A NONLINEAR SECOND INITIAL BOUNDARY VALUE PROBLEM FOR THE HEAT EQUATION*

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1. Introduction. Mann and Wolf [6] proved the existence and uniqueness of an initial boundary value problem of a one-dimensional heat equation with zero initial temperature and nonlinear second boundary condition. Their result was improved by Roberts and Mann [9], and later on by Padmavally [8]. Using Schauder's fixed point theorem [10], Friedman [2] considered an n-dimensional linear parabolic differential equation with linear initial condition and nonlinear boundary condition involving the conormal.

We use a completely different approach to establish the existence and uniqueness of a solution for a nonlinear second initial boundary value problem consisting of a semilinear parabolic differential equation with linear initial and quasilinear boundary conditions. The arguments, similar to those of Duff [1] for the elliptic case, give the solution by successive approximations; in each step of the construction, we make use of the solution of the corresponding linear problem. The method can be used for the more general parabolic differential equation,

$$\sum_{i,j=1}^{n} a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u - \frac{\partial u}{\partial t} = g(x, t; u),$$

since for this the strong maximum principle [7] holds, and the Neumann function exists [3, p. 155, 4, 5] under certain conditions on the coefficients and the domain of definition. For simplicity of discussion, we consider here an n-dimensional semilinear heat equation.

2. Statement of the problem. Let D be a bounded convex n-dimensional domain in the real n-dimensional Euclidean space, D^- its closure and ∂D its boundary. For every point $x = (x_1, x_2, \cdots x_n)$ of ∂D, there exists an n-dimensional neighborhood V such that $V \cap \partial D$ can be represented for some i $(1 \leq i \leq n)$ in the form

$$x_i = h(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$$

and the functions h, $D_x h$, $D_x^2 h$ are Hölder continuous of exponent α where $0 < \alpha < 1$. Let $D \times (0, T] = \Omega$, $\partial D \times (0, T] = S$, and

$$L = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \frac{\partial}{\partial t}.$$

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Our problem is to find $u(x, t)$ satisfying the semilinear heat equation

$$Lu = g(x, t; u) \quad \text{in } \Omega \tag{2.1}$$

under the initial condition

$$u(x, 0) = \phi(x) \quad \text{on } D^{-} \tag{2.2}$$

and the quasilinear boundary condition

$$\frac{\partial u(x, t)}{\partial n_{(x,t)}} + B(x, t; u) = f(x, t) \quad \text{on } S, \tag{2.3}$$

where $g(x, t; u)$, $\phi(x)$, $B(x, t; u)$ and $f(x, t)$ are given functions, and $n_{(x,t)}$ is the outward normal to S at the point (x, t). We impose the following conditions:

(i) $g(x, t; u)$ is twice continuously differentiable; $g_u(x, t; u)$ is Hölder continuous when $(x, t) \in \Omega^{-}$ and u varies in a bounded set;

$$0 \leq g_u(x, t; u) < \infty \tag{2.4}$$

and

$$g(x, t; 0) = 0; \tag{2.5}$$

(ii) $\phi(x)$ is continuous in D^{-};

(iii) $B(x, t; u)$ is twice continuously differentiable when $(x, t) \in S^{-}$ and u varies in a bounded set; moreover

$$B_u(x, t; u) > 0 \tag{2.6}$$

and

$$B(x, t; 0) = 0; \tag{2.7}$$

(iv) $f(x, t)$ is continuous on S^{-}.

For $n = 3$, the problem can be interpreted physically as finding the temperature $u(x, t)$ of a convex, sufficiently smooth, homogeneous and isotropic body having an arbitrary initial distribution of temperature $\phi(x)$. Heat is generated in it at a rate proportional to $-g(x, t; u)$, which is a nonincreasing function of u (condition (2.4)) and satisfies (2.5). Heat transfer between the body at a higher temperature and its surroundings at a lower constant temperature [6, pp. 163–164] is subject to a nonlinear condition (2.3). Thus $f(x, t) - B(x, t; u)$ is a monotone decreasing function of u (condition (2.6)) [6, pp. 163–164]. If $f(x, t) \equiv 0$ on S^{-}, then (2.7) implies that the temperature of the surroundings is zero [6, p. 164].

The main result of this work is the following theorem.

THEOREM. *There exists a unique solution of the nonlinear second initial-boundary value problem* (2.1)–(2.3).

In Sec. 3, we consider three auxiliary lemmas. The proof of the theorem is given in Sec. 4. If conditions (2.5) and (2.7) are replaced by $g(x, t; m) = 0$ and $B(x, t; m) = 0$ where m is a constant, then (4.1) is replaced by

$$u(x, 0; \lambda) - m = \lambda(\phi(x) - m) \quad \text{on } D^{-}.$$

Accordingly, we make the corresponding changes in the existence proof; for example, we start with

$$u_0(x, t; \lambda) \equiv u(x, t; 0) = m$$

in the successive approximations. In effect, the procedures of the proof remain the same.

3. Auxiliary lemmas. Let $L_c = L - c(x, t)$, where $c(x, t) \geq 0$ and $c(x, t)$ is Hölder continuous in Ω^-. Also let

$$B_\tau = (D \times [0, T]) \cap \{t = \tau\},$$

$\Omega^* = D \times [0, T)$, and

$$\psi_\beta = \frac{\partial}{\partial n_{(x,t)}} + \beta(x, t)$$

where $\beta(x, t)$ is a continuous function on S^-. To define a Neumann function, we follow Friedman [3, p. 155].

Definition. A function $N(x, t; \xi, \tau)$ defined and continuous for $(x, t; \xi, \tau) \in \Omega^- \times \Omega^*$, $t > \tau$, is called a Neumann function of $L_c w = 0$ in Ω if for any $0 \leq \tau < T$ and for any continuous function $\psi(x)$ on B_τ having a compact support, the function

$$w(x, t) = \int_{B_\tau} N(x, t; \xi, \tau)\psi(\xi)\, d\xi$$

is a solution of $L_c w = 0$ in $D \times (\tau, T]$ and satisfies

$$\lim_{t \downarrow \tau} w(x, t) = \psi(x) \qquad \text{for } x \in B_\tau^- ,$$

and $\psi_\beta w(x, t) = 0$ on $\partial D \times (\tau, T]$.

Let $N^*(x, t; \xi, \tau)$ denote the Neumann function of the adjoint equation $L_c^* w = 0$ in Ω^* corresponding to the boundary condition $\psi_\beta w = 0$ on $\partial D \times [0, \tau)$. By Friedman [3, p. 155, pp. 82–84] and Itô [4], $N(x, t; \xi, \tau)$ and $N^*(x, t; \xi, \tau)$ exist and are unique, $L_c N(x, t; \xi, \tau) = 0$ for $(x, t) \in \Omega$, $L_c^* N^*(x, t; \xi, \tau) = 0$ for $(x, t) \in \Omega^*$, $\psi_\beta N(x, t; \xi, \tau) = 0$ for $(x, t) \in \partial D \times (\tau, T]$, $\psi_\beta N^*(x, t; \xi, \tau) = 0$ for $(x, t) \in \partial D \times [0, \tau)$, and furthermore, $N(x, t; \xi, \tau)$, $N_x(x, t; \xi, \tau)$, $N_{xx}(x, t; \xi, \tau)$ and $N_t(x, t; \xi, \tau)$ are continuous functions of $(x, t; \xi, \tau)$ in $\Omega \times \Omega^*$, $t > \tau$ while $N^*(x, t; \xi, \tau)$, $N_x^*(x, t; \xi, \tau)$, $N_{xx}^*(x, t; \xi, \tau)$ and $N_t^*(x, t; \xi, \tau)$ are continuous functions of $(x, t; \xi, \tau)$ in $\Omega^* \times \Omega$, $t < \tau$. The Neumann function can be constructed by the parametrix method used by Itô [4, 5].

Let $N(x, t; \xi, \tau)$ be the Neumann function corresponding to the case when $c(x, t) \geq 0$ and $\beta(x, t) \geq 0$, and $N^0(x, t; \xi, \tau)$ be that corresponding to the case when $c(x, t)$ and $\beta(x, t)$ are identically zero. Then,

LEMMA 1. *$N(x, t; \xi, \tau) \leq N^0(x, t; \xi, \tau)$.*

Proof. In the Green's identity,

$$vL_c u - uL_c^* v = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ \sum_{j=1}^{n} \left(v \frac{\partial u}{\partial x_j} - u \frac{\partial v}{\partial x_j} \right) \right\} - \frac{\partial}{\partial t}(uv),$$

let $u(y, \sigma) = N(y, \sigma; \xi, \tau)$ and $v(y, \sigma) = N^*(y, \sigma; x, t)$. Integrating this over the domain $D \times (\tau + \epsilon, t - \epsilon)$ and letting $\epsilon \to 0$, we have by the boundary condition

$$N(x, t; \xi, \tau) = N^*(\xi, \tau; x, t) \tag{3.1}$$

for any two points (x, t) and (ξ, τ) in Ω with $t > \tau$. An argument similar to the proof of Theorem 11 of Friedman [3, pp. 44–45] gives for each (ξ, τ) in Ω^*,

$$N(x, t; \xi, \tau) > 0 \quad \text{in } D \times (\tau, T]. \tag{3.2}$$

From this and (3.1), it follows that

$$N^*(x, t; \xi, \tau) > 0 \quad \text{in } D \times [0, \tau) \tag{3.3}$$

for each (ξ, τ) in Ω.

Let $N_\lambda(x, t; \xi, \tau)$ be the Neumann function of $L_c w = 0$ corresponding to the boundary condition $\psi_\lambda N_\lambda(x, t; \xi, \tau) = 0$, where $\lambda(x, t) \geq 0$. Then the Green's identity gives

$$N_\lambda(x, t; \xi, \tau) - N(x, t; \xi, \tau) = -\int_\tau^t \int_{\partial D} N_\lambda^*(y, \sigma; x, t) N(y, \sigma, \xi, \tau) \cdot \{\lambda(y, \sigma) - \beta(y, \sigma)\}\, dA_y\, d\sigma, \tag{3.4}$$

which gives

$$\delta N(x, t; \xi, \tau) = -\int_\tau^t \int_{\partial D} N^*(y, \sigma; x, t) N(y, \sigma; \xi, \tau)\, \delta\beta(y, \sigma)\, dA_y\, d\sigma. \tag{3.5}$$

Similarly, let $N_b(x, t; \xi, \tau)$ be the Neumann function of $L_b w = 0$ corresponding to $\psi_\beta w = 0$ with $b(x, t) \geq 0$. Then

$$N_b(x, t; \xi, \tau) - N(x, t; \xi, \tau) = -\int_\tau^t \int_D N_b^*(y, \sigma; x, t) N(y, \sigma; \xi, \tau) \cdot \{b(y, \sigma) - c(y, \sigma)\}\, dV_y\, d\sigma, \tag{3.6}$$

which gives

$$\delta N(x, t; \xi, \tau) = -\int_\tau^t \int_D N^*(y, \sigma; x, t) N(y, \sigma; \xi, \tau)\, \delta c(y, \sigma)\, dV_y\, d\sigma. \tag{3.7}$$

Thus from (3.2), (3.3), (3.5) and (3.7), $N(x, t; \xi, \tau) \leq N^0(x, t; \xi, \tau)$ follows.

In what follows, let $k_1, k_2, k_3, \cdots, k_{11}$ denote appropriate positive constants. For convenience of reference, we state the following lemma, whose proof can be found in Friedman [3, p. 146].

LEMMA 2. *If w is a solution of $L_c w = 0$ in Ω, $\psi_\beta w = f(x, t)$ on S and $w(x, 0) = \phi(x)$ on D^-, then for all $(x, t) \in \Omega$,*

$$|w(x, t)| \leq k_1(\operatorname*{l.u.b.}_{S^-} |f| + \operatorname*{l.u.b.}_{D^-} |\phi|),$$

where k_1 is a constant depending only on L_c, β and Ω^-.

LEMMA 3. *Let*

$$\theta^*(\xi, \tau; x, t) = k_2 \int_\tau^t \int_D N^0(y, \sigma; \xi, \tau) N^{0*}(y, \sigma; x, t)\, dV_y\, d\sigma + k_3 \int_\tau^t \int_{\partial D} N^0(y, \sigma; \xi, \tau) N^{0*}(y, \sigma; x, t)\, dA_y\, d\sigma.$$

Then

$$\int_D \theta^*(\xi, 0, x, t)\, dV_\xi + \int_0^t \int_{\partial D} \theta^*(\xi, \tau; x, t)\, dA_\xi\, d\tau \leq k_4$$

where k_4 is independent of (x, t).

Proof. Let L^* be the adjoint of L. It follows from the Green's identity that $\theta^*(\xi, \tau; x, t)$ is the solution of

$$L^*\theta^*(\xi, \tau; x, t) = -k_2 N^{0^*}(\xi, \tau; x, t) \qquad \text{in } D \times [0, t),$$
$$\theta^*(\xi, t; x, t) = 0 \qquad \text{on } \Omega^- \cap \{t = t\},$$

and

$$\frac{\partial \theta^*(\xi, \tau; x, t)}{\partial n_{(\xi,\tau)}} = k_3 N^{0^*}(\xi, \tau; x, t) \qquad \text{on } \partial D \times [0, t).$$

Let $w(x, t)$ be the solution of $Lw = 0$ in Ω, $w(x, 0) = 1$ on D^-, and $\partial w(x, t)/\partial n_{(x,t)} = 1$ on S. From Lemma 2, $|w(x, t)| \leq k_5$, a constant.

In the Green's identity, let $v = \theta^*(y, \sigma; x, t)$ and $u = w(y, \sigma)$. Integrating this over the domain $D \times (\epsilon, t - \epsilon)$, and letting $\epsilon \to 0$, we have

$$\int_D \theta^*(\xi, 0; x, t)\, dV_\xi + \int_0^t \int_{\partial D} \theta^*(\xi, \tau; x, t)\, dA_\xi\, d\tau$$
$$= k_2 \int_0^t \int_D w(\xi, \tau) N^{0^*}(\xi, \tau; x, t)\, dV_\xi\, d\tau + k_3 \int_0^t \int_{\partial D} w(\xi, \tau) N^{0^*}(\xi, \tau; x, t)\, dA_\xi\, d\tau.$$

Hence

$$\int_D \theta^*(\xi, 0; x, t)\, dV_\xi + \int_0^t \int_{\partial D} \theta^*(\xi, \tau; x, t)\, dA_\xi\, d\tau$$
$$\leq k_2 k_5 \int_0^t \int_D N^{0^*}(\xi, \tau; x, t)\, dV_\xi\, d\tau + k_3 k_5 \int_0^t \int_{\partial D} N^{0^*}(\xi, \tau; x, t)\, dA_\xi\, d\tau.$$

The right-hand side of the inequality is the solution of $Lz = -k_2 k_5$ in Ω, $z(x, 0) = 0$ on D^- and $\partial z(x, t)/\partial n_{(x,t)} = k_3 k_5$ on S. Hence from Lemma 2

$$|z(x, t)| \leq k_6 k_5 (k_2 + k_3).$$

Thus the lemma is proved.

4. Proof of the theorem. *Uniqueness:* Suppose $u_1(x, t)$ and $u_2(x, t)$ are two distinct solutions of our problem. Without loss of generality, let us assume that $u_2(x, t) > u_1(x, t)$ at some point of Ω. Then the function, $u(x, t) = u_2(x, t) - u_1(x, t)$ satisfies

$$Lu - g_u(x, t; u_3)u = 0 \quad \text{in } \Omega,$$

where u_3 lies between u_1 and u_2. Since $u(x, 0) = 0$ on D^-, we have by the weak maximum principle [7] that it attains its maximum at some point, say (x_0, t_0), of S. Hence $\partial u(x_0, t_0)/\partial n_{(x_0,t_0)} \geq 0$, but

$$\frac{\partial u(x_0, t_0)}{\partial n_{(x_0,t_0)}} = B(x_0, t_0; u_1) - B(x_0, t_0; u_2) < 0$$

by (2.6). Therefore, the solution is unique.

Existence: Let λ be a parameter such that $0 \leq \lambda \leq 1$. If $u(x, t; \lambda)$ is the solution of

$$Lu(x, t; \lambda) = g(x, t; u(x, t; \lambda)) \quad \text{in } \Omega,$$

$$\frac{\partial u(x, t; \lambda)}{\partial n_{(x,t)}} + B(x, t; u(x, t; \lambda)) = \lambda f(x, t) \quad \text{on } S$$

and

$$u(x, 0; \lambda) = \lambda\phi(x) \quad \text{on } D^-, \tag{4.1}$$

then $v(x, t; \lambda) \equiv \partial u(x, t; \lambda)/\partial\lambda$ satisfies

$$L_{g_u} v(x, t; \lambda) = 0 \quad \text{in } \Omega,$$

$$\psi_{B_u} v(x, t; \lambda) = f(x, t) \quad \text{on } S \tag{4.2}$$

and

$$v(x, 0; \lambda) = \phi(x) \quad \text{on } D^-.$$

Now if $u(x, t; \lambda)$ is already known, then by the Green's identity

$$v(x, t; \lambda) = \int_D N(x, t; \xi, 0; \lambda)\, \phi(\xi)\, dV_\xi + \int_0^t \int_{\partial D} N(x, t; \xi, \tau; \lambda) f(\xi, \tau)\, dA_\xi\, d\tau,$$

where $N(x, t; \xi, \tau; \lambda)$ is the Neumann function of (4.2) corresponding to the boundary condition $\psi_{B_u} v(x, t; \lambda) = 0$ on S. But as λ varies, $u(x, t; \lambda)$ changes, and this in turn affects the Neumann function. By (3.5) and (3.7), we have

$$\delta N(x, t; \xi, \tau; \lambda) = -\int_\tau^t \int_D N^*(y, \sigma; x, t; \lambda) N(y, \sigma; \xi, \tau; \lambda)\, \delta g_u(y, \sigma, u(y, \sigma; \lambda))\, dV_y\, d\sigma$$

$$- \int_\tau^t \int_{\partial D} N^*(y, \sigma; x, t; \lambda) N(y, \sigma; \xi, \tau; \lambda)\, \delta B_u(y, \sigma; u(y, \sigma; \lambda))\, dA_y\, d\sigma. \tag{4.3}$$

Thus to determine $u(x, t; \lambda)$ and $N(x, t; \xi, \tau; \lambda)$, we have the following system of integro-differential equations:

$$\frac{\partial u(x, t; \lambda)}{\partial \lambda} = \int_D N(x, t; \xi, 0; \lambda)\phi(\xi)\, dV_\xi + \int_0^t \int_{\partial D} N(x, t; \xi, \tau; \lambda) f(\xi, \tau)\, dA_\xi\, d\tau \tag{4.4}$$

and

$$\frac{\partial N(x, t; \xi, \tau; \lambda)}{\partial \lambda} = -\int_\tau^t \int_D N^*(y, \sigma; x, t; \lambda) N(y, \sigma; \xi, \tau; \lambda) \frac{\partial g_u(y, \sigma; u(y, \sigma; \lambda))}{\partial \lambda}\, dV_y\, d\sigma$$

$$- \int_\tau^t \int_{\partial D} N^*(y, \sigma; x, t; \lambda) N(y, \sigma; \xi, \tau; \lambda) \frac{\partial B_u(y, \sigma; u(y, \sigma; \lambda))}{\partial \lambda}\, dA_y\, d\sigma \tag{4.5}$$

with $u(x, t; 0) \equiv 0$.

By Lemma 2,

$$|v(x, t; \lambda)| \le k_7(\text{l.u.b.}_{S^-} |f| + \text{l.u.b.}_{D^-} |\phi|).$$

Hence

$$u(x, t; \lambda) \le k_7(\text{l.u.b.}_{S^-} |f| + \text{l.u.b.}_{D^-} |\phi|)$$

since $0 \le \lambda \le 1$. We now prove the existence in the theorem by successive approximations.

Let $u_0(x, t; \lambda) \equiv u(x, t; 0) = 0$. For $n = 1, 2, 3, \cdots$, let $u_n(x, t; 0) \equiv 0$, and

$$\frac{\partial u_n(x, t; \lambda)}{\partial \lambda} = \int_D N_{n-1}(x, t; \xi, 0; \lambda)\phi(\xi)\, dV_\xi + \int_0^t \int_{\partial D} N_{n-1}(x, t; \xi, \tau; \lambda) f(\xi, \tau)\, dA_\xi\, d\tau \tag{4.7}$$

where $N_n(x, t; \xi, \tau; \lambda)$ is the Neumann function of the differential equation

$$Lv(x, t; \lambda) = g_u(x, t; u_n(x, t; \lambda))v(x, t; \lambda)$$

corresponding to the boundary condition

$$\frac{\partial v(x, t; \lambda)}{\partial n_{(x,t)}} + B_u(x, t; u_n(x, t; \lambda))v(x, t; \lambda) = 0.$$

Thus we can find $N_0(x, t; \xi, \tau; \lambda)$, $u_1(x, t; \lambda)$, $N_1(x, t; \xi, \tau; \lambda)$, and so on successively.

Since $g(x, t; u)$ and $B(x, t; u)$ are twice continuously differentiable, we have by (4.6) that g_{uu} and B_{uu} are bounded. Let $|g_{uu}| \leq k_2$ and $|B_{uu}| \leq k_3$. Also let

$$\rho_n(\lambda) = \max_{(x,t)\in\Omega} |u_n(x, t; \lambda) - u_{n-1}(x, t; \lambda)|. \tag{4.8}$$

Then

$$|g_u(x, t; u_n(x, t; \lambda)) - g_u(x, t; u_{n-1}(x, t; \lambda))| \leq k_2\rho_n(\lambda)$$

and

$$|B_u(x, t; u_n(x, t; \lambda)) - B_u(x, t; u_{n-1}(x, t; \lambda))| \leq k_3\rho_n(\lambda).$$

These together with (3.4), (3.6), Lemma 1 and the definition of $\theta^*(\xi, \tau; x, t)$ in Lemma 3 give

$$|N_n(x, t; \xi, \tau; \lambda) - N_{n-1}(x, t; \xi, \tau; \lambda)| \leq \rho_n(\lambda)\theta^*(\xi, \tau; x, t). \tag{4.9}$$

Let $|\phi(x)| \leq k_8$, $|f(x, t)| \leq k_9$ and $k_{10} = \max \{k_8 , k_9\}$. Then from (4.7) and (4.9), we have

$$\left|\frac{\partial u_{n+1}(x, t; \lambda)}{\partial\lambda} - \frac{\partial u_n(x, t; \lambda)}{\partial\lambda}\right|$$

$$\leq k_{10}\rho_n(\lambda)\left\{\int_D \theta^*(\xi, 0; x, t)\, dV_\xi + \int_0^t \int_{\partial D} \theta^*(\xi, \tau; x, t)\, dA_\xi\, d\tau\right\} \leq k_{10}\rho_n(\lambda)k_4 \tag{4.10}$$

by Lemma 3. Since $u_n(x, t; 0) = 0$, we have from (4.10)

$$|u_{n+1}(x, t; \lambda) - u_n(x, t; \lambda)| \leq k_4k_{10} \int_0^\lambda \rho_n(r)\, dr,$$

which is independent of (x, t). By (4.8)

$$\rho_{n+1}(\lambda) \leq k_4k_{10} \int_0^\lambda \rho_n(r)\, dr.$$

Since $u_0(x, t; \lambda) = 0$, we have

$$\rho_1(\lambda) = \max_{(x,t)\in\Omega^-} |u_1(x, t; \lambda)|.$$

By (4.6), $\rho_1(\lambda) \leq k_{11}$. It follows from induction that

$$\rho_n(\lambda) \leq \frac{k_{11}(k_4k_{10}\lambda)^{n-1}}{(n-1)!}. \tag{4.11}$$

Therefore, $\sum_{n=0}^\infty [u_{n+1}(x, t; \lambda) - u_n(x, t; \lambda)]$ converges absolutely and uniformly in (x, t). Let $u(x, t; \lambda)$ be the limit. Except at the point of singularity $(x, t) = (\xi, \tau)$ of $N^0(x, t; \xi, \tau)$, it follows from (4.9) that the sequence $\{N_n(x, t; \xi, \tau; \lambda)\}$ converges uniformly to a limit, say $N(x, t; \xi, \tau; \lambda)$. Thus for $(x, t) \neq (\xi, \tau)$, $N(x, t; \xi, \tau; \lambda)$ is continuous and furthermore, from (4.3), it depends continuously on the coefficient of the partial differential equation and on the boundary condition. Therefore $N(x, t; \xi, \tau; \lambda)$ is the Neumann function of (4.2) corresponding to $\psi_{B_u} v(x, t; \lambda) = 0$ on S. Hence from (4.3) $\partial N(x, t; \xi, \tau; \lambda)/\partial\lambda$ is given by (4.5). Since $u_0(x, t; \lambda) = 0$, we have from (4.10) and (4.11) that $\partial u_n(x, t; \lambda)/\partial\lambda$ converges uniformly and absolutely. As $n \to \infty$, (4.7) becomes (4.4). Thus $u(x, t; \lambda)$ and $N(x, t; \xi, \tau; \lambda)$ satisfy the integro-differential equations (4.4) and (4.5) with $u(x, t; 0) \equiv 0$. Hence $u(x, t; 1)$ is the solution to our problem.

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