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# A NONLINEAR WAVE EQUATION ASSOCIATED WITH A NONLINEAR INTEGRAL EQUATION

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**Abstract.** Motivated by the well-posedness results in [Nonlinear Anal. Ser. B: RWA. **4(3)** (2003), 483–501; Nonlinear Anal. Ser. B: RWA. **11(5)** (2010), 3453–3462] for the models describing the propagation of high frequency electromagnetic waves in nonlinear dielectric media, because of their mathematical context, we study a similar model and prove results about existence, uniqueness, the asymptotic behavior and an asymptotic expansion of the solution up to order N in a small parameter $\lambda$ with error $\lambda^{N+\frac{1}{2}}$.

## 1. Introduction

In this paper, we consider the following problem:

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Find a pair $(u, P)$ of functions satisfying

$$\begin{cases} u_{tt} - u_{xx} + \alpha(x)u_t + \beta(x)P_{tt}(x,t) = f(x,t),\ 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) = hu(0,t) + \lambda u_t(0,t),\ u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ \ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.1}$$

where $h \geq 0$, $\lambda > 0$ are given constants and $\tilde{u}_0$, $\tilde{u}_1$, $f$, $\alpha$, $\beta$ are given functions satisfying conditions specified later, and the unknown functions $u(x,t)$ and $P(x,t)$ satisfy the following integral equation

$$P(x,t) = \tilde{P}_0(x) + \int_0^t g(x, t-s)G(u(x,s), P(x,s))ds, \tag{1.2}$$

for $0 < x < 1$, $0 < t < T$, where $g$, $G$, $\tilde{P}_0$ are given functions. Problem (1.1), (1.2) may be considered as the generalizations of mathematical models of high frequency electromagnetic waves in nonlinear dielectric media given in [1], [4]. In [4], by using the Galerkin method, Y. Zaidan proved existence, uniqueness and continuous dependence of the following problem

$$\begin{cases} E_{tt} - E_{zz} + \alpha(z)E_t + \beta(z)P_{tt}(z,t) = f(z,t),\ 0 < z < 1,\ 0 < t < T, \\ P_t(z,t) = -G(P(z,t)) + \gamma E(z,t),\ 0 < z < 1,\ 0 < t < T, \\ E_z(0,t) = \lambda E_t(0,t),\ E(1,t) = 0, \\ E(z,0) = \widetilde{E}_0(z),\ \ E_t(z,0) = \widetilde{E}_1(z),\ P(z,0) = 0, \end{cases} \tag{1.3}$$

where $\lambda > 0, \gamma$ are given constants and $\widetilde{E}_0$, $\widetilde{E}_1$, $f$, $G$, $\alpha, \beta$ are given functions. Problem (1.3) is a mathematical model describing the propagation of high frequency electromagnetic pulses in dielectric materials. It is realistic model that includes a nonlinear function of the polarization $P$ given by the nonlinear Debye equation, the electric field $E$ is polarized with oscillations in the xz-plane only, an absorbing boundary condition is placed at $z = 0$ to prevent the reflection of waves. In [1], Banks and Pinter also established well-posedness results for the following model describing the propagation of high-intensity electromagnetic waves in a nonlinear medium

$$\begin{cases} E_{tt} - E_{zz} + \alpha(z)E_t + \beta(z)P_{tt}(z,t) = f(z,t),\ 0 < z < 1,\ 0 < t < T, \\ E_z(0,t) = \lambda E_t(0,t),\ E(1,t) = 0, \\ E(z,0) = \widetilde{E}_0(z),\ \ E_t(z,0) = \widetilde{E}_1(z), \end{cases} \tag{1.4}$$

and

$$P(z,t) = \int_0^t g(z, t-s)\left[E(x,s) + G(E(x,s))\right]ds, \tag{1.5}$$

where $\lambda > 0$ is given constant and $\widetilde{E}_0$, $\widetilde{E}_1$, $g$, $G$, $k$, $\alpha, \beta$ are given functions.

Eq (1.5) is a representation of the polarization $P$ by a nonlinear convolution. This formulation can be interpreted as a generalization of the Debye or Lorentz polarization models in the sense that the polarization dynamics is driven by a nonlinear function of the electric field $E$.

The original ideas in [1], [4] lead to the study of problem (1.1), (1.2) because of their mathematical context.

Applying the methods and techniques used in [5]–[8], we prove existence, uniqueness, asymptotic behavior and asymptotic expansion of the solution of problem (1.1), (1.2).

The structure of the paper is as follows. Section 2 presents some required preliminaries. The existence and uniqueness of a weak solution to problem (1.1), (1.2) are given in Section 3. At first, by techniques used in [6] and [8], we associate with problem (1.1), (1.2) a linear recurrent sequence $\{(u_m, P_m)\}$ which is bounded in a suitable space of functions. Next, the proof is done by using the Galerkin method associated to a priori estimates, weak convergence and compactness techniques. Furthermore, based on the methods as in [5] and [7], the asymptotic behavior of solutions as $\lambda \to 0_+$ and an asymptotic expansion of solutions up to order $N$ in a small parameter $\lambda$ with error $\lambda^{N+\frac{1}{2}}$ are also discussed in Sections 4 and 5, respectively. The results obtained here may be considered as the generalizations of those in [1], [4].

## 2. Preliminaries

Put $Q_T = (0,1) \times (0,T)$, $T > 0$. We denote the usual function spaces used in this paper by the notations $C^m[0,1]$, $W^{m,p} = W^{m,p}(0,1)$, $L^p = W^{0,p}(0,1)$, $H^m = W^{m,2}(0,1)$, $1 \leq p \leq \infty$, $m = 0, 1, \cdots$. Let $\langle \cdot, \cdot \rangle$ be either the scalar product in $L^2$ or the dual pairing of a continuous linear functional and an element of a function space. We denote by $\|\cdot\|_{L^p}$ the norm in $L^p$, with $1 \leq p \leq \infty$, $p \neq 2$. The notation $\|\cdot\|$ stands for the norm in $L^2$ and we denote by $||\cdot||_X$ the norm in the Banach space $X$. We call $X'$ the dual space of $X$. We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of real functions $u : (0,T) \to X$ measurable, such that $||u||_{L^p(0,T;X)} < +\infty$, with

$$||u||_{L^p(0,T;X)} = \begin{cases} \left( \int_0^T ||u(t)||_X^p dt \right)^{1/p}, & \text{if} \quad 1 \leq p < \infty, \\ \underset{0<t<T}{ess\sup} ||u(t)||_X, & \text{if} \quad p = \infty. \end{cases}$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \bigtriangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively. With $G \in C^k(\mathbb{R}^2)$, $G = G(y,z)$, we put $D_1^{\alpha_1} G = \frac{\partial^{\alpha_1} G}{\partial y^{\alpha_1}}$, $D_2^{\alpha_2} G = \frac{\partial^{\alpha_2} G}{\partial z^{\alpha_2}}$, and $D^\alpha G = D_1^{\alpha_1} D_2^{\alpha_2} G = \frac{\partial^{\alpha_1+\alpha_2} G}{\partial y^{\alpha_1} \partial z^{\alpha_2}}$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, $|\alpha| = \alpha_1 + \alpha_2 \leq k$; $D^{(0,0)} G = D^0 G = G$.

On $H^1$, we shall use the following norm

$$\|v\|_{H^1} = \left( \|v\|^2 + \|v_x\|^2 \right)^{1/2}. \tag{2.1}$$

We put

$$V = \{v \in H^1 : v(1) = 0\}, \tag{2.2}$$

$$a(u,v) = \int_0^1 u_x(x) v_x(x) dx + h u(0) v(0), \quad \text{for all } u,\, v \in V,\ h \geq 0. \tag{2.3}$$

We remark that $V$ is a closed subspace of $H^1$ and three norms $\|v\|_{H^1}$, $\|v_x\|$ and $\|v\|_V = \sqrt{a(v,v)}$ are equivalent norms on $V$. So are the norms $v \longmapsto \|v\|_{H^1}$, $v \longmapsto \|v\|_V$ and $v \longmapsto \|v_x\|$ on $H_0^1$. Then the following lemmas are known.

**Lemma 2.1.** *The imbedding $H^1 \hookrightarrow C^0[0,1]$ is compact and*

$$\|v\|_{C^0[0,1]} \leq \sqrt{2}\, \|v\|_{H^1} \ \text{ for all } v \in H^1, \tag{2.4}$$

*where* $\|v\|_{C^0[0,1]} = \sup_{x \in [0,1]} |v(x)|$.

**Lemma 2.2.** *The imbedding $V \hookrightarrow C^0[0,1]$ is compact and*

$$\begin{cases} \text{(i)} \quad \|v\|_{C^0[0,1]} \leq \|v_x\| \leq \|v\|_V\,, \\ \text{(ii)} \quad \frac{1}{\sqrt{2}} \|v\|_{H^1} \leq \|v_x\| \leq \|v\|_V \leq \sqrt{1+h}\, \|v_x\| \leq \sqrt{1+h}\, \|v\|_{H^1}\,, \end{cases} \tag{2.5}$$

*for all $v \in V$. On the other hand,*

$$||v||_{C^0[0,1]} \leq ||v_x|| \ \text{for all } v \in H_0^1. \tag{2.6}$$

**Lemma 2.3.** *Let $h \geq 0$. Then the symmetric bilinear form $a(\cdot,\cdot)$ defined by (2.3) is continuous on $V \times V$ and coercive on $V$.*

According to the definition of $a(\cdot,\cdot)$ and by

$$\begin{aligned} \frac{\partial^2 P}{\partial t^2}(x,t) &= g(x,0) \frac{\partial}{\partial t} G(u(x,t), P(x,t)) + g'(x,0) G(u(x,t), P(x,t)) \\ &\quad + \int_0^t g''(x,t-s) G(u(x,s), P(x,s))\, ds, \end{aligned} \tag{2.7}$$

we can define the weak solution of (1.1), (1.2) as follows.

**Definition 2.4.** We say that $(u,P)$ is a weak solution of (1.1), (1.2) if

$$\begin{aligned} &u,\, P \in L^\infty\left(0,T; V \cap H^2\right),\ u_t,\ P_t \in L^\infty(0,T;V), \\ &u_{tt},\ P_{tt} \in L^\infty(0,T;L^2),\ u_{tt}(0,\cdot) \in L^2(0,T), \end{aligned}$$

and a pair $(u,P)$ satisfies the following variational equation

$$\begin{cases} \langle u_{tt}(t), v\rangle + a(u(t), v) + \lambda u_t(0,t)v(0) + \langle \alpha u_t(t), v\rangle \\ \qquad +\langle \beta g(0)\frac{\partial}{\partial t}G(u,P), v\rangle + \langle \beta g'(0)G(u,P), v\rangle \\ \qquad +\langle \beta \int_0^t g''(t-s)G(u(s),P(s))ds, v\rangle = \langle f(t), v\rangle\,, \\ P(x,t) = \tilde{P}_0(x) + \int_0^t g(x,t-s)G(u(x,s),P(x,s))ds, \end{cases} \tag{2.8}$$

for all $v \in V$, a.e., $t \in (0,T)$ together with the initial conditions

$$u(0) = \tilde{u}_0,\ u_t(0) = \tilde{u}_1. \tag{2.9}$$

## 3. Existence and uniqueness of a weak solution

Let $T^* > 0$. We make the following assumptions:

$(H_0)$ $h \geq 0,\ \lambda > 0$;
$(H_1)$ $\alpha, \beta \in L^\infty$;
$(H_2)$ $(\tilde{u}_0,\ \tilde{u}_1,\ \tilde{P}_0) \in \left(V \cap H^2\right) \times V \times \left(V \cap H^2\right)$;
$(H_3)$ $f,\ f' \in L^2(0,T^*;L^2)$;
$(H_4)$ $g \in H^3(Q_{T^*}) \cap L^1(0,T^*;H^2) \cap L^2(0,T^*;L^\infty),\ g', g'' \in L^1(0,T^*;L^2)$;
$(H_5)$ $G \in C^2(\mathbb{R})$ satisfies $G(0,0) = 0$.

Let $M > 0$, we put

$$K_M(G) = \|G\|_{C^2([-M,M]^2)} = \sup_{(y,z)\in[-M,M]^2} \sum\nolimits_{|\alpha|\leq 2} |D^\alpha G(y,z)|\,. \tag{3.1}$$

For each $T \in (0,T^*]$, we get

$$X_T = \{u \in L^\infty(0,T;V) : u' \in L^\infty(0,T;V),\ u'' \in L^\infty(0,T;L^2)\}. \tag{3.2}$$

We note that $X_T$ is a Banach space with respect to the norm

$$\|v\|_{X_T} = \max\{\ \|v\|_{L^\infty(0,T;V)}\,,\ \|v'\|_{L^\infty(0,T;V)}\,,\ \|v''\|_{L^\infty(0,T;L^2)}\}. \tag{3.3}$$

For each $T \in (0,T^*]$ and $M > 0$, we set

$$B_T(M) = \{v \in X_T : \|v\|_{X_T} \leq M\}. \tag{3.4}$$

We shall choose the first initial term $(u_0,P_0) \equiv (\tilde{u}_0, \tilde{P}_0)$. Suppose that

$$\begin{cases} u_{m-1},\ P_{m-1} \in B_T(M) \cap L^\infty(0,T;V\cap H^2), \\ \sqrt{2\lambda}||u''_{m-1}(0,\cdot)||_{L^2(0,T)} \leq M, \end{cases} \tag{3.5}$$

and associate with problem (2.8), (2.9) the following problem:
Find $u_m,\ P_m \in B_T(M) \cap L^\infty(0,T;V\cap H^2)$ satisfying the following problem

$$\begin{cases} \text{(i)}\ P_m(t) = \tilde{P}_0 + \int_0^t g(t-s)G(u_{m-1}(s), P_{m-1}(s))ds, \\ \text{(ii)}\ \langle u''_m(t),v\rangle + a(u_m(t),v) + \lambda u'_m(0,t)v(0) + \langle \alpha u'_m(t),v\rangle = \langle F_m(t),v\rangle\,, \\ \qquad \text{for all } v \in V, \text{ a.e., } t \in (0,T), \end{cases} \tag{3.6}$$

together with the initial conditions

$$u_m(0) = \tilde{u}_0, \;\; u_m'(0) = \tilde{u}_1, \tag{3.7}$$

where

$$\begin{aligned} F_m(t) = f(t) - \beta g(0)\tfrac{\partial}{\partial t} G(u_{m-1}, P_{m-1}) - \beta g'(0) G(u_{m-1}, P_{m-1}) \\ -\beta \textstyle\int_0^t g''(t-s) G(u_{m-1}(s), P_{m-1}(s))ds. \end{aligned} \tag{3.8}$$

Then, we have the following theorem.

**Theorem 3.1.** *Suppose that* $(H_0) - (H_5)$ *hold and the initial data* $(\tilde{u}_0, \tilde{u}_1) \in (V \cap H^2) \times V$ *satisfy the compatibility condition*

$$\tilde{u}_{0x}(0) = h\tilde{u}_0(0) + \lambda \tilde{u}_1(0). \tag{3.9}$$

*Then there exist positive constants* $M, T > 0$ *such that, for* $(u_0, P_0) \equiv (\tilde{u}_0, \tilde{P}_0)$, *there exists a recurrent sequence* $\{(u_m, P_m)\}$ *defined by* (3.6)-(3.8) *and satisfying*

$$u_m, \, P_m \in B_T(M) \cap L^\infty(0, T; V \cap H^2), \; \sqrt{2\lambda} ||u_m''(0, \cdot)||_{L^2(0,T)} \leq M. \tag{3.10}$$

*Proof.* The proof consists of two parts.
**Part 1.** We show that there exist positive constants $M, T > 0$ such that

$$P_m \in B_T(M) \cap L^\infty(0, T; V \cap H^2). \tag{3.11}$$

So, we need the following lemma, its proof will be presented in the appendix.

**Lemma 3.2.** *Suppose that* (3.5) *holds. Then*

(i) $\|G(u_{m-1}(t), P_{m-1}(t))\|_{L^\infty} \leq K_M(G)$,

(ii) $\|G(u_{m-1}(t), P_{m-1}(t))\|_{L^\infty} \leq \left\| G(\tilde{u}_0, \tilde{P}_0) \right\|_{L^\infty} + 2TMK_M(G)$,

(iii) $\left\| \dfrac{\partial}{\partial t} G(u_{m-1}(t), P_{m-1}(t)) \right\|_{L^\infty} \leq 2MK_M(G)$,

(iv) $\left\| \dfrac{\partial}{\partial t} G(u_{m-1}(t), P_{m-1}(t)) \right\|$

$\quad \leq \left\| D_1 G(\tilde{u}_0, \tilde{P}_0)\tilde{u}_1 + D_2 G(\tilde{u}_0, \tilde{P}_0) g(0) G(\tilde{u}_0, \tilde{P}_0) \right\|$

$\quad\quad + 2TM\left(1 + 2M\right) K_M(G)$,

$$
\begin{aligned}
&\text{(v)} \left\| \frac{\partial}{\partial x} G(u_{m-1}(t), P_{m-1}(t)) \right\| \leq 2MK_M(G), \\
&\text{(vi)} \left\| \frac{\partial}{\partial x} G(u_{m-1}(t), P_{m-1}(t)) \right\| \\
&\quad \leq \left\| \frac{\partial}{\partial x} G(\tilde{u}_0, \tilde{P}_0) \right\| + 2TM(1+2M)K_M(G), \\
&\text{(vii)} \left\| \frac{\partial^2}{\partial t^2} G(u_{m-1}(t), P_{m-1}(t)) \right\| \leq 2M\left(1+2M\right)K_M(G), \\
&\text{(viii)} \left\| \frac{\partial^2}{\partial x^2} G(u_{m-1}(t), P_{m-1}(t)) \right\| \\
&\quad \leq K_M(G) \Big[ 4\sqrt{2}M^2 + \Big(1+2\sqrt{2}M\Big) \left( \|\Delta u_{m-1}(t)\| + \|\Delta P_{m-1}(t)\| \right) \Big].
\end{aligned} \tag{3.12}
$$

Next, we computing partial derivatives of $P_m(x,t) : P_{mx}(t), P'_m(t), P''_m(t), P'_{mx}(t), P_{mxx}(t)$ and note

$$
\begin{aligned}
&u_{m-1}(1,s) = P_{m-1}(1,s) = G(0,0) = 0, \\
&P_m(1,t) = \tilde{P}_0(1) + \int_0^t g(1,t-s)G(u_{m-1}(1,s), P_{m-1}(1,s))ds = 0, \\
&P'_m(1,t) = g(1,0)G(u_{m-1}(1,t), P_{m-1}(1,t)) \\
&\qquad\qquad + \int_0^t g'(1,t-s)G(u_{m-1}(1,s), P_{m-1}(1,s))ds = 0.
\end{aligned}
$$

Therefore, it is clear that $(H_4)$, $(H_5)$ and (3.5) lead to

$$
P_m \in X_T \cap L^\infty(0,T; V \cap H^2). \tag{3.13}
$$

Furthermore, the following estimates are valid

$$
\begin{aligned}
&\text{(ix)} \left\| P_{mx} \right\|_{L^\infty(0,T;L^2)} \\
&\quad \leq \left\| \tilde{P}_{0x} \right\| + K_M(G) \left[ \|g_x\|_{L^1(0,T;L^2)} + 2M \|g\|_{L^1(0,T;L^\infty)} \right], \\
&\text{(x)} \left\| P'_{mx} \right\|_{L^\infty(0,T;L^2)} \\
&\quad \leq \|g_x(0)\| \left\| G(\tilde{u}_0, \tilde{P}_0) \right\|_{L^\infty} + \|g(0)\|_{L^\infty} \left\| \frac{\partial}{\partial x} G(\tilde{u}_0, \tilde{P}_0) \right\| \\
&\qquad + 2TMK_M(G) \left( \|g_x(0)\| + (1+2M) \|g(0)\|_{L^\infty} \right), \\
&\text{(xi)} \left\| P''_m \right\|_{L^\infty(0,T;L^2)} \\
&\quad \leq \|g(0)\|_{L^\infty} \left\| D_1 G(\tilde{u}_0, \tilde{P}_0)\tilde{u}_1 + D_2 G(\tilde{u}_0, \tilde{P}_0) g(0) G(\tilde{u}_0, \tilde{P}_0) \right\| \\
&\qquad + \left\| g'(0) \right\| \left\| G(\tilde{u}_0, \tilde{P}_0) \right\|_{L^\infty} \\
&\qquad + K_M(G) \Big[ 2TM \Big( (1+2M) \|g(0)\|_{L^\infty} + \left\| g'(0) \right\| \Big) + \left\| g'' \right\|_{L^1(0,T;L^2)} \Big],
\end{aligned} \tag{3.14}
$$

hence we can choose $T > 0$ small enough and $M > 0$ sufficiently large such that $\|P_m\|_{X_T} \leq M$. Thus $P_m \in B_T(M) \cap L^\infty(0,T;V \cap H^2)$.

**Part 2.** We prove that there exists $u_m \in B_T(M) \cap L^\infty(0,T;V \cap H^2)$ satisfying $\sqrt{2\lambda}||u''_m(0,\cdot)||_{L^2(0,T)} \leq M$. It consists of three steps.

**Step 1:** *The Faedo-Galerkin approximation* (introduced by Lions [3]).

Let $\{w_j\}$ be a denumerable base of $V \cap H^2$. We find an approximate solution of problem (2.8), (2.9) in the form

$$u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t) w_j, \tag{3.15}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the following system of linear differential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j \rangle + a(u_m^{(k)}(t), w_j) + \lambda \dot{u}_m^{(k)}(0,t) w_j(0) + \langle \alpha \dot{u}_m^{(k)}(t), w_j \rangle \\ \quad = \langle F_m(t), w_j \rangle \,,\, 1 \leq j \leq k, \\ u_m^{(k)}(0) = \tilde{u}_0, \;\; \dot{u}_m^{(k)}(0) = \tilde{u}_1. \end{cases} \tag{3.16}$$

By (3.5), system (3.16) has a unique solution $c_{mj}^{(k)}(t)$, $1 \leq j \leq k$ on $[0,T]$, let us omit the details (see [2]).

**Step 2**. *A priori estimates.*

For all $j = 1, 2, ...k$, multiplying $(3.16)_1$ by $\dot{c}_{mj}^{(k)}(t)$, summing on $j$, and integrating with respect to the time variable from 0 to $t$, we have

$$X_m^{(k)}(t) = -2 \int_0^t \langle \alpha \dot{u}_m^{(k)}(s), \dot{u}_m^{(k)}(s) \rangle ds + 2 \int_0^t \left\langle F_m(s), \dot{u}_m^{(k)}(s) \right\rangle ds, \tag{3.17}$$

where

$$X_m^{(k)}(t) = \left\| \dot{u}_m^{(k)}(t) \right\|^2 + a(u_m^{(k)}(t), u_m^{(k)}(t)) + 2\lambda \int_0^t \left| \dot{u}_m^{(k)}(0,s) \right|^2 ds. \tag{3.18}$$

Next, by differentiating $(3.16)_1$ with respect to $t$ and substituting $w_j = \ddot{u}_m^{(k)}(t)$, after integrating with respect to the time variable from 0 to $t$, we have

$$Y_m^{(k)}(t) = -2 \int_0^t \langle \alpha \ddot{u}_m^{(k)}(s), \ddot{u}_m^{(k)}(s) \rangle ds + 2 \int_0^t \left\langle F'_m(s), \ddot{u}_m^{(k)}(s) \right\rangle ds, \tag{3.19}$$

where

$$Y_m^{(k)}(t) = \left\| \ddot{u}_m^{(k)}(t) \right\|^2 + a\left( \dot{u}_m^{(k)}(t), \dot{u}_m^{(k)}(t) \right) + 2\lambda \int_0^t \left| \ddot{u}_m^{(k)}(0,s) \right|^2 ds. \tag{3.20}$$

We define

$$S_m^{(k)}(t) = X_m^{(k)}(t) + Y_m^{(k)}(t), \tag{3.21}$$

then, it follows from (3.17)-(3.21), that

$$\begin{aligned} S_m^{(k)}(t) &= S_m^{(k)}(0) - 2\int_0^t \left[\langle \alpha \dot{u}_m^{(k)}(s), \dot{u}_m^{(k)}(s)\rangle + \langle \alpha \ddot{u}_m^{(k)}(s), \ddot{u}_m^{(k)}(s)\rangle\right] ds \\ &\quad + 2\int_0^t \left[\left\langle F_m(s), \dot{u}_m^{(k)}(s)\right\rangle ds + \langle F_m'(s), \ddot{u}_m^{(k)}(s)\rangle\right] ds \\ &= S_m^{(k)}(0) + I_1 + I_2. \end{aligned} \tag{3.22}$$

We shall estimate the integrals on the right hands of (3.22) as follows. Using $(H_1)$, (3.18), (3.20) and (3.21) lead to

$$\begin{aligned} I_1 &= -2\int_0^t \left[\langle \alpha \dot{u}_m^{(k)}(s), \dot{u}_m^{(k)}(s)\rangle + \langle \alpha \ddot{u}_m^{(k)}(s), \ddot{u}_m^{(k)}(s)\rangle\right] ds \\ &\leq 2\|\alpha\|_{L^\infty} \int_0^t \left(\left\|\dot{u}_m^{(k)}(s)\right\|^2 + \left\|\ddot{u}_m^{(k)}(s)\right\|^2\right) ds \\ &\leq 2\|\alpha\|_{L^\infty} \int_0^t S_m^{(k)}(s) ds. \end{aligned} \tag{3.23}$$

We have

$$\begin{aligned} I_2 &= 2\int_0^t \left[\left\langle F_m(s), \dot{u}_m^{(k)}(s)\right\rangle ds + \langle F_m'(s), \ddot{u}_m^{(k)}(s)\rangle\right] ds \\ &\leq \int_0^t \|F_m(s)\|^2 ds + \int_0^t \|F_m'(s)\| ds + \int_0^t (1 + \|F_m'(s)\|) S_m^{(k)}(s) ds. \end{aligned} \tag{3.24}$$

We need estimate $\int_0^t \|F_m(s)\|^2 ds$. By (3.8) and (3.12), we obtain

$$\begin{aligned} &\|F_m(t)\| \\ &\leq \|f(t)\| + K_M(G)\|\beta\|_{L^\infty}\left[2M\|g(0)\|_{L^\infty} + \|g'(0)\| + \|g''\|_{L^1(0.T;L^2)}\right]. \end{aligned} \tag{3.25}$$

Thus

$$\int_0^t \|F_m(s)\|^2 ds \leq \Phi_M^{(1)}(T), \tag{3.26}$$

where

$$\begin{aligned} \Phi_M^{(1)}(T) &= 2\|f\|_{L^2(Q_T)}^2 + 2T\|\beta\|_{L^\infty}^2 K_M^2(G) \\ &\quad \times \left[2M\|g(0)\|_{L^\infty} + \|g'(0)\|_{L^\infty} + \|g''\|_{L^1(0.T^*;L^2)}\right]^2. \end{aligned} \tag{3.27}$$

We estimate $\int_0^t \|F_m'(s)\| ds$. By (3.8), we have

$$\begin{aligned} &F_m'(t) \\ &= f'(t) - \beta g(0)\tfrac{\partial^2}{\partial t^2} G(u_{m-1}(t), P_{m-1}(t)) - \beta g'(0)\tfrac{\partial}{\partial t} G(u_{m-1}(t), P_{m-1}(t)) \\ &\quad - \beta g''(0) G(u_{m-1}(t), P_{m-1}(t)) - \beta\int_0^t g'''(t-s) G(u_{m-1}(s), P_{m-1}(s)) ds. \end{aligned} \tag{3.28}$$

So

$$\begin{aligned} \|F_m'(t)\| \leq \|f'(t)\| + \|\beta\|_{L^\infty} K_M(G)\Big[&2M(1+2M)\|g(0)\|_{L^\infty} \\ &+ 2M\|g'(0)\| + \|g''(0)\| + \|g'''\|_{L^1(0.T^*;L^2)}\Big]. \end{aligned} \tag{3.29}$$

Thus

$$\int_0^t \|F_m'(s)\|\, ds \le \Phi_M^{(2)}(T), \tag{3.30}$$

where

$$\Phi_M^{(2)}(T) = \|f'\|_{L^1(0,T;L^2)} + T\, \|\beta\|_{L^\infty}\, K_M(G) \Big[ 2M\,(1+2M)\, \|g(0)\|_{L^\infty} + 2M\, \|g'(0)\| + \|g''(0)\| + \|g'''\|_{L^1(0.T^*;L^2)} \Big].$$

Consequently

$$I_2 \le \Phi_M^{(1)}(T) + \Phi_M^{(2)}(T) + \int_0^t \left(1 + \|F_m'(s)\|\right) S_m^{(k)}(s) ds. \tag{3.31}$$

It remains to estimate $S_m^{(k)}(0)$. We have

$$S_m^{(k)}(0) = \|\tilde{u}_1\|^2 + a(\tilde{u}_0, \tilde{u}_0) + a\left(\tilde{u}_1, \tilde{u}_1\right) + \left\|\ddot{u}_m^{(k)}(0)\right\|^2. \tag{3.32}$$

On the other hand, letting $t \to 0_+$ in $(3.16)_1$, multiplying the result by $\ddot{c}_{mj}^{(k)}(0)$ and using the compatibility (3.9), we get

$$\left\|\ddot{u}_m^{(k)}(0)\right\|^2 + \left\langle -\tilde{u}_{0xx} + \alpha\tilde{u}_1, \ddot{u}_m^{(k)}(0)\right\rangle = \left\langle F_m(0), \ddot{u}_m^{(k)}(0)\right\rangle, \tag{3.33}$$

so

$$\left\|\ddot{u}_m^{(k)}(0)\right\| \le \|-\tilde{u}_{0xx} + \alpha\tilde{u}_1\| + \|F_m(0)\|. \tag{3.34}$$

We also have

$$\begin{aligned} &\|F_m(0)\| \\ &\le \|f(0)\| + \|\beta\|_{L^\infty} \|g(0)\|_{L^\infty} \Big\| D_1G(\tilde{u}_0, \tilde{P}_0)\tilde{u}_1 \\ &\quad + D_2G(\tilde{u}_0,\tilde{P}_0)g(0)G(\tilde{u}_0,\tilde{P}_0)\Big\| + \|\beta\|_{L^\infty} \|g'(0)\| \Big\|G(\tilde{u}_0,\tilde{P}_0)\Big\|_{L^\infty}. \end{aligned} \tag{3.35}$$

Therefore

$$\left\|\ddot{u}_m^{(k)}(0)\right\| \le \|-\tilde{u}_{0xx} + \alpha\tilde{u}_1\| + \|F_m(0)\| \le \overline{C}_{01} \text{ for all } m, \tag{3.36}$$

where $\overline{C}_{01}$ is a constant depending only on $\tilde{u}_0,\ \tilde{u}_1,\ \tilde{P}_0,\ \alpha,\ \beta,\ g,\ f,\ G$.

By (3.32) and (3.36) then there exists a positive constant $\overline{C}_{02}$ depending only on $\tilde{u}_0$, $\tilde{u}_1$, $\tilde{P}_0$, $\alpha$, $\beta$, $f$, $g$, $h$ and $G$, such that

$$S_m^{(k)}(0) \le \overline{C}_{02}, \text{ for all } m. \tag{3.37}$$

It follows from (3.22), (3.23), (3.31) and (3.37), that

$$\begin{aligned} S_m^{(k)}(t) \le \overline{C}_{02} + \Phi_M^{(1)}(T) + \Phi_M^{(2)}(T) \\ + \int_0^t \left(1 + 2\,\|\alpha\|_{L^\infty} + \|F_m'(s)\|\right) S_m^{(k)}(s) ds. \end{aligned} \tag{3.38}$$

Assumptions $(H_1)$, $(H_3)-(H_5)$ and (3.27), (3.30) yield

$$\lim_{T\to 0_+}\Phi_M^{(1)}(T)=\lim_{T\to 0_+}\Phi_M^{(2)}(T)=0. \tag{3.39}$$

Thus, with $M$, $T>0$ chosen in Part 1, it can be seen that $M^2\geq 2\overline{C}_{02}$ and $T\in(0,T^*]$ such that

$$\left(\tfrac{1}{2}M^2+\Phi_M^{(1)}(T)+\Phi_M^{(2)}(T)\right)\leq M^2\exp\left[-T\left(1+2\left\|\alpha\right\|_{L^\infty}\right)-\Phi_M^{(2)}(T)\right] \tag{3.40}$$

and

$$k_T=5d(M,T)\exp\left[\tfrac{1}{2}T\left(1+2\left\|\alpha\right\|_{L^\infty}\right)\ \right]<1, \tag{3.41}$$

where

$$\begin{cases} d(M,T)=\sqrt{Td_1^2(M,T)+d_2^2(M,T)+d_3^2(M,T)},\\ d_1(M,T)=\left\|\beta\right\|_{L^\infty}K_M(G)\Big[(1+2M)\left\|g(0)\right\|_{L^\infty}+\left\|g'(0)\right\|+\left\|g''\right\|_{L^1(0,T;L^2)}\Big],\\ d_2(M,T)=K_M(G)\left[T\left\|g(0)\right\|_{L^\infty}+\left\|g'\right\|_{L^1(0,T;L^2)}\right],\\ d_3(M,T)=K_M(G)\left[\left\|g_x\right\|_{L^1(0,T;L^2)}+(1+2M)\left\|g_x\right\|_{L^1(0,T;L^\infty)}\right]. \end{cases}$$

According to (3.38) and (3.40), we get

$$\begin{aligned} S_m^{(k)}(t)\leq M^2\exp&\left[-T\left(1+2\left\|\alpha\right\|_{L^\infty}\right)-\Phi_M^{(2)}(T)\right]\\ &+\int_0^t\left(1+2\left\|\alpha\right\|_{L^\infty}+\left\|F_m'(s)\right\|\right)S_m^{(k)}(s)ds. \end{aligned} \tag{3.42}$$

By using Gronwall's lemma, the result is

$$S_m^{(k)}(t)\leq M^2,\quad \text{for all } t\in[0,T],\quad \text{for all } m \text{ and } k. \tag{3.43}$$

Therefore, for all $m$ and $k$,

$$u_m^{(k)}\in B_T(M)\cap L^\infty(0,T;V\cap H^2),\quad \sqrt{2\lambda}\left\|\ddot{u}_m^{(k)}(0,\cdot)\right\|_{L^2(0,T)}\leq M, \tag{3.44}$$

**Step 3**. *Limiting process.*

We deduce from (3.44) that

$$\begin{cases} \left\|u_m^{(k)}\right\|_{L^\infty(0,T;V)}\leq M,\quad \left\|\dot{u}_m^{(k)}\right\|_{L^\infty(0,T;V)}\leq M,\\ \left\|\ddot{u}_m^{(k)}\right\|_{L^\infty(0,T;L^2)}\leq M,\\ \left\|\ddot{u}_m^{(k)}(0,\cdot)\right\|_{L^2(0,T)}\leq \frac{M}{\sqrt{2\lambda}},\quad \text{for all } m \text{ and } k. \end{cases} \tag{3.45}$$

From (3.46), there exists a subsequence of $\{u_m^{(k)}\}_k$, it is still so denoted, such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty(0,T;V) \text{ weak*}, \\ \dot{u}_m^{(k)} \to w_m^{(1)} & \text{in} \quad L^\infty(0,T;V) \text{ weak*}, \\ \ddot{u}_m^{(k)} \to w_m^{(2)} & \text{in} \quad L^\infty(0,T;L^2) \text{ weak*}, \\ \ddot{u}_m^{(k)}(0,\cdot) \to \bar{w}_m(\cdot) & \text{in} \quad L^2(0,T) \text{ weak}, \end{cases} \tag{3.46}$$

and

$$\begin{cases} \|u_m\|_{L^\infty(0,T;V)} \leq M, \quad \left\|w_m^{(1)}\right\|_{L^\infty(0,T;V)} \leq M, \\ \left\|w_m^{(2)}\right\|_{L^\infty(0,T;L^2)} \leq M, \\ \|\bar{w}_m(\cdot)\|_{L^2(0,T)} \leq \frac{M}{\sqrt{2\lambda}}, \text{ for all } m \text{ and } k. \end{cases} \tag{3.47}$$

First we show that $w_m^{(1)} = u_m'$, $w_m^{(2)} = u_m''$, in $V$ and $\bar{w}_m(\cdot) = u_m''(0,\cdot)$ in $L^2(0,T)$.

For each $m$, $k$ we have that

$$\begin{cases} u_m^{(k)}(t) = u_m^{(k)}(0) + \int_0^t \dot{u}_m^{(k)}(s)ds, \\ \dot{u}_m^{(k)}(t) = \dot{u}_m^{(k)}(0) + \int_0^t \ddot{u}_m^{(k)}(s)ds, \\ \dot{u}_m^{(k)}(0,t) = \dot{u}_m^{(k)}(0,0) + \int_0^t \ddot{u}_m^{(k)}(0,s)ds. \end{cases} \tag{3.48}$$

By (3.46), passing to the limit in $(3.48)_{1,2}$ with sense of "weak*" and in $(3.48)_3$ with sense of "weak", we obtain

$$\begin{cases} u_m(t) = \tilde{u}_0 + \int_0^t w_m^{(1)}(s)ds, \\ u_m'(t) = \tilde{u}_1 + \int_0^t w_m^{(2)}(s)ds, \\ u_m'(0,t) = \tilde{u}_1(0) + \int_0^t \bar{w}_m(s)ds. \end{cases} \tag{3.49}$$

where $(3.49)_{1,2}$ hold in $V$ for each $t \in [0,T]$. Thus $(3.49)_{1,2}$ imply that $w_m^{(1)} = u_m'$, $w_m^{(2)} = u_m''$, while from $(3.49)_3$ we can conclude that $u_m'(0,t)$ exists and it is continuous in $t$. Therefore $u_m'(0,t)$ is absolutely continuous in $[0,T]$, so $\bar{w}_m(t) = u_m''(0,t)$ for a.e. $t \in [0,T]$.

Consequently, (3.46) and (3.47) lead to

$$u_m \in B_T(M), \ \sqrt{2\lambda}\,\|u_m''(0,\cdot)\|_{L^2(0,T)} \leq M, \tag{3.50}$$

and

$$\begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty(0,T;V) \text{ weak*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in} \quad L^\infty(0,T;V) \text{ weak*}, \\ \ddot{u}_m^{(k)} \to u_m'' & \text{in} \quad L^\infty(0,T;L^2) \text{ weak*}, \\ \ddot{u}_m^{(k)}(0,\cdot) \to u_m''(0,\cdot) & \text{in} \quad L^2(0,T) \text{ weak}. \end{cases} \tag{3.51}$$

Passing to limit in (3.16), we have $u_m$ satisfying $(3.6)_{(ii)}$, (3.7) in $L^2(0,T)$.

On the other hand, it follows from $(3.6)_{(ii)}$, (3.8) and $(3.51)_3$ that

$$u_{mxx} = u_m'' + \alpha(x)u_m' - F_m(t) \in L^\infty(0,T;L^2),$$

hence $u_m \in L^\infty(0,T;V \cap H^2)$, Theorem 3.1 follows. □

**Theorem 3.3.** *Let assumptions* $(H_0)-(H_5)$ *and* (3.9) *hold. Then*

(i) *There exist positive constants* $M$ *and* $T$ *such that problem* (2.8), (2.9) *has a unique solution* $(u,P)$ *satisfying*

$$u, P \in B_T(M) \cap L^\infty(0,T;V\cap H^2), \sqrt{2\lambda}||u''(0,\cdot)||_{L^2(0,T)} \le M. \quad (3.52)$$

(ii) *On the other hand, the linear recurrent sequence* $\{(u_m,P_m)\}$ *defined by* (3.6)-(3.8) *converges to the solution* $(u,P)$ *of problem* (2.8), (2.9) *strongly in the space*

$$W_1(T) = \{(u,P) \in L^\infty(0,T;V\times V) : (u',P') \in L^\infty(0,T;L^2\times L^2)\}. \quad (3.53)$$

*Furthermore, we have the estimate*

$$\begin{aligned}&\|u_m - u\|_{L^\infty(0,T;V)} + \|P_m - P\|_{L^\infty(0,T;V)} + \|u_m' - u'\|_{L^\infty(0,T;L^2)}\\ &+ \|P_m' - P'\|_{L^\infty(0,T;L^2)} + \sqrt{2\lambda}\,\|u_m'(0,\cdot) - u'(0,\cdot)\|_{L^2(0,T)} \le Ck_T^m,\end{aligned} \quad (3.54)$$

*for all* $m\in\mathbb{N}$, *where the constant* $k_T \in (0,1)$ *is defined as in* (3.41) *and* $C$ *is a constant depending only on* $T$, $\tilde{u}_0$, $\tilde{u}_1$, $\tilde{P}_0$, $\alpha$, $\beta$, $f$, $g$, $G$ *and* $k_T$.

*Proof.* (i) *Existence of the solution.*

First, we note that $W_1(T)$ is a Banach space with respect to the norm (see Lions [3]) below

$$\begin{aligned}\|(u,P)\|_{W_1(T)} &= \|u\|_{L^\infty(0,T;V)} + \|P\|_{L^\infty(0,T;V)}\\ &\quad + \|u'\|_{L^\infty(0,T;L^2)} + \|P'\|_{L^\infty(0,T;L^2)}\,.\end{aligned} \quad (3.55)$$

We shall prove that $\{(u_m,P_m)\}$ is a Cauchy sequence in $W_1(T)$. Let $v_m = u_{m+1} - u_m$, $Q_m = P_{m+1} - P_m$. Then $(v_m,Q_m)$ satisfies the problem

$$\begin{cases} Q_m(t) = P_{m+1}(t) - P_m(t)\\ = \int_0^t g(t-s)\left[G(u_m(s),P_m(s)) - G(u_{m-1}(s),P_{m-1}(s))\right]ds,\\ \langle v_m''(t),v\rangle + a(v_m(t),v) + \lambda v_m'(0,t)v(0) + \langle \alpha v_m'(t),v\rangle\\ = \langle F_{m+1}(t) - F_m(t),v\rangle\,,\ \forall v\in V,\\ v_m(0) = v_m'(0) = 0,\end{cases} \quad (3.56)$$

where

$$\begin{aligned} &F_{m+1}(t) - F_m(t) \\ &= -\beta g(0) \tfrac{\partial}{\partial t} \left[ G(u_m, P_m) - G(u_{m-1}, P_{m-1}) \right] \\ &\quad -\beta g'(0) \left[ G(u_m, P_m) - G(u_{m-1}, P_{m-1}) \right] \\ &\quad -\beta \textstyle\int_0^t g''(t-s) \left[ G(u_m(s), P_m(s)) - G(u_{m-1}(s), P_{m-1}(s)) \right] ds. \end{aligned} \tag{3.57}$$

Taking $v = v_m'$ in $(3.56)_2$, after integrating in $t$, we get

$$Z_m(t) \le (1 + 2\|\alpha\|_{L^\infty}) \textstyle\int_0^t \|v_m'(s)\|^2 ds + \int_0^t \|F_{m+1}(s) - F_m(s)\|^2 ds, \tag{3.58}$$

where

$$Z_m(t) = \|v_m'(t)\|^2 + a(v_m(t), v_m(t)) + 2\lambda \textstyle\int_0^t |v_m'(0,s)|^2 ds. \tag{3.59}$$

Put

$$\begin{cases} \eta_m(t) = Z_m(t) + \|Q_m'(t)\|^2 + \|Q_{mx}(t)\|^2, \\ \bar{\eta}_m(t) = \|v_m'(t)\|^2 + \|v_{mx}(t)\|^2 + \|Q_m'(t)\|^2 + \|Q_{mx}(t)\|^2 \\ \qquad\qquad +2\lambda \int_0^t |v_m'(0,s)|^2 ds, \\ \gamma_m = \|(v_m, Q_m)\|_{W_1(T)} + \sqrt{2\lambda} \|v_m'(0,\cdot)\|_{L^2(0,T)}, \end{cases} \tag{3.60}$$

we have

$$\eta_m(t) = \bar{\eta}_m(t) + h v_m^2(0,t) \ge \bar{\eta}_m(t). \tag{3.61}$$

We need estimate $\int_0^t \|F_{m+1}(s) - F_m(s)\|^2 ds$. We have

$$\begin{aligned} &\|F_{m+1}(t) - F_m(t)\| \\ &\le \|\beta\|_{L^\infty} \|g(0)\|_{L^\infty} \left\| \tfrac{\partial}{\partial t} \left[ G(u_m, P_m) - G(u_{m-1}, P_{m-1}) \right] \right\| \\ &\quad + \|\beta\|_{L^\infty} \|g'(0)\| \, \|G(u_m, P_m) - G(u_{m-1}, P_{m-1})\|_{L^\infty} \\ &\quad + \|\beta\|_{L^\infty} \textstyle\int_0^t \|g''(t-s)\| \\ &\quad \times \left\| G(u_m(s), P_m(s)) - G(u_{m-1}(s), P_{m-1}(s)) \right\|_{L^\infty} ds. \end{aligned} \tag{3.62}$$

We shall estimate the terms on the right hands of (3.62) as follows. From the equation

$$\begin{aligned} &\tfrac{\partial}{\partial t} \left[ G(u_m, P_m) - G(u_{m-1}, P_{m-1}) \right] \\ &= D_1 G(u_m, P_m) v_{m-1}' + \left[ D_1 G(u_m, P_m) - D_1 G(u_{m-1}, P_{m-1}) \right] u_{m-1}' \\ &\quad + D_2 G(u_m, P_m) Q_{m-1}' + \left[ D_2 G(u_m, P_m) - D_2 G(u_{m-1}, P_{m-1}) \right] P_{m-1}', \end{aligned} \tag{3.63}$$

it follows that

$$\begin{aligned}
&\left\|\tfrac{\partial}{\partial t}\left[G(u_m,P_m)-G(u_{m-1},P_{m-1})\right]\right\| \\
&\le K_M(G)\left\|v'_{m-1}\right\|+\|D_1G(u_m,P_m)-D_1G(u_{m-1},P_{m-1})\|\left\|u'_{m-1}\right\|_{L^\infty} \\
&\quad +K_M(G)\left\|Q'_{m-1}\right\|+\|D_2G(u_m,P_m)-D_2G(u_{m-1},P_{m-1})\|\left\|P'_{m-1}\right\|_{L^\infty} \\
&\le K_M(G)\left\|v'_{m-1}\right\|+MK_M(G)\left[\|v_{m-1}\|+\|Q_{m-1}\|\right] \\
&\quad +K_M(G)\left\|Q'_{m-1}\right\|+MK_M(G)\left[\|v_{m-1}\|+\|Q_{m-1}\|\right] \\
&= K_M(G)\left[\left\|v'_{m-1}\right\|+\left\|Q'_{m-1}\right\|\right]+2MK_M(G)\left[\|v_{m-1}\|+\|Q_{m-1}\|\right] \\
&\le (1+2M)K_M(G)\left\|(v_{m-1},Q_{m-1})\right\|_{W_1(T)}.
\end{aligned} \tag{3.64}$$

On the other hand, we have

$$\begin{aligned}
&\|G(u_m,P_m)-G(u_{m-1},P_{m-1})\|_{L^\infty} \\
&\le K_M(G)\left[\|v_{m-1}\|_{L^\infty}+\|Q_{m-1}\|_{L^\infty}\right] \\
&\le K_M(G)\left\|(v_{m-1},Q_{m-1})\right\|_{W_1(T)}.
\end{aligned} \tag{3.65}$$

Hence

$$\begin{aligned}
&\textstyle\int_0^t \|g''(t-s)\|\,\|G(u_m(s),P_m(s))-G(u_{m-1}(s),P_{m-1}(s))\|_{L^\infty}\,ds \\
&\le K_M(G)\left\|(v_{m-1},Q_{m-1})\right\|_{W_1(T)}\textstyle\int_0^t\|g''(t-s)\|\,ds \\
&= K_M(G)\left\|(v_{m-1},Q_{m-1})\right\|_{W_1(T)}\textstyle\int_0^t\|g''(s)\|\,ds \\
&\le K_M(G)\left\|(v_{m-1},Q_{m-1})\right\|_{W_1(T)}\|g''\|_{L^1(0,T;L^2)}.
\end{aligned} \tag{3.66}$$

Thus, we deduce from (3.62)-(3.66) that

$$\begin{aligned}
&\|F_{m+1}(t)-F_m(t)\| \\
&\le \|\beta\|_{L^\infty}\|g(0)\|_{L^\infty}\left\|\tfrac{\partial}{\partial t}\left[G(u_m,P_m)-G(u_{m-1},P_{m-1})\right]\right\| \\
&\quad+\|\beta\|_{L^\infty}\|g'(0)\|\,\|G(u_m,P_m)-G(u_{m-1},P_{m-1})\|_{L^\infty} \\
&\quad+\|\beta\|_{L^\infty}\textstyle\int_0^t\|g''(t-s)\|\,\|G(u_m(s),P_m(s))-G(u_{m-1}(s),P_{m-1}(s))\|_{L^\infty}\,ds \\
&\le \|\beta\|_{L^\infty}\|g(0)\|_{L^\infty}(1+2M)K_M(G)\left\|(v_{m-1},Q_{m-1})\right\|_{W_1(T)} \\
&\quad+\|\beta\|_{L^\infty}\|g'(0)\|\,K_M(G)\left\|(v_{m-1},Q_{m-1})\right\|_{W_1(T)} \\
&\quad+\|\beta\|_{L^\infty}K_M(G)\left\|(v_{m-1},Q_{m-1})\right\|_{W_1(T)}\|g''\|_{L^1(0,T;L^2)} \\
&= \|\beta\|_{L^\infty}K_M(G)\left[\|g(0)\|_{L^\infty}(1+2M)+\|g'(0)\|+\|g''\|_{L^1(0,T;L^2)}\right] \\
&\quad\times\left\|(v_{m-1},Q_{m-1})\right\|_{W_1(T)} \\
&\equiv d_1(M,T)\left\|(v_{m-1},Q_{m-1})\right\|_{W_1(T)},
\end{aligned} \tag{3.67}$$

where

$$d_1(M,T)=\|\beta\|_{L^\infty}K_M(G)\left[(1+2M)\|g(0)\|_{L^\infty}+\left\|g'(0)\right\|+\left\|g''\right\|_{L^1(0,T;L^2)}\right].$$

Thus, we deduce from (3.58) and (3.67) that

$$\begin{aligned} Z_m(t) \leq (1+2\|\alpha\|_{L^\infty}) \int_0^t \|v_m'(s)\|^2 ds \\ +T d_1^2(M,T) \|(v_{m-1}, Q_{m-1})\|_{W_1(T)}^2 . \end{aligned} \tag{3.68}$$

Now, we shall estimate $\|Q_m'(t)\|^2 + \|Q_{mx}(t)\|^2$.

From the following equation

$$\begin{aligned} Q_m'(t) &= g(0)\left[G(u_m(t), P_m(t)) - G(u_{m-1}(t), P_{m-1}(t))\right] \\ &\quad + \int_0^t g'(t-s)\left[G(u_m(s), P_m(s)) - G(u_{m-1}(s), P_{m-1}(s))\right] ds, \end{aligned} \tag{3.69}$$

it follows that

$$\|Q_m'(t)\| \leq d_2(M,T) \|(v_{m-1}, Q_{m-1})\|_{W_1(T)}, \tag{3.70}$$

where $d_2(M,T) = K_M(G)\left[T\|g(0)\|_{L^\infty} + \|g'\|_{L^1(0,T;L^2)}\right]$.

Similarly, by

$$\begin{aligned} Q_{mx}(t) &= \int_0^t g_x(t-s)\left[G(u_m(s), P_m(s)) - G(u_{m-1}(s), P_{m-1}(s))\right] ds \\ &\quad + \int_0^t g(t-s)\frac{\partial}{\partial x}\left[G(u_m(s),P_m(s)) - G(u_{m-1}(s),P_{m-1}(s))\right] ds, \end{aligned} \tag{3.71}$$

it follows that

$$\|Q_{mx}(t)\| \leq d_3(M,T) \|(v_{m-1}, Q_{m-1})\|_{W_1(T)}, \tag{3.72}$$

where $d_3(M,T) = K_M(G)\left[\|g_x\|_{L^1(0,T;L^2)} + (1+2M)\|g_x\|_{L^1(0,T;L^\infty)}\right]$.

Combining (3.60), (3.61), (3.68), (3.70) and (3.72) we obtain

$$\begin{aligned} \bar{\eta}_m(t) &\leq \eta_m(t) = Z_m(t) + \|Q_m'(t)\|^2 + \|Q_{mx}(t)\|^2 \\ &\leq d^2(M,T)\|(v_{m-1}, Q_{m-1})\|_{W_1(T)}^2 + (1+2\|\alpha\|_{L^\infty}) \int_0^t \bar{\eta}_m(s) ds, \end{aligned} \tag{3.73}$$

where $d(M,T) = \sqrt{T d_1^2(M,T) + d_2^2(M,T) + d_3^2(M,T)}$.

Using Gronwall's lemma, we deduce from (3.73) that

$$\begin{aligned} \bar{\eta}_m(t) &\leq d^2(M,T)\|(v_{m-1}, Q_{m-1})\|_{W_1(T)}^2 \exp[T(1+2\|\alpha\|_{L^\infty})] \\ &\leq d^2(M,T)\exp[T(1+2\|\alpha\|_{L^\infty})]\gamma_{m-1}^2, \ \forall m \in \mathbb{N}, \ \forall t \in [0,T]. \end{aligned} \tag{3.74}$$

On the other hand

$$\begin{cases} \|v_m'(t)\| \leq \sqrt{\bar{\eta}_m(t)} \leq d(M,T)\exp[\frac{1}{2}T(1+2\|\alpha\|_{L^\infty})]\gamma_{m-1}, \\ \|v_{mx}(t)\| \leq \sqrt{\bar{\eta}_m(t)} \leq d(M,T)\exp[\frac{1}{2}T(1+2\|\alpha\|_{L^\infty})]\gamma_{m-1}, \\ \|Q_m'(t)\| \leq \sqrt{\bar{\eta}_m(t)} \leq d(M,T)\exp[\frac{1}{2}T(1+2\|\alpha\|_{L^\infty})]\gamma_{m-1}, \\ \|Q_{mx}(t)\| \leq \sqrt{\bar{\eta}_m(t)} \leq d(M,T)\exp[\frac{1}{2}T(1+2\|\alpha\|_{L^\infty})]\gamma_{m-1}, \\ \sqrt{2\lambda}\|v_m'(0,\cdot)\|_{L^2(0,T)} \leq \sqrt{\bar{\eta}_m(t)} \leq d(M,T)\exp[\frac{1}{2}T(1+2\|\alpha\|_{L^\infty})]\gamma_{m-1}, \end{cases}$$

and

$$\begin{aligned}\gamma_m &= \|(v_m, Q_m)\|_{W_1(T)} + \sqrt{2\lambda}\, \|v_m'(0,\cdot)\|_{L^2(0,T)} \\ &= \|v_m'\|_{L^\infty(0,T;L^2)} + \|v_m\|_{L^\infty(0,T;V)} + \|Q_m'\|_{L^\infty(0,T;L^2)} \\ &\quad + \|Q_m\|_{L^\infty(0,T;V)} + \sqrt{2\lambda}\, \|v_m'(0,\cdot)\|_{L^2(0,T)} \,,\end{aligned}$$

we deduce that

$$\gamma_m \leq k_T \gamma_{m-1}, \quad \forall m \in \mathbb{N}, \tag{3.75}$$

with $k_T = 5d(M,T) \exp\left[\frac{1}{2}T\left(1 + 2\|\alpha\|_{L^\infty}\right)\right] < 1$ defined in (3.41), which implies that for all $m$, $p \in \mathbb{N}$,

$$\begin{aligned}&\|(u_m, P_m) - (u_{m+p}, P_{m+p})\|_{W_1(T)} + \sqrt{2\lambda}\left\|u_m'(0,\cdot) - u_{m+p}'(0,\cdot)\right\|_{L^2(0,T)} \\ &\leq \gamma_0 (1-k_T)^{-1} k_T^m.\end{aligned} \tag{3.76}$$

It follows that $\{(u_m, P_m, u_m'(0,\cdot))\}$ is a Cauchy sequence in $W_1(T) \times L^2(0,T)$. Then there exists $(u, P, \xi) \in W_1(T) \times L^2(0,T)$ such that

$$\begin{cases} (u_m, P_m) \to (u, P) & \text{strongly in} \quad W_1(T), \\ u_m'(0,\cdot) \to \xi & \text{strongly in} \quad L^2(0,T). \end{cases} \tag{3.77}$$

On the other hand, from (3.50), there exists a subsequence $\{(u_{m_j}, P_{m_j})\}$ of $\{(u_m, P_m)\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in} \quad L^\infty(0,T;V) \text{ weak*}, \\ u_{m_j}' \to u' & \text{in} \quad L^\infty(0,T;V) \text{ weak*}, \\ u_{m_j}'' \to u'' & \text{in} \quad L^\infty(0,T;L^2) \text{ weak*}, \\ u_{m_j}''(0,\cdot) \to u''(0,\cdot) & \text{in} \quad L^2(0,T) \text{ weak}, \end{cases} \tag{3.78}$$

and

$$u,\ P \in B_T(M),\ \sqrt{2\lambda}\|u''(0,\cdot)\|_{L^2(0,T)} \leq M. \tag{3.79}$$

It follows from $(3.77)_2$ and $(3.78)_4$, that $\xi = u'(0,\cdot)$.

On the other hand, by the compactness lemma of Lions ([3], p.57) and the imbedding $H^2(0,T) \hookrightarrow C^1([0,T])$, (3.78) leads to the existence of a subsequence still denoted by $\{(u_{m_j}, P_{m_j})\}$, such that

$$\begin{cases} u_{m_j} \to u & \text{strongly in} \quad L^2(Q_T), \\ u_{m_j}' \to u' & \text{strongly in} \quad L^2(Q_T), \\ u_{m_j}(0,\cdot) \to u(0,\cdot) & \text{strongly in} \quad C^1([0,T]). \end{cases} \tag{3.80}$$

□

In order to obtain the result $(3.80)_{1,2}$, we use the following.

**Theorem 3.4.** (The compactness Lemma of Lions, [3], p.57) *Let $B_0$, $B$, $B_1$ be three Banach spaces, with*

(i) $B_0 \hookrightarrow B \hookrightarrow B_1$, *with $B_0$, $B_1$ are reflection;*
(ii) *The imbedding $B_0 \hookrightarrow B$ is compact.*

*Let $1 < p_0$, $p_1$, $T < +\infty$, then*

$$W(0,T) = \{v \in L^{p_0}(0,T;B_0) : v' \in L^{p_1}(0,T;B_1)\}$$

*is the Banach space with respect the norm*

$$\|v\| = \|v\|_{L^{p_0}(0,T;B_0)} + \|v'\|_{L^{p_1}(0,T;B_1)}.$$

*Therefore, the imbedding $W(0,T) \hookrightarrow L^{p_0}(0,T;B)$ is compact.*

Consider $p_0 = p_1 = 2$, $B_0 = V$, $B = B_1 = L^2$. In this case, $L^2(0,T;L^2) = L^2(Q_T)$ and the imbedding

$$W(0,T) = \{v \in L^2(0,T;V) : v' \in L^2(Q_T)\} \hookrightarrow L^2(Q_T)$$

is compact. Hence, it follows that $X_T \hookrightarrow L^2(Q_T)$ with the imbedding is compact.

Putting

$$\begin{aligned} F(t) = f(t) - \beta g(0)\tfrac{\partial}{\partial t}G(u,P) - \beta g'(0)G(u,P) \\ -\beta \textstyle\int_0^t g''(t-s)G(u(s),P(s))ds. \end{aligned} \tag{3.81}$$

By

$$\begin{cases} \|G(u_m,P_m) - G(u,P)\| \le K_M(G)\,\|(u_m,P_m)-(u,P)\|_{W_1(T)}, \\ \left\|\tfrac{\partial}{\partial t}\left[G(u_m,P_m) - G(u,P)\right]\right\| \le (1+2M)K_M(G)\|(u_m,P_m)-(u,P)\|_{W_1(T)}, \end{cases} \tag{3.82}$$

(3.8) and (3.81) imply

$$\begin{aligned} &\left\|F_{m_j}(t) - F(t)\right\| \\ &\le \|\beta\|_{L^\infty} K_M(G)\left[(1+2M)\,\|g(0)\|_{L^\infty} + \|g'(0)\|_{L^\infty} + \|g''\|_{L^1(0,T;L^\infty)}\right] \\ &\quad\times \left\|(u_{m_j-1},P_{m_j-1}) - (u,P)\right\|_{W_1(T)}. \end{aligned} \tag{3.83}$$

Hence, combining $(3.77)_1$ and (3.83) yield

$$F_{m_j}(t) \to F(t) \text{ strongly in } L^\infty(0,T;L^2). \tag{3.84}$$

On the other hand, by $(3.77)_1$, we deduce that

$$\begin{aligned} &\left\|P(t) - \tilde{P}_0 - \textstyle\int_0^t g(t-s)G(u(s),P(s))ds\right\| \\ &\le \|P-P_m\|_{L^\infty(0,T;V)} \\ &\quad + K_M(G)\,\|g\|_{L^1(0,T;L^\infty)}\,\|(u_{m-1},P_{m-1})-(u,P)\|_{W_1(T)} \\ &\to 0. \end{aligned} \tag{3.85}$$

Thus

$$P(t) - \tilde{P}_0 - \int_0^t g(t-s)G(u(s),P(s))ds = 0. \tag{3.86}$$

Finally, passing to limit in (3.6)-(3.8) as $m = m_j \to \infty$, it implies from (3.77), (3.78), (3.84) and (3.86) that there exists $(u,P)$ satisfying

$$\begin{cases} u,\, P \in B_T(M),\ \sqrt{2\lambda}||u''(0,\cdot)||_{L^2(0,T)} \le M, \\ P(t) = \tilde{P}_0 + \int_0^t g(t-s)G(u(s),P(s))ds, \\ \langle u''(t),v\rangle + a(u(t),v) + \lambda u'(0,t)v(0) + \langle \alpha u'(t), v\rangle = \langle F(t), v\rangle\,, \end{cases} \tag{3.87}$$

for all $v \in V$ and the initial conditions

$$u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1. \tag{3.88}$$

Furthermore, by $(H_1)$, we obtain from $(3.78)_{2,3}$, (3.84) and $(3.87)_2$ that

$$u_{xx} = u'' + \alpha(x)u' - F(t) \in L^\infty(0,T;L^2), \tag{3.89}$$

hence $u \in L^\infty(0,T;V\cap H^2)$. Thus $u \in B_T(M) \cap L^\infty(0,T;V\cap H^2)$. We also have $P \in L^\infty(0,T;V\cap H^2)$. Indeed,

$$\begin{aligned} &\|P_{xx}(t)\| \\ &\le \left\|\tilde{P}_{0xx}\right\| + K_M(G)\int_0^t \|g_{xx}(s)\|\,ds + 4MK_M(G)\int_0^t \|g_x(s)\|_{L^\infty}\,ds \\ &\quad + \|g\|_{L^\infty(Q_T)} K_M(G)\int_0^t \left[\|u_{xx}(s)\| + \|P_{xx}(s)\|\right]ds \\ &\quad + \|g\|_{L^\infty(Q_T)} K_M(G)\int_0^t \left[\left\|u_x^2(s)\right\| + 2\left\|u_x(s)P_x(s)\right\| + \left\|P_x^2(s)\right\|\right]ds \\ &\le D_T^{(1)}(M) + D_T^{(2)}(M)\int_0^t \|P_{xx}(s)\|\,ds, \end{aligned} \tag{3.90}$$

where

$$\begin{cases} D_T^{(1)}(M) \\ = \left\|\tilde{P}_{0xx}\right\| + K_M(G)\left[\|g_{xx}\|_{L^1(0,T;L^2)} + 4M\|g_x\|_{L^1(0,T;L^\infty)}\right] \\ \quad + K_M(G)T\|g\|_{L^\infty(Q_T)}\left[\left(1+3\sqrt{2}M\right)\|u_{xx}\|_{L^\infty(0,T;L^2)} + 4\sqrt{2}M^2\right], \\ D_T^{(2)}(M) = K_M(G)\left(\|g\|_{L^\infty(Q_T)} + \sqrt{2}M\|g\|_{L^\infty(Q_T)}\right). \end{cases} \tag{3.91}$$

By Gronwall's inequality we obtain that

$$\|P_{xx}(t)\| \le D_T^{(1)}(M)\exp(TD_T^{(2)}(M)). \tag{3.92}$$

Thus $P_{xx} \in L^\infty(0,T;L^2)$, hence $P \in L^\infty(0,T;V\cap H^2)$. It follows that $P \in B_T(M)\cap L^\infty(0,T;V\cap H^2)$. The existence proof is completed.

(ii) *Uniqueness of the solution.*

Let $(u_i, P_i)$, $i = 1, 2$ be two solutions of problem (2.8), (2.9). Then $(u, P)$, with $u = u_1 - u_2$, $P = P_1 - P_2$ satisfies the problem

$$\begin{cases} u_i,\ P_i \in B_T(M) \cap L^\infty(0,T; V \cap H^2), \\ \sqrt{2\lambda}||u_i''(0,\cdot)||_{L^2(0,T)} \le M,\ i = 1,\ 2, \\ P(t) = \int_0^t g(t-s)\bar{G}(s)ds, \\ \langle u''(t), v\rangle + a(u(t), v) + \lambda u'(0,t)v(0) + \langle \alpha u'(t), v\rangle = \langle F(t), v\rangle, \end{cases} \tag{3.93}$$

for all $v \in V$, a.e., $t \in (0,T)$, together with the initial conditions

$$u(0) = u'(0) = 0, \tag{3.94}$$

where

$$\begin{cases} F(t) = -\beta g(0)\bar{G}'(t) - \beta g'(0)\bar{G}(t) - \beta \int_0^t g''(t-s)\bar{G}(s)ds, \\ \bar{G}(t) = G(u_1(t), P_1(t)) - G(u_2(t), P_2(t)),\ \bar{G}(0) = 0. \end{cases} \tag{3.95}$$

We take $v = u'$ in $(3.93)_2$ and integrate in $t$ to get

$$Z(t) \le (1 + 2\left\|\alpha\right\|_{L^\infty}) \int_0^t \left\|u'(s)\right\|^2 ds + \int_0^t \left\|F(s)\right\|^2 ds, \tag{3.96}$$

where

$$\begin{aligned} Z(t) &= \left\|u'(t)\right\|^2 + a(u(t), u(t)) + 2\lambda \int_0^t \left|u'(0,s)\right|^2 ds \\ &\ge \left\|u'(t)\right\|^2 + \left\|u_x(t)\right\|^2 \equiv \bar{Z}(t). \end{aligned} \tag{3.97}$$

We set

$$\begin{aligned} \rho(t) &= \bar{Z}(t) + \left\|P'(t)\right\|^2 + \left\|P_x(t)\right\|^2 \\ &= \left\|u'(t)\right\|^2 + \left\|u_x(t)\right\|^2 + \left\|P'(t)\right\|^2 + \left\|P_x(t)\right\|^2 \end{aligned} \tag{3.98}$$

and $M = \max_{i=1,2} \left\|(u_i, P_i)\right\|_{W_1(T)}$, we estimate all terms of (3.95) as follows

$$\begin{aligned} &\text{(i)}\ \ \left\|\bar{G}(t)\right\| \le K_M(G)\left[\left\|u(t)\right\| + \left\|P(t)\right\|\right] \le 2K_M(G) \int_0^t \sqrt{\rho(s)}ds, \\ &\text{(ii)}\ \ \left\|\bar{G}(t)\right\|_{L^\infty} \le K_M(G)\left[\left\|u_x(t)\right\| + \left\|P_x(t)\right\|\right] \le 2K_M(G)\sqrt{\rho(t)}, \\ &\text{(iii)}\ \left\|\bar{G}'(t)\right\| \le (1+2M)\, K_M(G)\left[\left\|u'\right\| + \left\|P'\right\| + \left\|u\right\| + \left\|P\right\|\right] \\ &\qquad\qquad \le 2\,(1+2M)\, K_M(G)\sqrt{\rho(t)}, \\ &\text{(iv)}\ \left\|\bar{G}_x(t)\right\| \le (1+2M)K_M(G)\left(\left\|u_x(t)\right\| + \left\|P_x(t)\right\|\right) \\ &\qquad\qquad \le 2\,(1+2M)\, K_M(G)\sqrt{\rho(t)}, \\ &\text{(v)}\ \ \left\|P'(t)\right\|^2 \le 8K_M^2(G)\left[\left\|g(0)\right\|_{L^\infty}^2 T + \left\|g'\right\|_{L^2(0,T;L^2)}^2\right] \int_0^t \rho(s)ds \\ &\qquad\qquad \equiv \eta_1(M,T) \int_0^t \rho(s)ds, \\ &\text{(vi)}\ \left\|P_x(t)\right\|^2 \le 8K_M^2(G)\left[1 + (1+2M)^2\right] \left\|g_x\right\|_{L^2(0,T;L^\infty)}^2 \int_0^t \rho(s)ds \\ &\qquad\qquad \equiv \eta_2(M,T) \int_0^t \rho(s)ds. \end{aligned} \tag{3.99}$$

It follows from $(3.95)_1$, that

$$\begin{aligned} \|F(t)\| &\leq \|\beta g(0)\|_{L^\infty} \left\|\bar{G}'(t)\right\| + \|\beta g'(0)\| \left\|\bar{G}(t)\right\|_{L^\infty} \\ &\quad + \|\beta\|_{L^\infty} \int_0^t \|g''(t-s)\| \left\|\bar{G}(s)\right\|_{L^\infty} ds \\ &\leq \eta_3(M)\sqrt{\rho(t)} + \eta_4(M) \int_0^t \|g''(t-s)\| \sqrt{\rho(s)} ds, \end{aligned} \tag{3.100}$$

where

$$\begin{aligned} \eta_3(M) &= 2K_M(G) \|\beta\|_{L^\infty} \left[(1+2M) \|g(0)\|_{L^\infty} + \|g'(0)\|\right], \\ \eta_4(M) &= 2K_M(G) \|\beta\|_{L^\infty} . \end{aligned} \tag{3.101}$$

Hence

$$\begin{aligned} \int_0^t \|F(s)\|^2 ds &\leq 2 \left(\eta_3^2(M) + \eta_4^2(M)T \|g''\|^2_{L^2(0,T;L^2)}\right) \int_0^t \rho(s)ds \\ &\equiv \eta_5(M,T) \int_0^t \rho(s)ds. \end{aligned} \tag{3.102}$$

It follows from (3.96), (3.97) and (3.102), that

$$\bar{Z}(t) \leq Z(t) \leq 2\left(1 + \|\alpha\|_{L^\infty} + \eta_5(M,T)\right) \int_0^t \rho(s)ds. \tag{3.103}$$

From (3.98), $(3.99)_{v,vi}$ and (3.103), we get

$$\rho(t) \leq \left[2\left(1+\|\alpha\|_{L^\infty}+\eta_5(M,T)\ \right)+\eta_1(M,T)+\eta_2(M,T)\right] \int_0^t \rho(s)ds. \tag{3.104}$$

By Gronwall's inequality we obtain that $\rho(t) = 0$ on $(0,T)$, i.e., $u = u_1 - u_2 \equiv 0$, $P = P_1 - P_2 \equiv 0$, and hence the solution is unique. Passing to the limit as $p \to +\infty$ for $m$ fixed, we obtain estimate (3.54) from (3.76). This completes the proof of Theorem 3.3. □

**Remark 3.5.** Under assumptions of Theorem 3.1, the existence and uniqueness of a local weak solution are established. If we strengthen assumption $(H_5)$ by $(\widehat{H}_5)$ as below, it means that $G(\cdot,\cdot)$ is global Lipschitz which allows for applicability of the methods used as above, with less complicated techniques in order to get existence and uniqueness of a global weak solution. This is also an extension of the result obtained in [4].

$(\widehat{H}_5)$ $G \in C^1(\mathbb{R}^2)$ satisfies the following conditions:

(i) $|G(y,z)| \leq m_1\left(1 + |y| + |z|\right),\ \forall y,z \in \mathbb{R}, \quad m_1 > 0;$

(ii) $|D_1G(y,z)| + |D_2G(y,z)| \leq L,\ \forall y,z \in \mathbb{R}, \quad L > 0.$

## 4. Asymptotic behavior of a weak solution as $\lambda \to 0_+$

In this section, we let $h \geq 0$ and $\alpha$, $\beta$, $f$, $g$ and $G$ satisfy assumptions $(H_1)$, $(H_3) - (H_5)$. We also assume that

$(H_2')$ $(\tilde{u}_0, \tilde{u}_1, \tilde{P}_0) \in \left(V \cap H^2\right) \times H_0^1 \times \left(V \cap H^2\right)$ satisfy the compatibility condition $\tilde{u}_{0x}(0) = h\tilde{u}_0(0)$.

We consider the following perturbed problem, where $\lambda > 0$ is a small parameter:

$$(L_\lambda)\begin{cases} \langle u_{tt}(t), v\rangle + a(u(t), v) + \lambda u_t(0,t)v(0) + \langle \alpha u_t(t), v\rangle \\ +\langle \beta g(0)\frac{\partial}{\partial t}G(u,P), v\rangle + \langle \beta g'(0)G(u,P), v\rangle \\ +\langle \beta \int_0^t g''(t-s)G(u(s),P(s))ds, v\rangle = \langle f(t), v\rangle\,,\ \ \forall v\in V, \\ u(0)=\tilde{u}_0,\ u_t(0)=\tilde{u}_1, \\ P(t)=\tilde{P}_0+\int_0^t g(t-s)G(u(s),P(s))ds. \end{cases}$$

Then, for every $\lambda > 0$, by Theorem 3.1, problem $(L_\lambda)$ has a unique solution

$$u_\lambda,\ \ P_\lambda \in B_T(M)\cap L^\infty(0,T;V\cap H^2),\quad \sqrt{2\lambda}||u''_\lambda(0,\cdot)||_{L^2(0,T)} \le M. \tag{4.1}$$

depending on $\lambda$. We shall consider asymptotic behavior of this solution as $\lambda \to 0_+$.

**Theorem 4.1.** *Let $h \ge 0$ and $(H_1)$, $(H_2')$, $(H_3)-(H_5)$ hold. Then*

(i) *Problem $(L_0)$ corresponding to $\lambda = 0$ has a unique solution $(u_0, P_0)$ satisfying*

$$u_0,\ P_0 \in B_T(M)\cap L^\infty\left(0,T;V\cap H^2\right). \tag{4.2}$$

(ii) *The solution $(u_\lambda, P_\lambda)$ converges strongly in $W_1(T)$ to $(u_0, P_0)$, as $\lambda \to 0_+$. Furthermore, we have the estimate*

$$\begin{aligned}&\|(u_\lambda - u_0,\ P_\lambda - P_0)\|_{W_1(T)} + \sqrt{\lambda}\,\|u'_\lambda(0,\cdot) - u'_0(0,\cdot)\|_{L^2(0,T)} \\ &\le C\sqrt{\lambda},\end{aligned} \tag{4.3}$$

*where $C$ is a positive constant independent of $\lambda$.*

*Proof.* Let $\lambda \in (0,1]$. First, we note that a priori estimates of the linear recurrent sequence $\{(u_m, P_m)\}$ for problem $(L_\lambda)$ satisfy

$$u_m,\ P_m \in B_T(M)\cap L^\infty(0,T;V\cap H^2),\ \sqrt{2\lambda}||u''_m(0,\cdot)||_{L^2(0,T)} \le M, \tag{4.4}$$

where $M$ is a constant independent of $\lambda$ as in the proof of Theorem 3.1. Hence, the limit $(u_\lambda, P_\lambda)$ of the sequence $\{(u_m, P_m)\}$ as $m \to +\infty$, in suitable function spaces is a unique solution of problem $(L_\lambda)$ satisfying

$$u_\lambda,\ \ P_\lambda \in B_T(M)\cap L^\infty(0,T;V\cap H^2),\quad \sqrt{2\lambda}\,\|u''_\lambda(0,\cdot)\|_{L^2(0,T)} \le M. \tag{4.5}$$

It follows from (4.5) that

$$\begin{cases} \|u_\lambda(0,\cdot)\|_{H^1(0,T)} = \sqrt{\|u_\lambda(0,\cdot)\|^2 + \|u'_\lambda(0,\cdot)\|^2} \\ \qquad\qquad \leq \sqrt{\|u_{\lambda x}\|^2_{L^\infty(0,T;L^2)} + \|u'_{\lambda x}\|^2_{L^\infty(0,T;L^2)}} \leq M_1, \\ \sqrt{\lambda}\,\|u_\lambda(0,\cdot)\|_{H^2(0,T)} = \sqrt{\lambda}\sqrt{\|u_\lambda(0,\cdot)\|^2 + \|u'_\lambda(0,\cdot)\|^2 + \|u''_\lambda(0,\cdot)\|^2} \\ \qquad\qquad \leq M_1, \\ \|G(u_\lambda, P_\lambda)\|_{H^1(Q_T)} \leq M_1;\ \|D_1 G(u_\lambda, P_\lambda)\|_{H^1(Q_T)} \leq M_1; \\ \|D_2 G(u_\lambda, P_\lambda)\|_{H^1(Q_T)} \leq M_1, \end{cases} \tag{4.6}$$

where $M_1$ always indicates a constant independent of $\lambda$.

Let $\lambda_m$ be a sequence such that $\lambda_m \to 0^+$ as $m \to \infty$. From (4.5), (4.6), there exists a subsequence of $\{(u_{\lambda_m}, P_{\lambda_m})\}$, it is still so denoted, such that

$$\begin{cases} (u_{\lambda_m}, P_{\lambda_m}) \to (u_0, P_0) & \text{in} & L^\infty(0,T; V\times V) & \text{weakly*}, \\ (u'_{\lambda_m}, P'_{\lambda_m}) \to (u'_0, P'_0) & \text{in} & L^\infty(0,T; V\times V) & \text{weakly*}, \\ (u''_{\lambda_m}, P''_{\lambda_m}) \to (u''_0, P''_0) & \text{in} & L^\infty(0,T; L^2\times L^2) & \text{weakly*}, \\ u_{\lambda_m}(0,\cdot) \to u_0(0,\cdot) & \text{in} & H^1(0,T) & \text{weakly}, \\ \sqrt{\lambda_m} u_{\lambda_m}(0,\cdot) \to \eta_0 & \text{in} & H^2(0,T) & \text{weakly}, \\ G(u_{\lambda_m}, P_{\lambda_m}) \to \chi_0 & \text{in} & H^1(Q_T) & \text{weakly}, \\ D_1 G(u_{\lambda_m}, P_{\lambda_m}) \to \chi_1 & \text{in} & H^1(Q_T) & \text{weakly}, \\ D_2 G(u_{\lambda_m}, P_{\lambda_m}) \to \chi_2 & \text{in} & H^1(Q_T) & \text{weakly}. \end{cases} \tag{4.7}$$

By the compactness lemma of Lions ([3], p.57) and the imbeddings $H^1(Q_T) \hookrightarrow L^2(Q_T)$, $H^1(0,T) \hookrightarrow C^0([0,T])$, $H^2(0,T) \hookrightarrow C^1([0,T])$, we can deduce from (4.7) the existence of a subsequence still denoted by $\{(u_{\lambda_m}, P_{\lambda_m})\}$, such that

$$\begin{cases} (u_{\lambda_m}, P_{\lambda_m}) \to (u_0, P_0) & \text{strongly in } L^2(Q_T) \times L^2(Q_T), \\ (u'_{\lambda_m}, P'_{\lambda_m}) \to (u'_0, P'_0) & \text{strongly in } L^2(Q_T) \times L^2(Q_T), \\ u_{\lambda_m}(0,\cdot) \to u_0(0,\cdot) & \text{strongly in } C^0([0,T]), \\ \sqrt{\lambda_m} u_{\lambda_m}(0,\cdot) \to \eta_0 & \text{strongly in } C^1([0,T]), \\ G(u_{\lambda_m}, P_{\lambda_m}) \to \chi_0 & \text{strongly in } L^2(Q_T), \\ D_1 G(u_{\lambda_m}, P_{\lambda_m}) \to \chi_1 & \text{strongly in } L^2(Q_T), \\ D_2 G(u_{\lambda_m}, P_{\lambda_m}) \to \chi_2 & \text{strongly in } L^2(Q_T). \end{cases} \tag{4.8}$$

By $\sqrt{\lambda_m} u_{\lambda_m}(0,\cdot) \to \eta_0$ strongly in $C^1([0,T])$, we deduce from $(4.8)_3$ that

$$\eta_0 = 0. \tag{4.9}$$

Then, $(4.8)_4$ and (4.9) imply

$$\sqrt{\lambda_m} u'_{\lambda_m}(0,\cdot) \to 0 \ \text{ strongly in } C^0([0,T]). \tag{4.10}$$

Similarly, by $(4.8)_{1,\,2,\,5-7}$, we can to prove that

$$\chi_0 = G\left(u_0, P_0\right), \ \chi_1 = D_1 G\left(u_0, P_0\right), \ \chi_2 = D_2 G\left(u_0, P_0\right). \tag{4.11}$$

By passing to the limit, as in the proof of Theorem 3.1, we conclude that $(u_0, P_0)$ is a unique solution of problem $(L_0)$ corresponding to $\lambda = 0$ satisfying the a priori estimates (4.2). Put

$$u = u_\lambda - u_0, \ \ P = P_\lambda - P_0,$$

then $(u, P)$ satisfy the variational problem

$$\begin{cases} P(t) = \int_0^t g(t-s) H_\lambda(s) ds, \\ \langle u''(t), v\rangle + a(u(t), v) + \lambda u_\lambda'(0,t) v(0) + \langle \alpha u'(t), v\rangle \\ = \langle F_\lambda(t), v\rangle, \ \ \forall v \in V, \\ u(0) = u'(0) = 0, \end{cases} \tag{4.12}$$

where

$$\begin{cases} F_\lambda(t) = -\beta g(0) H_\lambda'(t) - \beta g'(0) H_\lambda(t) - \beta \int_0^t g''(t-s) H_\lambda(s) ds, \\ H_\lambda(t) = G(u_\lambda(t), P_\lambda(t)) - G(u_0(t), P_0(t)). \end{cases} \tag{4.13}$$

We take $w = u'$ in $(4.12)_2$ and integrate over $t$ to get

$$\begin{aligned} S(t) \leq & \left(1 + 2\left\|\alpha\right\|_{L^\infty}\right) \int_0^t \left\|u'(s)\right\|^2 ds - 2\lambda \int_0^t u_0'(0,s) u'(0,s) ds \\ & + \int_0^t \left\|F_\lambda(s)\right\|^2 ds, \end{aligned} \tag{4.14}$$

where

$$S(t) = \left\|u'(t)\right\|^2 + a(u(t), u(t)) + 2\lambda \int_0^t \left|u'(0,s)\right|^2 ds. \tag{4.15}$$

Note that

$$S(t) \geq \left\|u'(t)\right\|^2 + \left\|u_x(t)\right\|^2 + 2\lambda \int_0^t \left|u'(0,s)\right|^2 ds \equiv \bar{S}(t). \tag{4.16}$$

Set

$$X(t) = \bar{S}(t) + \left\|P'(t)\right\|^2 + \left\|P_x(t)\right\|^2. \tag{4.17}$$

By similar argument as in proof of Theorem 3.1, we can estimate $X(t)$ and the results are

$$\begin{aligned} \bar{S}(t) \leq & \ 2\lambda \left\|u_0'(0,\cdot)\right\|_{L^2(0,T)}^2 \\ & +2\left(1 + 2\left\|\alpha\right\|_{L^\infty} + 2\xi_1^2(M) + 2T\xi_2^2(M,T)\right) \int_0^t X(s) ds, \end{aligned} \tag{4.18}$$

where

$$\begin{cases} \xi_1(M) = K_M(G)\left[2(1+2M)\left\|\beta g(0)\right\|_{L^\infty} + \sqrt{2}\left\|\beta g'(0)\right\|\right], \\ \xi_2(M,T) = \sqrt{2} K_M(G)\left\|\beta\right\|_{L^\infty}\left\|g''\right\|_{L^2(0,T;L^2)}, \end{cases}$$

$$\begin{aligned}
&\|P'(t)\|^2 \\
&\le 2K_M^2(G)\left[4T(1+2M)^2\|g(0)\|_{L^\infty}^2+\|g'\|_{L^2(0,T;L^2)}^2\right]\int_0^t X(s)ds \\
&\le 2K_M^2(G)\left[4T(1+2M)^2\|g(0)\|_{L^\infty}^2+\|g'\|_{L^2(0,T;L^2)}^2\right]\int_0^t X(s)ds,
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
&\|P_x(t)\|^2 \\
&\le \left(\int_0^t \|g_x(t-s)\|\,\|H_\lambda(s)\|_{L^\infty}ds+\int_0^t\|g(t-s)\|_{L^\infty}\left\|\frac{\partial}{\partial x}H_\lambda(s)\right\|ds\right)^2 \\
&\le 2K_M^2(G)\left[\|g_x\|_{L^2(0,T;L^2)}^2+(1+2M)^2\|g\|_{L^2(0,T;L^\infty)}^2\right]\int_0^t X(s)ds.
\end{aligned} \tag{4.20}$$

Combining (4.17)-(4.20) yield

$$X(t)\le 2\lambda\|u_0'(0,\cdot)\|_{L^2(0,T)}^2+\xi(M,T)\int_0^t X(s)ds, \tag{4.21}$$

where $\xi(M,T)$ is a positive constant that depends only on $M$, $T$. Using Gronwall's lemma, we obtain $X(t)\le C\lambda$ and the estimate (4.3) follows. Theorem 4.1 is proved. □

## 5. An asymptotic expansion of a weak solution

In this section, we assume that $h\ge 0$ and $\alpha$, $\beta$, $f$, $g$ and $G$ satisfy assumptions $(H_1)$, $(H_2')$, $(H_3)-(H_5)$. The next result gives an asymptotic expansion of the solution $(u_\lambda,P_\lambda)$ up to order $N$ in $\lambda$ with error $\lambda^{N+\frac{1}{2}}$, for $\lambda$ sufficiently small. We make the following assumptions:

$$(H_5^{(N)})\quad G\in C^{N+2}(\mathbb{R}^2)\ \text{ satisfies }\ G(0,0)=0.$$

We use the following notation. For a multi-index $\alpha=(\alpha_1,...,\alpha_N)\in\mathbb{Z}_+^N$, and $x=(x_1,...,x_N)\in\mathbb{R}^N$, we put

$$|\alpha|=\alpha_1+...+\alpha_N,\quad \alpha!=\alpha_1!...\alpha_N!,\quad x^\alpha=x_1^{\alpha_1}...x_N^{\alpha_N}. \tag{5.1}$$

First, we need the following lemma.

**Lemma 5.1.** *Suppose $m$, $N\in\mathbb{N}$, $x=(x_1,...,x_N)\in\mathbb{R}^N$, and $\lambda\in\mathbb{R}$. Then*

$$\left(\sum_{i=1}^N x_i\lambda^i\right)^m=\sum_{i=m}^{mN}\Psi_i^{[m]}[x]\lambda^i, \tag{5.2}$$

*where the coefficients $\Psi_i^{[m]}[x]$, $m\le i\le mN$ depending on $x=(x_1,...,x_N)$ are defined by the formula*

$$\begin{cases}\Psi_i^{[m]}[x]=\sum\limits_{\alpha\in A_i^{(m)}}\frac{m!}{\alpha!}x^\alpha,\ m\le i\le mN,\\[2mm] A_i^{(m)}=\{\alpha\in\mathbb{Z}_+^N:|\alpha|=m,\ \sum_{j=1}^N j\alpha_j=i\}.\end{cases} \tag{5.3}$$

*Proof.* The proof of this lemma is not difficult, hence we omit the details. □

Let $(u_0, P_0)$ be a solution of problem $(L_0)$ as in Theorem 4.1.

$$(L_0)\begin{cases} P_0(t) = \tilde{P}_0 + \int_0^t g(t-s)G(u_0(s), P_0(s))ds, \\ \langle u_0''(t), w\rangle + a(u_0(t), w) + \langle \alpha u_0'(t), w\rangle = \langle \Phi_0(t), w\rangle\,,\ \forall w \in V, \\ u_0(0) = \tilde{u}_0,\ u_0'(0) = \tilde{u}_1, \\ \langle \Phi_0(t), w\rangle = \langle f(t), w\rangle \\ \qquad\qquad -\langle \beta \frac{\partial^2}{\partial t^2} \int_0^t g(t-s)G(u_0(s), P_0(s))ds, w\rangle,\ \ \forall w \in V, \\ u_0,\ P_0 \in B_T(M) \cap L^\infty(0,T; V\cap H^2). \end{cases} \tag{5.4}$$

Let us consider solutions $(u_i, P_i)\,,\ i = 1, 2, ..., N,$ defined by the following problems:

$$(\overline{L}_i)\ \begin{cases} P_i(t) = \int_0^t g(t-s)C_i(s)ds, \\ \langle u_i''(t), w\rangle + a(u_i(t), w) + \langle \alpha u_i'(t), w\rangle = \langle \Phi_i(t), w\rangle\,,\ \ \forall w \in V, \\ u_i(0) = u_i'(0) = 0, \\ u_i,\ P_i \in B_T(M) \cap L^\infty(0,T; V\cap H^2),\ i = 2, ..., N, \end{cases} \tag{5.5}$$

where

$$\begin{cases} \langle \Phi_1(t), w\rangle = -\left\langle \beta \frac{\partial^2}{\partial t^2}\left(\int_0^t g(t-s)C_1(s)ds\right), w\right\rangle, \\ \langle \Phi_i(t), w\rangle = -u_{i-1}'(0,t)w(0) \\ \qquad\qquad -\left\langle \beta \frac{\partial^2}{\partial t^2}\left(\int_0^t g(t-s)C_i(s)ds\right), w\right\rangle,\ i = 2, ..., N, \end{cases} \tag{5.6}$$

$$\begin{cases} C_i(t) = \sum_{|\gamma|=1}^{i} \frac{1}{\gamma!} D^\gamma G(u_0, P_0) \sum\limits_{j\in A_i(\gamma)} \Psi_j^{[\gamma_1]}[u]\Psi_{i-j}^{[\gamma_2]}[P],\ i = 1, ..., N, \\ A_i(\gamma) \equiv A_i(\gamma_1, \gamma_2) = \{j \in \mathbb{Z}_+ : \gamma_1 \le j \le N\gamma_1,\ \gamma_2 \le i - j \le N\gamma_2\}, \end{cases} \tag{5.7}$$

with $u = (u_1, ..., u_N)\,,\ P = (P_1, ..., P_N)\,.$ Then, we have the following theorem.

**Theorem 5.2.** *Let $(H_1)\,,\ (H_2'),\ (H_3)\,,\ (H_4)\,,\ (H_5^{(N)})$ hold. Then, there exist positive constants $M$ and $T$ such that, for every $\lambda$ with $0 < \lambda \le 1$, problem $(L_\lambda)$ has a unique solution $(u_\lambda, P_\lambda)$ satisfying the asymptotic estimation up to order $N$ as follows*

$$\begin{aligned} &\left\|\left(u_\lambda - \textstyle\sum_{i=0}^N u_i\lambda^i,\ P_\lambda - \sum_{i=0}^N P_i\lambda^i\right)\right\|_{W_1(T)} \\ &+\sqrt{\lambda}\left\|u_\lambda'(0,\cdot) - \textstyle\sum_{i=0}^N u_i'(0,\cdot)\lambda^i\right\|_{L^2(0,T)} \le C\lambda^{N+\frac{1}{2}}, \end{aligned} \tag{5.8}$$

*where $C$ is a positive constant independent of $\lambda$ and $(u_i, P_i)\,,\ i = 0,\ 1, ..., N,$ are the solutions of problems $(L_0)$, $(\overline{L}_i)$, $i = 1, ..., N,$ respectively.*

*Proof.* Let $(u,P)\equiv(u_\lambda,P_\lambda)$ be a unique solution of $(L_\lambda)$. Then $(v,Q)$, with

$$\begin{cases} v=u-\sum_{i=0}^{N}u_i\lambda^i\equiv u-U\equiv u-u_0-U_1, \\ Q=P-\sum_{i=0}^{N}P_i\lambda^i\equiv P-\eta\equiv P-P_0-\eta_1, \end{cases} \tag{5.9}$$

satisfies the problem

$$\begin{cases} Q(t)=\int_0^t g(t-s)\left[G(v+U,Q+\eta)-G(U,\eta)\right]ds+\bar{E}_\lambda(t), \\ \langle v''(t),w\rangle+a(v(t),w)+\langle\alpha v'(t),w\rangle \\ =-\lambda v'(0,t)w(0) \\ \quad -\left\langle\beta\frac{\partial^2}{\partial t^2}\left(\int_0^t g(t-s)\left[G(v+U,Q+\eta)-G(U,\eta)\right]ds\right),w\right\rangle \\ \quad +\langle E_\lambda(t),w\rangle,\ \forall w\in V, \\ v(0)=v'(0)=0, \end{cases} \tag{5.10}$$

where

$$\begin{cases} \langle E_\lambda(t),w\rangle \\ =-\lambda U_1'(0,t)w(0)-\sum_{i=1}^{N}\lambda^i\langle\Phi_i(t),w\rangle \\ \quad -\left\langle\beta\frac{\partial^2}{\partial t^2}\left(\int_0^t g(t-s)\left[G(u_0+U_1,P_0+\eta_1)-G(u_0,P_0)\right]ds\right),w\right\rangle, \\ \bar{E}_\lambda(t)=\int_0^t g(t-s)\left[G(u_0+U_1,P_0+\eta_1)-G(u_0,P_0)\right]ds \\ \qquad -\sum_{i=1}^{N}P_i(t)\lambda^i. \end{cases} \tag{5.11}$$

Then, we have the following lemma.

**Lemma 5.3.** *Let* $(H_1)$, $(H_2')$, $(H_3)$, $(H_4)$, $(H_5^{(N)})$ *hold. Then*

$$\begin{aligned} &\text{(i) } 2\int_0^t\langle E_\lambda(s),v'(s)\rangle ds\le D_T\lambda^{2N+1}+\lambda\int_0^t|v'(0,s)|^2ds \\ &\qquad\qquad +3\int_0^t\|v'(s)\|^2ds, \\ &\text{(ii) } \left\|\bar{E}_{\lambda x}\right\|_{L^\infty(0,T;L^2)}\le\overline{C}_{1N}\lambda^{N+1}, \\ &\text{(iii) } \left\|\bar{E}_\lambda'\right\|_{L^\infty(0,T;L^2)}\le\overline{C}_{2N}\lambda^{N+1}, \end{aligned} \tag{5.12}$$

*for all* $\lambda\in(0,1]$, *where* $D_T$, $\overline{C}_{1N}$, $\overline{C}_{2N}$, $\overline{C}_{3N}$ *are constants depending only on* $N$, $T$, $G$ *and* $\|u_i\|_{L^\infty(0,T;H^2)}$, $\|u_i'\|_{L^\infty(0,T;H^1)}$, $\|P_i\|_{L^\infty(0,T;H^2)}$, $\|P_i'\|_{L^\infty(0,T;H^1)}$, $(i=0,1,...,N)$.

*Proof of Lemma* 5.3. (i) In the case of $N=1$, the proof of Lemma 5.3 is easy, hence we omit the details.

Now, we consider $N\ge2$. Putting

$$\begin{cases} U=u_0+U_1,\ \ U_1=\sum_{i=1}^{N}u_i\lambda^i, \\ \eta\equiv P_0+\eta_1,\ \ \eta_1=\sum_{i=1}^{N}P_i\lambda^i. \end{cases} \tag{5.13}$$

By using Taylor's expansion of the function $G(U,\eta) = G(u_0+U_1, P_0+\eta_1)$ around the point $(u_0, P_0)$ up to order $N$, we obtain

$$\begin{aligned} G(u_0+U_1, P_0+\eta_1) = G(u_0,P_0) + \sum_{1\leq|\gamma|\leq N} \tfrac{1}{\gamma!} D^\gamma G(u_0,P_0) U_1^{\gamma_1}\eta_1^{\gamma_2} \\ +\lambda^{N+1} R_N^{(1)}[G, u_0, P_0, U_1, \eta_1], \end{aligned} \tag{5.14}$$

where

$$\begin{aligned} &\lambda^{N+1} R_N^{(1)}[G, u_0, P_0, U_1, \eta_1] \\ &= \sum_{|\gamma|=N+1} \tfrac{N+1}{\gamma!} U_1^{\gamma_1}\eta_1^{\gamma_2} \int_0^1 (1-\theta)^N D^\gamma G(u_0+\theta U_1, P_0+\theta\eta_1) d\theta. \end{aligned} \tag{5.15}$$

By Lemma 5.1, we obtain from (5.14), after some rearrangements in the order of $\lambda$, that

$$G(u_0+U_1, P_0+\eta_1) - G(u_0,P_0) = \textstyle\sum_{i=1}^N C_i(t)\lambda^i + \lambda^{N+1} R_N^{(2)}(t), \tag{5.16}$$

where $C_i(t),\ i=1,2,...,N$, defined by (5.7) and

$$\begin{aligned} &\lambda^{N+1} R_N^{(2)}(t) \equiv \lambda^{N+1} R_N^{(2)}[G, u_0, P_0, U_1, \eta_1] \\ &= \lambda^{N+1} R_N^{(1)}[G, u_0, P_0, U_1, \eta_1] \\ &\quad + \sum_{1\leq|\gamma|\leq N} \tfrac{1}{\gamma!} D^\gamma G(u_0,P_0) \textstyle\sum_{i=N+1}^{N|\gamma|} \displaystyle\sum_{j\in A_i(\gamma)} \Psi_j^{[\gamma_1]}[u]\Psi_{i-j}^{[\gamma_2]}[P]\lambda^i. \end{aligned} \tag{5.17}$$

Combining $(L_0)$, $(\overline{L}_i)$, (5.6), (5.7), (5.11) and (5.16) yield

$$\begin{aligned} &\langle E_\lambda(t), w\rangle \\ &= -\lambda^{N+1} u_N'(0,t)w(0) - \lambda^{N+1} \left\langle \beta \tfrac{\partial^2}{\partial t^2} \left( \textstyle\int_0^t g(t-s) R_N^{(2)}(s) ds \right), w \right\rangle, \end{aligned} \tag{5.18}$$

$$\bar{E}_\lambda(t) = \lambda^{N+1} \textstyle\int_0^t g(t-s) R_N^{(2)}(s) ds. \tag{5.19}$$

By the boundedness of the functions $(u_i, P_i)$, $(u_i', P_i')$, $i=0,1,...,N$, in the function space $W_1(T)$, we obtain after some lengthy calculation from (5.15) and (5.17), that

$$\begin{aligned} &\left\| R_N^{(2)} \right\|_{L^\infty(0,T;L^\infty)} + \left\| \tfrac{\partial}{\partial t} R_N^{(2)} \right\|_{L^\infty(0,T;L^\infty)} + \left\| \tfrac{\partial}{\partial x} R_N^{(2)} \right\|_{L^\infty(0,T;L^\infty)} \\ &\leq \overline{C}_{0N}, \end{aligned} \tag{5.20}$$

where $\overline{C}_{0N}$ is a constant depending only on $N$, $T$, $G$ and $\|u_i\|_{L^\infty(0,T;H^1)}$, $\|P_i\|_{L^\infty(0,T;H^1)}$, $(i=0,1,...,N)$, $\sup\limits_{|y|,\ |z|\leq M} |D^\gamma G(y,z)|$, $|\gamma| \leq N+2$. By (5.18) and (5.20), we deduce that

$$\begin{aligned} &2\int_0^t \langle E_\lambda(s), v'(s)\rangle ds \\ &\le \lambda^{2N+1}\left\|u_N'(0,\cdot)\right\|_{L^2(0,T)}^2 + \lambda\int_0^t |v'(0,s)|^2\,ds \\ &\quad +\lambda^{2N+2}\|\beta\|_{L^\infty}^2\,\overline{C}_{0N}^2\left[\|g(0)\|_{L^\infty}^2 + \|g'(0)\|_{L^\infty}^2 + \|g''\|_{L^1(0,T;L^2)}^2\right] \\ &\quad +3\int_0^t \|v'(s)\|^2\,ds \\ &\le D_T\lambda^{2N+1} + \lambda\int_0^t |v'(0,s)|^2\,ds + 3\int_0^t\|v'(s)\|^2\,ds, \end{aligned} \tag{5.21}$$

where

$$\begin{aligned} D_T &= \left\|u_N'(0,\cdot)\right\|_{L^2(0,T)}^2 \\ &\quad + \|\beta\|_{L^\infty}^2\,\overline{C}_{0N}^2\left[\|g(0)\|_{L^\infty}^2 + \|g'(0)\|_{L^\infty}^2 + \|g''\|_{L^1(0,T;L^2)}^2\right]. \end{aligned} \tag{5.22}$$

(ii) By (5.19), we deduce that

$$\bar{E}_{\lambda x}(t) = \lambda^{N+1}\int_0^t g_x(t-s)R_N^{(2)}(s)ds + \lambda^{N+1}\int_0^t g(t-s)\frac{\partial}{\partial x}R_N^{(2)}(s)ds. \tag{5.23}$$

Thus

$$\begin{aligned} \left\|\bar{E}_{\lambda x}(t)\right\| &\le \lambda^{N+1}\int_0^t \|g_x(t-s)\|\left\|R_N^{(2)}\right\|_{L^\infty(0,T;L^\infty)} ds \\ &\quad +\lambda^{N+1}\int_0^t\|g(t-s)\|\left\|\frac{\partial}{\partial x}R_N^{(2)}\right\|_{L^\infty(0,T;L^\infty)} ds \\ &\le \overline{C}_{0N}\left[\|g\|_{L^1(0,T;L^2)} + \|g_x\|_{L^1(0,T;L^2)}\right]\lambda^{N+1} \equiv \overline{C}_{1N}\lambda^{N+1}. \end{aligned} \tag{5.24}$$

(iii) Similary, by (5.19) we have

$$\bar{E}_\lambda'(t) = \lambda^{N+1}\left[g(0)R_N^{(2)}(t) + \int_0^t g'(t-s)R_N^{(2)}(s)ds\right]. \tag{5.25}$$

Thus

$$\begin{aligned} \left\|\bar{E}_\lambda'(t)\right\| &\le \lambda^{N+1}\left\|R_N^{(2)}\right\|_{L^\infty(0,T;L^\infty)}\left[\|g(0)\| + \int_0^t\|g'(t-s)\|\,ds\right] \\ &\le \overline{C}_{0N}\lambda^{N+1}\left[\|g(0)\| + \|g'\|_{L^1(0,T;L^2)}\right] \equiv \overline{C}_{2N}\lambda^{N+1}. \end{aligned} \tag{5.26}$$

This implies (5.12), Lemma 5.3 follows. □

Lemma 5.3 is the key to obtain the asymptotic expansion of a weak solution $(u_\lambda, P_\lambda)$ of order $N+1$ in a small parameter $\lambda$. Indeed, we take $w = v'$ in $(5.10)_1$ and after integration over $t$, we find without difficulty from Lemma 5.3, that

$$\bar{S}(t) \le D_T\lambda^{2N+1} + (3 + 2\|\alpha\|_{L^\infty})\int_0^t\|v'(s)\|^2\,ds + J, \tag{5.27}$$

where

$$\begin{aligned} &\bar{S}(t) = \|v'(t)\|^2 + \|v_x(t)\|^2 + \lambda\int_0^t|v'(0,s)|^2\,ds, \\ &J = -2\int_0^t\left\langle \beta\frac{\partial^2}{\partial s^2}\left(\int_0^s g(s-r)\left[G(v+U,Q+\eta) - G(U,\eta)\right]dr\right), v'(s)\right\rangle ds. \end{aligned} \tag{5.28}$$

Put

$$\sigma(t) = \bar{S}(t) + \|Q'(t)\|^2 + \|Q_x(t)\|^2. \tag{5.29}$$

Apply similar methods as in above sections, we can estimate all the terms of $\sigma(t)$ and obtain

$$\sigma(t) \leq \eta_1(M,T)\lambda^{2N+1} + \eta_2(M,T)\int_0^t \sigma(s)ds, \tag{5.30}$$

where $\eta_1(M,T)$, $\eta_2(M,T)$ are positive constant depending only on $M, T$. Using Gronwall's lemma, we get (5.8). Theorem 5.2 is proved. □

**Appendix**. *Proof of Lemma* 3.2.
(i) Prove that $\|G(u_{m-1}(t), P_{m-1}(t))\|_{L^\infty} \leq K_M(G)$. By

$$\begin{aligned} &\|u_{m-1}(t)\|_{L^\infty} \leq \|u_{m-1}(t)\|_V \leq \|u_{m-1}\|_{L^\infty(0,T;V)} \leq M \\ &\text{and } \|P_{m-1}(t)\|_{L^\infty} \leq \|P_{m-1}(t)\|_V \leq \|P_{m-1}\|_{L^\infty(0,T;V)} \leq M, \end{aligned} \tag{a1}$$

we deduce that

$$|G(u_{m-1}(t), P_{m-1}(t))| \leq \|G\|_{C^0([-M,M]^2)} \leq K_M(G), \text{ a.e. } x \in \Omega. \tag{a2}$$

Thus (i) holds.
(ii) Prove that $\|G(u_{m-1}(t), P_{m-1}(t))\|_{L^\infty} \leq \left\|G(\tilde{u}_0, \tilde{P}_0)\right\|_{L^\infty} + 2TMK_M(G)$.
Let (iii) holds. Then

$$G(u_{m-1}(t), P_{m-1}(t)) = G(\tilde{u}_0, \tilde{P}_0) + \int_0^t \frac{\partial}{\partial s} G(u_{m-1}(s), P_{m-1}(s))ds. \tag{a3}$$

Hence, by (iii) and (a3), we obtain

$$\begin{aligned} &\|G(u_{m-1}(t), P_{m-1}(t))\|_{L^\infty} \\ &\leq \left\|G(\tilde{u}_0, \tilde{P}_0)\right\|_{L^\infty} + \int_0^t \left\|\frac{\partial}{\partial s} G(u_{m-1}(s), P_{m-1}(s))\right\|_{L^\infty} ds \\ &\leq \left\|G(\tilde{u}_0, \tilde{P}_0)\right\|_{L^\infty} + \int_0^t 2MK_M(G)ds \\ &\leq \left\|G(\tilde{u}_0, \tilde{P}_0)\right\|_{L^\infty} + 2TMK_M(G). \end{aligned} \tag{a4}$$

Thus (ii) holds.
(iii) Prove that $\left\|\frac{\partial}{\partial t} G(u_{m-1}(t), P_{m-1}(t))\right\|_{L^\infty} \leq 2MK_M(G)$.
We have

$$\begin{aligned} &\frac{\partial}{\partial t} G(u_{m-1}(t), P_{m-1}(t)) \\ &= D_1 G(u_{m-1}(t), P_{m-1}(t))u'_{m-1}(t) + D_2 G(u_{m-1}(t), P_{m-1}(t))P'_{m-1}(t). \end{aligned} \tag{a5}$$

By

$$\begin{aligned}&\left\|u'_{m-1}(t)\right\|_{L^\infty} \le \left\|u'_{m-1}(t)\right\|_V \le \left\|u'_{m-1}\right\|_{L^\infty(0,T;V)} \le M,\\ &\left\|P'_{m-1}(t)\right\|_{L^\infty} \le \left\|P'_{m-1}(t)\right\|_V \le \left\|P'_{m-1}\right\|_{L^\infty(0,T;V)} \le M,\\ &\left|D_1G(u_{m-1}(t),P_{m-1}(t))\right| \le K_M(G),\\ &\left|D_2G(u_{m-1}(t),P_{m-1}(t))\right| \le K_M(G),\end{aligned} \tag{a6}$$

we deduce that

$$\begin{aligned}\left|\tfrac{\partial}{\partial t}G(u_{m-1}(t),P_{m-1}(t))\right| &\le K_M(G)\left[\left|u'_{m-1}(t)\right| + \left|P'_{m-1}(t)\right|\right]\\ &\le 2MK_M(G).\end{aligned} \tag{a7}$$

Thus (iii) holds.
(iv) Prove that

$$\begin{aligned}\left\|\frac{\partial}{\partial t}G(u_{m-1}(t),P_{m-1}(t))\right\| \le{}& \left\|D_1G(\tilde{u}_0,\tilde{P}_0)\tilde{u}_1 + D_2G(\tilde{u}_0,\tilde{P}_0)g(0)G(\tilde{u}_0,\tilde{P}_0)\right\|\\ &+ 2TM\left(1+2M\right)K_M(G).\end{aligned}$$

Let (vii) holds. We have

$$\begin{aligned}&\tfrac{\partial}{\partial t}G(u_{m-1}(t),P_{m-1}(t))\\ &= \tfrac{\partial}{\partial t}G(u_{m-1}(t),P_{m-1}(t))\Big|_{t=0} + \textstyle\int_0^t \tfrac{\partial^2}{\partial s^2}G(u_{m-1}(s),P_{m-1}(s))ds\\ &= D_1G(\tilde{u}_0,\tilde{P}_0)\tilde{u}_1 + D_2G(\tilde{u}_0,\tilde{P}_0)g(0)G(\tilde{u}_0,\tilde{P}_0)\\ &\quad + \textstyle\int_0^t \tfrac{\partial^2}{\partial s^2}G(u_{m-1}(s),P_{m-1}(s))ds.\end{aligned} \tag{a8}$$

Hence, by (vii) and (a8), we obtain

$$\begin{aligned}&\left\|\tfrac{\partial}{\partial t}G(u_{m-1}(t),P_{m-1}(t))\right\|\\ &\le \left\|D_1G(\tilde{u}_0,\tilde{P}_0)\tilde{u}_1 + D_2G(\tilde{u}_0,\tilde{P}_0)g(0)G(\tilde{u}_0,\tilde{P}_0)\right\|\\ &\quad + \textstyle\int_0^t \left\|\tfrac{\partial^2}{\partial s^2}G(u_{m-1}(s),P_{m-1}(s))\right\|ds\\ &\le \left\|D_1G(\tilde{u}_0,\tilde{P}_0)\tilde{u}_1 + D_2G(\tilde{u}_0,\tilde{P}_0)g(0)G(\tilde{u}_0,\tilde{P}_0)\right\|\\ &\quad + \textstyle\int_0^t 2M\left(1+2M\right)K_M(G)ds\\ &\le \left\|D_1G(\tilde{u}_0,\tilde{P}_0)\tilde{u}_1 + D_2G(\tilde{u}_0,\tilde{P}_0)g(0)G(\tilde{u}_0,\tilde{P}_0)\right\|\\ &\quad +2TM\left(1+2M\right)K_M(G)ds.\end{aligned} \tag{a9}$$

Thus (iv) holds.
(v) Prove that $\left\|\frac{\partial}{\partial x}G(u_{m-1}(t),P_{m-1}(t))\right\| \le 2MK_M(G)$. We have

$$\begin{aligned}&\tfrac{\partial}{\partial x}G(u_{m-1},P_{m-1})\\ &= D_1G(u_{m-1},P_{m-1})\tfrac{\partial u_{m-1}}{\partial x} + D_2G(u_{m-1},P_{m-1})\tfrac{\partial P_{m-1}}{\partial x}.\end{aligned} \tag{a10}$$

By

$$\begin{aligned} \left\| \tfrac{\partial u_{m-1}}{\partial x}(t) \right\| &= \|u_{m-1}(t)\|_V \le \|u_{m-1}\|_{L^\infty(0,T;V)} \le M, \\ \left\| \tfrac{\partial P_{m-1}}{\partial x}(t) \right\| &= \|P_{m-1}(t)\|_V \le \|P_{m-1}\|_{L^\infty(0,T;V)} \le M, \end{aligned} \tag{a11}$$

we deduce that

$$\begin{aligned} \left\| \tfrac{\partial}{\partial x} G(u_{m-1}(t), P_{m-1}(t)) \right\| &\le K_M(G) \left[ \left\| \tfrac{\partial u_{m-1}}{\partial x}(t) \right\| + \left\| \tfrac{\partial P_{m-1}}{\partial x}(t) \right\| \right] \\ &\le 2MK_M(G). \end{aligned} \tag{a12}$$

Thus (v) holds.
(vi) Prove that

$$\left\| \frac{\partial}{\partial x} G(u_{m-1}(t), P_{m-1}(t)) \right\| \le \left\| \frac{\partial}{\partial x} G(\tilde{u}_0, \tilde{P}_0) \right\| + 2TM(1+2M)K_M(G).$$

We have

$$\begin{aligned} & \tfrac{\partial}{\partial x} G(u_{m-1}(t), P_{m-1}(t)) \\ & = \tfrac{\partial}{\partial x} G(\tilde{u}_0, \tilde{P}_0) + \int_0^t \tfrac{\partial}{\partial s} \left[ \tfrac{\partial}{\partial x} G(u_{m-1}(s), P_{m-1}(s)) \right] ds; \\ & \\ & \left\| \tfrac{\partial}{\partial x} G(u_{m-1}(t), P_{m-1}(t)) \right\| \\ & \le \left\| \tfrac{\partial}{\partial x} G(\tilde{u}_0, \tilde{P}_0) \right\| + \int_0^t \left\| \tfrac{\partial}{\partial s} \left[ \tfrac{\partial}{\partial x} G(u_{m-1}(s), P_{m-1}(s)) \right] \right\| ds; \\ & \\ & \tfrac{\partial}{\partial t} \left[ \tfrac{\partial}{\partial x} G(u_{m-1}, P_{m-1}) \right] \\ & = \tfrac{\partial}{\partial t} \left[ D_1 G(u_{m-1}, P_{m-1}) \tfrac{\partial u_{m-1}}{\partial x} \right] + \tfrac{\partial}{\partial t} \left[ D_2 G(u_{m-1}, P_{m-1}) \tfrac{\partial P_{m-1}}{\partial x} \right] \\ & = D_1 G(u_{m-1}, P_{m-1}) \tfrac{\partial u'_{m-1}}{\partial x} + D_{11} G(u_{m-1}, P_{m-1}) u'_{m-1} \tfrac{\partial u_{m-1}}{\partial x} \\ & \quad + D_{12} G(u_{m-1}, P_{m-1}) P'_{m-1} \tfrac{\partial u_{m-1}}{\partial x} \\ & \quad + D_2 G(u_{m-1}, P_{m-1}) \tfrac{\partial P'_{m-1}}{\partial x} + D_{21} G(u_{m-1}, P_{m-1}) u'_{m-1} \tfrac{\partial P_{m-1}}{\partial x} \\ & \quad + D_{22} G(u_{m-1}, P_{m-1}) P'_{m-1} \tfrac{\partial P_{m-1}}{\partial x}. \end{aligned} \tag{a13}$$

$$\begin{aligned} & \left\| \tfrac{\partial}{\partial t} \left[ \tfrac{\partial}{\partial x} G(u_{m-1}, P_{m-1}) \right] \right\| \\ & \le K_M(G) \left[ \left\| \tfrac{\partial u'_{m-1}}{\partial x} \right\| + \left\| u'_{m-1} \tfrac{\partial u_{m-1}}{\partial x} \right\| + \left\| P'_{m-1} \tfrac{\partial u_{m-1}}{\partial x} \right\| \right] \\ & \qquad + K_M(G) \left[ \left\| \tfrac{\partial P'_{m-1}}{\partial x} \right\| + \left\| u'_{m-1} \tfrac{\partial P_{m-1}}{\partial x} \right\| + \left\| P'_{m-1} \tfrac{\partial P_{m-1}}{\partial x} \right\| \right] \\ & \le 2M(1+2M)K_M(G). \end{aligned} \tag{a14}$$

Hence, by (a13) and (a14), we obtain

$$\begin{aligned}
&\left\|\tfrac{\partial}{\partial x}G(u_{m-1}(t),P_{m-1}(t))\right\| \\
&\leq \left\|\tfrac{\partial}{\partial x}G(\tilde{u}_0,\tilde{P}_0)\right\| + \textstyle\int_0^t \left\|\tfrac{\partial}{\partial s}\left[\tfrac{\partial}{\partial x}G(u_{m-1}(s),P_{m-1}(s))\right]\right\| ds \\
&\leq \left\|\tfrac{\partial}{\partial x}G(\tilde{u}_0,\tilde{P}_0)\right\| + 2TM(1+2M)K_M(G).
\end{aligned} \tag{a15}$$

Thus (vi) holds.
(vii) Prove that $\left\|\frac{\partial^2}{\partial t^2}G(u_{m-1}(t),P_{m-1}(t))\right\| \leq 2M\,(1+2M)\,K_M(G)$. We have

$$\begin{aligned}
&\tfrac{\partial^2}{\partial t^2}G(u_{m-1},P_{m-1}) \\
&= D_1G(u_{m-1},P_{m-1})u''_{m-1} + D_{11}G(u_{m-1},P_{m-1})\left|u'_{m-1}\right|^2 \\
&\quad + D_{12}G(u_{m-1},P_{m-1})P'_{m-1}u'_{m-1} \\
&\quad + D_2G(u_{m-1},P_{m-1})P''_{m-1} + D_{21}G(u_{m-1},P_{m-1})u'_{m-1}P'_{m-1} \\
&\quad + D_{22}G(u_{m-1},P_{m-1})\left|P'_{m-1}\right|^2,
\end{aligned} \tag{a16}$$

we deduce that

$$\begin{aligned}
&\left\|\tfrac{\partial^2}{\partial t^2}G(u_{m-1},P_{m-1})\right\| \\
&\leq K_M(G)\left[\left\|u''_{m-1}\right\| + \left\|\left|u'_{m-1}\right|^2\right\| + \left\|P'_{m-1}u'_{m-1}\right\|\right] \\
&\quad + K_M(G)\left[\left\|P''_{m-1}\right\| + \left\|u'_{m-1}P'_{m-1}\right\| + \left\|\left|P'_{m-1}\right|^2\right\|\right] \\
&\leq 2M(1+2M)K_M(G).
\end{aligned} \tag{a17}$$

Thus (vii) holds.
(viii) Prove that

$$\begin{aligned}
&\left\|\frac{\partial^2}{\partial x^2}G(u_{m-1}(t),P_{m-1}(t))\right\| \\
&\leq K_M(G)\left[4\sqrt{2}M^2 + \left(1+2\sqrt{2}M\right)\left(\|\Delta u_{m-1}(t)\| + \|\Delta P_{m-1}(t)\|\right)\right].
\end{aligned}$$

We have

$$\begin{aligned}
&\tfrac{\partial^2}{\partial x^2}G(u_{m-1},P_{m-1}) \\
&= D_1G(u_{m-1},P_{m-1})\Delta u_{m-1} + D_{11}G(u_{m-1},P_{m-1})\left|\tfrac{\partial u_{m-1}}{\partial x}\right|^2 \\
&\quad + D_{12}G(u_{m-1},P_{m-1})\tfrac{\partial P_{m-1}}{\partial x}\tfrac{\partial u_{m-1}}{\partial x} \\
&\quad + D_2G(u_{m-1},P_{m-1})\Delta P_{m-1} + D_{21}G(u_{m-1},P_{m-1})\tfrac{\partial u_{m-1}}{\partial x}\tfrac{\partial P_{m-1}}{\partial x} \\
&\quad + D_{22}G(u_{m-1},P_{m-1})\left|\tfrac{\partial P_{m-1}}{\partial x}\right|^2,
\end{aligned} \tag{a18}$$

we deduce that

$$\begin{aligned}
&\left\| \frac{\partial^2}{\partial x^2} G(u_{m-1}, P_{m-1}) \right\| \\
&\le K_M(G) \left[ \|\Delta u_{m-1}\| + \left\| \left| \frac{\partial u_{m-1}}{\partial x} \right|^2 \right\| + \left\| \frac{\partial P_{m-1}}{\partial x} \frac{\partial u_{m-1}}{\partial x} \right\| \right] \\
&\quad + K_M(G) \left[ \|\Delta P_{m-1}\| + \left\| \frac{\partial u_{m-1}}{\partial x} \frac{\partial P_{m-1}}{\partial x} \right\| + \left\| \left| \frac{\partial P_{m-1}}{\partial x} \right|^2 \right\| \right] \\
&\le 2M(1+2M) K_M(G).
\end{aligned} \tag{a19}$$

Thus (viii) holds. The Lemma 3.2 is proved completely. □

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