly, the tuning is set to extreme values.

Tuning of C presents a very interesting case. The symmetry property observed in the pure tolerance problem is preserved. Due to this symmetry, a 5-percent tuning range of C produces an increase of 90 percent in the tolerances of L_1 and L_2.

Suppose the designer has no prior knowledge of the choice of the tuning component. We consider an objective function of the form

$$C = \sum_{i=1}^{3} \left[\frac{\phi_i{}^0}{\varepsilon_i} + c\, \frac{t_i}{\phi_i{}^0} \right]. \tag{85}$$

One additional vertex ϕ^3 is considered in order to bound the solution during optimization. We omit details of the constraints, and summarize the final results in Table IV for different c. There are 21 variables and 36 constraints, hence, the computational effort has substantially increased over the previous case. The advantage gained in the general formulation is that the optimization will automatically choose the most appropriate component for tuning, which is C in the objective of (85).

The same designs can be obtained by the reduced formulation using C as a tuned and toleranced component and L_1 and L_2 as toleranced components.

High-Pass Filter

This problem was suggested by Pinel and Roberts [13]. The circuit diagram is shown in Fig. 6 and the basic specifications for the design are listed in Table V. The insertion loss relative to the loss at 990 Hz is to be constrained as indicated with resistances R_5 and R_7 related to $L_5{}^0$ and

TABLE IV
OPTIMAL TUNING

Parameters	c = 10	c = 20	c = 50
$L_1^0 = L_2^0$	1.8440	1.9221	2.0492
C^0	1.1730	1.0486	0.9069
$100\, \varepsilon_1/L_1^0 = 100\, \varepsilon_3/L_2^0$	31.62 %	23.84 %	16.15 %
$100\, \varepsilon_2/C^0$	31.62 %	22.36 %	14.14 %
$100\, t_1/L_1^0 = 100\, t_3/L_2^0$	2.54 %	0.00 %	0.00 %
$100\, t_2/C^0$	54.31 %	35.89 %	14.14 %
$\rho_1(6)$	-1.0000	-0.7165	0.9743
$\rho_2(6)$	0.1645	0.2466	1.0000
$\rho_3(6)$	-1.0000	-0.9992	-0.9846
$\rho_1(8)$	-1.0000	-1.0000	-0.8813
$\rho_2(8)$	-1.0000	-1.0000	-1.0000
$\rho_3(8)$	-1.0000	-1.0000	-0.9876
$\rho_1(1)$	1.0000	0.9887	0.9933
$\rho_2(1)$	1.0000	1.0000	1.0000
$\rho_3(1)$	1.0000	0.9989	0.9029
$\rho_1(3)$	1.0000	0.8433	-0.6051
$\rho_2(3)$	-0.1645	-0.1468	0.6434
$\rho_3(3)$	1.0000	0.8944	0.6441
$100\, \varepsilon_1'/L_1^0 = 100\, \varepsilon_3'/L_2^0$	29.08 %	23.84 %	14.14 %
$100\, t_2'/C^0$	22.69 %	13.53 %	0.00 %
	n = 21	m = 36	

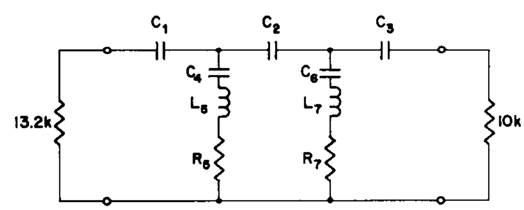


Fig. 6. High-pass filter.

TABLE V
SPECIFICATIONS FOR HIGH-PASS FILTER

Frequency Range (Hz)	Basic Sample Points (Hz)	Relative Insertion Loss (dB)	Weight w
170	170	45.	-1
360	360	49.	-1
440	440	42.	-1
630 - 680	630	4.	+1
680 - 1800	680 710 725 740	1.75	+1
630 - 1800	630 650 680 860 910 930 1050	-0.05	-1

Reference Frequency: 990 Hz

R_5, R_7 related to L_5^0 and L_7^0 through $Q = \dfrac{2\pi 990 L_5^0}{R_5} = \dfrac{2\pi\, 990 L_7^0}{R_7} = 1456$

L_7^0 with constant Q. The terminations are fixed, the designable parameters being C_1, C_2, C_3, C_4, L_5, C_6, and L_7.

The objective function throughout was taken as

$$C = \sum_{i=1}^{7} \frac{\phi_i^0}{\varepsilon_i} \tag{86}$$

where

$$\phi^0 = \begin{bmatrix} C_1^0 \\ C_2^0 \\ C_3^0 \\ C_4^0 \\ L_5^0 \\ C_6^0 \\ L_7^0 \end{bmatrix} \qquad \varepsilon = \begin{bmatrix} \varepsilon_{C_1} \\ \varepsilon_{C_2} \\ \varepsilon_{C_3} \\ \varepsilon_{C_4} \\ \varepsilon_{L_5} \\ \varepsilon_{C_6} \\ \varepsilon_{L_7} \end{bmatrix}.$$

The optimization package used here is DISOPT [14], which has been previously employed in worst-case tolerance problems [6]. The same quasi-Newton unconstrained minimization procedure as for the work described in the previous section is incorporated into DISOPT. The extrapolation feature [15] was chosen to accelerate convergence to the constrained optimum.

Verification of the designs to be described was carried out using all 2^7 vertices plus the nominal point at 170, 360, 440, 630–680, and 680–1800 Hz. Forty-two logarithmically spaced points were taken for the latter interval, and 8 for the former interval.

Table VI indicates the effort required to obtain the results of Table VII. Because of the complexity of the problems preliminary runs of the program were required before the final number of constraints were established. This information along with a realistic assessment of cost is given.

Case 1: No Tuning ($t = 0$)

Table VI summarizes the particular frequencies, specifications and the particular vertex number employed to obtain the final tolerances listed in Table VII. Table VII also lists the shifts in nominal parameter values with respect to those of an uncentered design [7], [13].

Case 2: 3 Percent Tuning for L_5

Results corresponding to the ones for Case 1 are tabulated in Tables VI and VII. Note that all the tolerances have increased. Fig. 7 shows the nominal response as well as the worst upper and lower outcomes based on all 2^7 vertices.

A more detailed verification of the results was made. Sixty logarithmically spaced points were taken from the critical region 630–680 Hz as well as 40 from 600–630 Hz. All the vertices were checked plus the nominal point, followed by 4000 Monte Carlo simulations uniformly distributed in the effective tolerance region. No violations were detected, and the upper and lower limits of response given by the vertices bounded the results from the Monte Carlo analysis except at 638.2 Hz, where the lowest relative loss obtained from the vertices was -0.0243 dB, whereas the Monte Carlo analysis yielded -0.0246 dB.

As a further check on the optimality of these results, L_5 was allowed to be both toleranced and tuned as distinct from being effectively toleranced from the point of view of optimization. The same vertices, an additional 25-ρ variables and 50 additional constraints on the ρ variables were used without any significant improvement in the results. The values of the ρ variables confirmed the assumption that L_5 should be effectively toleranced for 3 percent tuning.

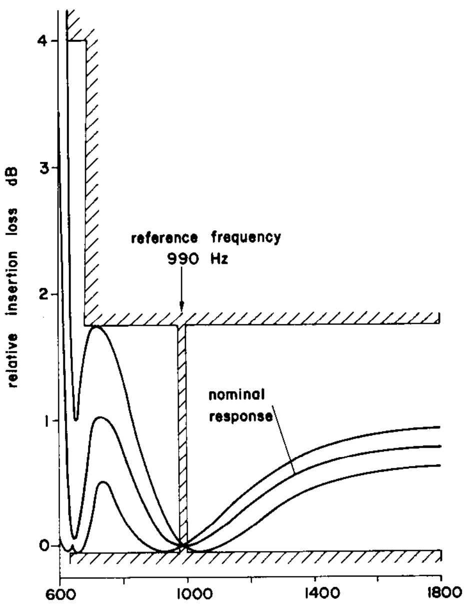
Case 3: 3 Percent Tuning for L_5 *and* L_7

As indicated by Table VII a further improvement in all tolerances has been obtained.

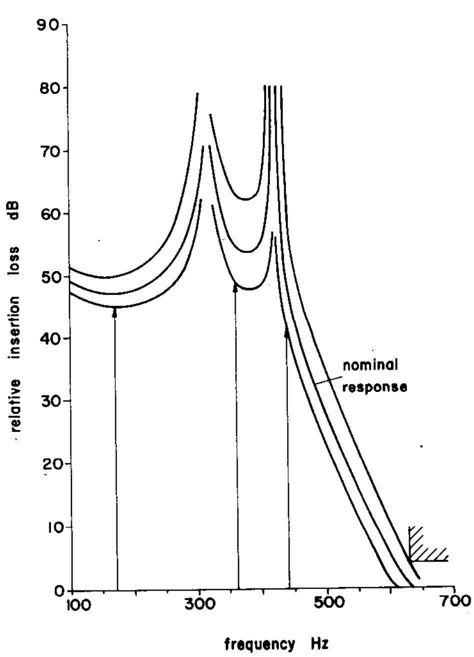
TABLE VI
DATA FOR OPTIMIZATION OF HIGH-PASS FILTER

Frequency (Hz)	S (dB)	w	Vertex Number: Case 1 No Tuning	Case 2 L_5 Tuned	Case 3 L_5 and L_7 Tuned	Case 4 L_7 Tuned
170	45	-1	8	8	8	8
360	49	-1	48	48	48	48
440	42	-1	128	128	128	128
630	4	+1	1	1	1	1
630	-0.05	-1	60,100,104, 108,120,126	58,60,100, 104,108,120 126	60,108,120	60,87,95, 100,104,108, 120,126
637	-0.05	-1	-	-	-	87
640	-0.05	-1	-	58	108	52,58,60
643	-0.05	-1	-	-	-	85,93,117
650	-0.05	-1	nominal,12, 50,58,102	nominal,12, 34,42,50,58, 102,106,126	nominal,12,34, 42,44,58,106, 126	nominal,12, 36,42,50,58, 85,93,94, 102,106,126
658	-0.05	-1	-	-	42	58,69,85
665	-0.05	-1	-	-	34,42	34,58
670	-0.05	-1	-	-	-	2
680	1.75	+1	123	123	123	123
680	-0.05	-1	2,6	2,6	2,6	2,6
710	1.75	+1	43,83	43,83	43,83,123	43,83
725	1.75	+1	43,83	43,83	43,83	43,83
730	1.75	+1	-	-	43,83	43
740	1.75	+1	43,83	43,83	43,83	43,83
860	-0.05	-1	118,126	118,126	118,126	118,126
910	-0.05	-1	118,126	118,126	118,126	118,126
930	-0.05	-1	118,126	118,126	118,126	118,126
1040	-0.05	-1	-	-	-	3
1050	-0.05	-1	3	3	3	3
Number of Response Constraints			31	37	37	55
Total Number of Constraints m			45	51	51	69
Number of Variables n			14	14	14	14
Starting Point			Given by Pinel [13]	Optimum	for	Case1
Number of Trial Runs			3	1	2	1
Total Computing Effort (min)†			15	5	6	7
Computing Cost Including Trials†			$94	$31	$37	$44

† On a CDC 6400.



(a)



(b)

Fig. 7. (a) Passband details of optimized high-pass filter (Case 2). (b) Stopband details of optimized high-pass filter (Case 2).

Case 4: 3 Percent Tuning for L_7

The results for this problem (Table VII) are slightly worse than those for Case 2. A slight violation of the specification at 658 Hz was detected. We conclude that if only one inductor is to be tuned, L_5 should be chosen.

IX. Conclusions

A theory of optimal worst-case design embodying centering, tolerancing and tuning has been presented. The concept of a tunable constraint region that allows variable specifications as set by the customer has also been incorporated. This may find application, for example, in tunable filters. The purely toleranced and purely tuned problems become special cases. Further simplification has been discussed in the light of one-dimensional convexity.

TABLE VII
Results for High-Pass Filter

	Parameters	Case 1 No Tuning	Case 2 L_5 Tuned	Case 3 L_5 and L_7 Tuned	Case 4 L_7 Tuned
C_1	tolerance (%)	5.71	6.77	7.90	6.63
	nom. shift(%)	+18.1	+17.8	+18.3	+17.6
C_2	tolerance (%)	4.33	4.97	5.32	4.77
	nom. shift(%)	+16.2	+15.2	+14.4	+15.3
C_3	tolerance (%)	4.72	5.81	7.23	5.83
	nom. shift(%)	+16.6	+18.0	+18.8	+17.8
C_4	tolerance (%)	4.54	5.03	5.15	4.78
	nom. shift(%)	-3.8	-2.2	-1.2	-3.1
L_5	tolerance (%)	3.29	3.95	4.44	3.82
	nom. shift(%)	-3.0	-3.0	-4.3	-4.1
C_6	tolerance (%)	6.32	7.05	7.27	6.66
	nom. shift(%)	-7.3	-5.1	-3.6	-6.0
L_7	tolerance (%)	3.64	4.34	5.04	4.32
	nom. shift(%)	-6.4	-7.9	-7.9	-6.3
	Cost	157	135	121	138*

* Violation of specifications. Relative loss = -0.052 dB at 658 Hz.

As expected, the inclusion of tunable elements can increase the tolerances on the components. The results seem to justify the reduction of the general tolerance–tuning problem into one containing effectively toleranced and effectively tuned components, where appropriate. If the separation of the components is not decided in advance, the general problem with the cost function reflecting both tolerances and tuning ranges is appropriate, since an optimization program requires an explicit number of variables and constraints in advance.

A component may be both tuned and toleranced simultaneously. Thus, one can represent the effects of an uncertainty of a tuned component if the tuning range is larger than the tolerance. On the other hand, if the tolerance is larger than the tuning range (see, for example, Table VII), it may be considered to be a toleranced component with some small tuning capacity. The tuning range may or may not appear in the objective function. The different weightings of tuning and tolerancing in the objective exhibit the flexibility of the formulation. With a very heavy weighting in the tuning, we will obtain a solution equivalent to a pure tolerance problem. Zero tuning is automatically indicated by the result of the formulation. Reducing the weighting will increase the tolerance as well as the tuning with a net effect of reducing the effective tolerance $\varepsilon_i' = \varepsilon_i - t_i$ until a crossover occurs from effective tolerance to effective tuning. Beyond that, the effective tuning value will continue to increase until a threshold value occurs. Below the threshold, the solution in terms of effective tuning and tolerance problem is unaffected. The tolerances of other components will continue to increase with decreasing weighting on the tuning.

A cost function tending to maximize tolerances and minimizing tuning has been implemented successfully in this context. For the high-pass filter the 3-percent tuning range on the inductors was considered free, thus tuning did not enter into the objective function. A reduced problem involving effective tolerances was found adequate since, as shown in Table VII, the tolerances exceed the tuning ranges. A good starting point for the tuning problem is a worst-case toleranced solution. The small tuning ranges in the high-pass filter problem meant that relatively small nominal shifts were obtained.

It may be added that, as far as the authors are aware, this seems to be the most general formulation to date dealing with the centering, tolerancing and tuning problems at the design stage.

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