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A NONLOCAL MODEL FOR THE EXCHANGE ENERGY IN FERROMAGNETIC MATERIALS

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Dedicated to John Nohel on the occasion of his 65th birthday.

ABSTRACT. When the exchange energy term of micromagnetics is replaced with a nonlocal energy term, the balance laws for a ferromagnetic material change from a system of second-order partial differential equations to a system of integral equations. The new system admits "measure-valued" magnetizations which describe the oscillations of the material's domain structure. A general existence theory is established for minimizers of the new energy, and multiple solutions for specific problems are found.

1. Introduction. The theory of micromagnetics (the most widely accepted mathematical model of ferromagnetism) was developed by Brown (cf. [**3**]) as a generalization of a model of Landau and Lifschitz [**12**] for the energy of domain walls. The goal of the theory is to describe the magnetization of a ferromagnetic body $\mathcal{B}$ placed in an applied magnetic field. The mathematical problem consists in finding a magnetic field $\mathbf{m}$ that minimizes the energy functional

$$\mathcal{E}(\mathbf{m}) = \frac{1}{2}\int_{\mathbf{R}^3} |\hat{\mathbf{h}}(\mathbf{m})|^2 + \int_{\mathcal{B}} \{\mathcal{W}(\mathbf{m}) - \mathbf{m}\cdot\mathbf{h}_0 + \varepsilon|\nabla\mathbf{m}|^2\}$$

subject to the constraint

$$|\mathbf{m}(\mathbf{x})| = \mathcal{M}_0 \quad \text{a.e. in } \mathcal{B}.$$

Here, $\hat{\mathbf{h}}(\mathbf{m})$ is the resultant magnetic field induced by $\mathbf{m}$, $\mathcal{W}$ is the anisotropy energy density, and $\mathbf{h}_0$ is the applied magnetic field. To overstate the situation somewhat, the successes of the model so far have been in describing the small-scale features of a magnetized body:

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fields at the onset of nucleation, fields inside domain walls, etc. Micromagnetics is much less successful in describing macroscopic phenomena such as hysteresis subloops and the Barkhausen effect.

In this paper, I propose a new model designed to describe macroscopic effects in ferromagnetic materials. The crux of the model is the introduction of a new version of the exchange energy. The $|\nabla \mathbf{m}|^2$ term in the old model is replaced by a nonlocal energy density designed to cause neighboring points to have parallel magnetization. The new total magnetization energy becomes

$$\tilde{\mathcal{E}}(\mathbf{m}) = \frac{1}{2}\int_{\mathbf{R}^3} |\hat{\mathbf{h}}(\mathbf{m})|^2 + \int_{\mathcal{B}} \{\mathcal{W}(\mathbf{m}) - \mathbf{m}\cdot\mathbf{h}_0\} \\ - \int_{\mathcal{B}}\int_{\mathcal{B}} \mathbf{m}(\mathbf{x})\cdot\mathbf{m}(\mathbf{y}) k(\mathbf{x}-\mathbf{y})\, d\mathbf{x}\, d\mathbf{y}.$$

Here, k is a symmetric kernel concentrated at the origin and decaying at infinity.

Nonlocal constitutive equations have been proposed before, both for electromagnetism and elasticity (cf. e.g., Eringen [**9**]), but their main application has been to problems with some sort of convexity assumption. However, to my mind, the major mathematical advantage of a nonlocal version of the exchange energy is that it allows us to expand the class of admissible magnetizations to include discontinuous and even "measure-valued" magnetizations, and this is of great use in nonconvex problems such as those of ferromagnetism. Such magnetizations can be used to model the highly oscillatory "domain structures" observed in ferromagnetic materials. The use of measure-valued magnetizations is inspired by recent work on fine phase mixtures in nonconvex problems in crystals (cf. Ball and James [**1**], Chipot and Kinderlehrer [**6**], and Fonseca [**10**]).

Much of this paper is dedicated to describing measure-valued magnetizations and showing how they arise in physical situations. Once measure-valued magnetizations are defined, we go on to describe a general existence theory for minimization problems for the energy defined above. A key to the existence theory is a theorem involving the construction of a weakly convergent sequence of magnetizations satisfying differential constraints and having Young's measures with prescribed first and second moments. In addition to the general existence theory for minimizers, some specific problems that show how multiple solutions arise are studied.

The rest of this paper is organized as follows. Section 2 is a review of some basic concepts of magnetostatics, with particular attention to the reaction of the magnetic field to weakly convergent sequences of classical magnetization fields. Section 3 reviews the formulation of the theory of micromagnetics and some of the basic existence results of the theory. In Section 4, the new nonlocal model of the exchange energy is described. In Section 5, the "Young's measure" of a weakly convergent sequence of magnetizations is defined and the problem of a ball of ferromagnetic material in a uniform applied field is worked out in the case where the exchange energy is zero. In Section 6, we expand the notion of magnetizations to include "measure-valued magnetizations." We show that, under certain hypotheses, this leads to an existence theory for measure-valued minimizers of energies with nonlocal exchange energy. In Section 7, we work out several model problems that demonstrate multiple measure-valued relative minimizers of the magnetic energy. Section 8 contains the proof of a technical theorem from Section 6. Finally, Section 9 contains some conclusions and comments.

2. Magnetostatics. Let $\mathcal{B} \subset \mathbf{R}^3$ be the position of a stationary, rigid body, which we assume is compact with suitably regular boundary. Let $\mathbf{m} \in L^2(\mathcal{B})$ be the *magnetization* of the body. The *resultant magnetic field* is defined to be the unique function $\mathbf{h}$, in the set

$$\mathcal{A} = \{\mathbf{h} \in L^2(\mathbf{R}^3) | \operatorname{curl} \mathbf{h} = \mathbf{0} \text{ in } H^{-1}(\mathbf{R}^3)\}, \tag{2.1}$$

satisfying

$$\int_{\mathbf{R}^3} \mathbf{h}_r \cdot \mathbf{h}^\sharp = -\int_{\mathcal{B}} \mathbf{m} \cdot \mathbf{h}^\sharp \quad \forall\, \mathbf{h}^\sharp \in \mathcal{A}. \tag{2.2}$$

We denote this solution by $\mathbf{h}_r = \hat{\mathbf{h}}(\mathbf{m})$. The existence and uniqueness of solutions of (2.2) and their continuous dependence on $\mathbf{m}$ is guaranteed by the Lax-Milgramm lemma.

If $\mathbf{m}$ is piecewise differentiable, $\mathbf{h}_r$ satisfies the differential equations

$$\operatorname{curl} \mathbf{h}_r = \mathbf{0}, \tag{2.3}$$

$$\operatorname{div} \mathbf{h}_r = \begin{cases} -\operatorname{div} \mathbf{m}, & \text{in } \mathcal{B}, \\ \mathbf{0}, & \text{in } \mathcal{B}^c, \end{cases} \tag{2.4}$$

and the jump conditions

$$[|\mathbf{h}_r|] \times \mathbf{n} = 0, \tag{2.5}$$

$$[|\mathbf{h}_r|] \cdot \mathbf{n} = -[|\mathbf{m}|] \cdot \mathbf{n}, \tag{2.6}$$

on any surface of discontinuity of $\mathbf{m}$. Here, $\mathcal{B}^c$ indicates the complement of the body in $\mathbf{R}^3$, $\mathbf{n}$ denotes a unit normal to the surface of discontinuity and $[|\cdot|]$ indicates the jump of a piecewise continuous function in the direction $\mathbf{n}$. In this case, the solution is given by the following familiar version of Coulomb's law:

$$\hat{\mathbf{h}}(\mathbf{m})(\mathbf{y}) = \frac{1}{4\pi}\left[\int_{\mathcal{B}} \frac{-\operatorname{div}\mathbf{m}(\mathbf{x})(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3}\,dv_x + \sum_i \int_{S_i} \frac{[|\mathbf{m}|](\mathbf{x})\cdot\mathbf{n}(\mathbf{x})(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3}\,da_x\right]. \tag{2.7}$$

Here, S_i are surfaces of discontinuity of $\mathbf{m}$. Note that $\hat{\mathbf{h}}$ is a nonlocal operator: At every point $\mathbf{y}$, the quantity $\hat{\mathbf{h}}(\mathbf{m})(\mathbf{y})$ depends on the *global* values of $\mathbf{m}$.

We now examine some basic properties of the solution operator. The first is that $\hat{\mathbf{h}}$ is *weakly continuous*, i.e.,

THEOREM 2.1. *For any sequence of magnetizations* $\{\mathbf{m}^j\}$ *such that*

$$\mathbf{m}^j \rightharpoonup \overline{\mathbf{m}}, \quad \text{in } L^2(\mathcal{B}),$$

it follows that

$$\hat{\mathbf{h}}(\mathbf{m}^j) \rightharpoonup \hat{\mathbf{h}}(\overline{\mathbf{m}}), \quad \text{in } L^2(\mathbf{R}^3).$$

Here, the half arrow $\rightharpoonup$ *indicates weak convergence, i.e.,*

$$f^j \rightharpoonup f \quad \text{in } L^2(\Omega) \Longleftrightarrow \int_\Omega f^j\phi \to \int_\Omega f\phi \quad \forall\, \phi \in L^2(\Omega).$$

PROOF. For any $\phi \in L^2(\mathbf{R}^3)$ we use the standard Hodge orthogonal decomposition and write

$$\phi = \psi + \gamma,$$

where $\psi \in \mathcal{A}$ and $\gamma \in \mathcal{A}^{\perp}$. Thus, since $\hat{\mathbf{h}}(\mathbf{m}^j) \in \mathcal{A}$, for each j, we have

$$\int_{\mathbf{R}^3} \hat{\mathbf{h}}(\mathbf{m}^j) \cdot \phi = \int_{\mathbf{R}^3} \hat{\mathbf{h}}(\mathbf{m}^j) \cdot (\psi + \gamma) \tag{2.8}$$

$$= \int_{\mathbf{R}^3} \hat{\mathbf{h}}(\mathbf{m}^j) \cdot \psi \tag{2.9}$$

$$= -\int_{\mathcal{B}} \mathbf{m}^j \cdot \psi \tag{2.10}$$

$$\to -\int_{\mathcal{B}} \overline{\mathbf{m}} \cdot \psi \tag{2.11}$$

$$= \int_{\mathbf{R}^3} \hat{\mathbf{h}}(\overline{\mathbf{m}}) \cdot \psi \tag{2.12}$$

$$= \int_{\mathbf{R}^3} \hat{\mathbf{h}}(\overline{\mathbf{m}}) \cdot (\psi + \gamma). \tag{2.13}$$

Here, we have used (2.2) and the fact that $\hat{\mathbf{h}} \in \mathcal{A}$ and is, therefore, orthogonal to γ. □

It follows that the L^2 norm of the resultant field (which we refer to as the field energy below) is weak lower semicontinuous as a function of $\mathbf{m}$.

COROLLARY 2.2. *Suppose that*

$$\mathbf{m}^j \rightharpoonup \overline{\mathbf{m}}, \quad \text{in } L^2(\mathcal{B}),$$

then

$$\|\hat{\mathbf{h}}(\overline{\mathbf{m}})\|_{L^2(\mathbf{R}^3)} \leq \liminf_{j\to\infty} \|\hat{\mathbf{h}}(\mathbf{m}_2^j)\|_{L^2(\mathbf{R}^3)}. \tag{2.14}$$

PROOF. The result follows from the previous theorem and the weak lower semicontinuity of the L^2 norm (a direct result of Tonelli's theorem). □

The following theorem on strong convergence and the continuity of the field energy employs the ideas of compensated compactness.

THEOREM 2.3. *For any sequence of magnetizations $\{m^j\}$ such that*

$$m^j \rightharpoonup \overline{\mathbf{m}}, \quad \textit{in } L^2(\mathcal{B}),$$

and

$$\operatorname{div} \mathbf{m}^j \ \textit{are compact in } H^{-1}_{\mathrm{loc}}(\mathbf{R}^3),$$

(*where we have assumed m^j to be extended by zero into all of $\mathbf{R}^3$*) *it follows that* (*at least for a subsequence*)

$$\hat{\mathbf{h}}(\mathbf{m}^j) \to \hat{\mathbf{h}}(\overline{\mathbf{m}}), \quad (\textit{strongly}) \textit{ in } L^2(\mathbf{R}^3),$$

and, hence,

$$\|\hat{\mathbf{h}}(\overline{\mathbf{m}})\|_{L^2(\mathbf{R}^3)} = \lim_{j\to\infty} \|\hat{\mathbf{h}}(\mathbf{m}_2^j)\|_{L^2(\mathbf{R}^3)}. \tag{2.15}$$

PROOF. Since, by definition, $\hat{\mathbf{h}} \in \mathcal{A}$, we have $\operatorname{curl} \hat{\mathbf{h}}(\mathbf{m}^j) = 0$ and, hence, contained in a compact set in $H^{-1}_{\mathrm{loc}}(\mathbf{R}^3)$. Thus, using the weak continuity of $\hat{\mathbf{h}}$ and the Div-Curl lemma (cf. [**18**]), we have (at least for a subsequence)

$$\int_{\mathbf{R}^3} \hat{\mathbf{h}}(\mathbf{m}^j) \cdot \mathbf{m}^j \varphi \to \int_{\mathbf{R}^3} \hat{\mathbf{h}}(\overline{\mathbf{m}}) \cdot \overline{\mathbf{m}} \varphi \tag{2.16}$$

for every $\varphi \in C_0^\infty(\mathbf{R}^3)$. Since $\mathcal{B}$, the support of the extension of m^j, is compact, there exists a test function φ that is equal to one on $\mathcal{B}$. Thus, (2.16) holds with $\varphi = 1$. Using (2.2), we have

$$\|\hat{\mathbf{h}}(\mathbf{m}^j)\|^2_{L^2(\mathbf{R}^3)} = \int_{\mathbf{R}^3} \hat{\mathbf{h}}(\mathbf{m}^j) \cdot \hat{\mathbf{h}}(\mathbf{m}^j) \tag{2.17}$$

$$= \int_{\mathbf{R}^3} \hat{\mathbf{h}}(\mathbf{m}^j) \cdot \mathbf{m}^j \tag{2.18}$$

$$\to \int_{\mathbf{R}^3} \hat{\mathbf{h}}(\overline{\mathbf{m}}) \cdot \overline{\mathbf{m}} \tag{2.19}$$

$$= \|\hat{\mathbf{h}}(\overline{\mathbf{m}})\|^2_{L^2(\mathbf{R}^3)}. \tag{2.20}$$

Since weak convergence and convergence of norm implies strong convergence, the proof is complete. $\square$

We now consider an example that will be useful in the problems considered below. Let $B_1(O)$ be the unit ball centered at the origin. Let $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ be a fixed orthonormal basis for $\mathbf{R}^3$, and let $\underline{\mathbf{x}} = (r, \theta, \phi)$ be spherical coordinates for $\mathbf{R}^3$ defined by

$$\mathbf{x} = \hat{\mathbf{x}}(\underline{\mathbf{x}}) = r\mathbf{k}_r(\theta, \phi), \tag{2.21}$$

where

$$\mathbf{k}_r(\theta, \phi) = (\sin\phi(\cos\theta\mathbf{i}_1 + \sin\theta\mathbf{i}_2) + \cos\phi\mathbf{i}_3), \tag{2.22}$$
$$\mathbf{k}_\theta(\theta, \phi) = (\sin\phi(-\sin\theta\mathbf{i}_1 + \cos\theta\mathbf{i}_2) + \cos\phi\mathbf{i}_3), \tag{2.23}$$
$$\mathbf{k}_\phi(\theta, \phi) = (\cos\phi(\cos\theta\mathbf{i}_1 + \sin\theta\mathbf{i}_2) - \sin\phi\mathbf{i}_3). \tag{2.24}$$

If the unit ball $B_1(O)$ has a uniform magnetization $\mathbf{m} = \mathcal{M}\mathbf{i}_3$, then the resultant magnetic field $\hat{\mathbf{h}}(\mathbf{m})$ is given by

$$\hat{\mathbf{h}}(\mathbf{m}; \hat{\mathbf{x}}(\underline{\mathbf{x}})) = \begin{cases} -\frac{\mathcal{M}}{3}(\cos\phi\mathbf{k}_r - \sin\phi\mathbf{k}_\phi) = -\frac{\mathcal{M}}{3}\mathbf{i}_3, & r < 1, \\ \frac{\mathcal{M}}{3}\left(\frac{2\cos\phi}{r^3}\mathbf{k}_r + \frac{\sin\phi}{r^3}\mathbf{k}_\phi\right), & r > 1. \end{cases} \tag{2.25}$$

It is easy to check that this satisfies (2.3)–(2.6). Note that the uniform magnetization induces a uniform resultant field in the interior of the body. For future use, we compute the field energy of a unit ball with uniform magnetization.

$$\begin{aligned} \int_{\mathbf{R}^3} |\hat{\mathbf{h}}(\mathcal{M}\mathbf{i}_3)|^2 &= -\int_{B_1(O)} \mathcal{M}\mathbf{i}^3 \cdot \hat{\mathbf{h}}(\mathcal{M}\mathbf{i}_3) \\ &= \frac{4\pi\mathcal{M}^2}{9}. \end{aligned} \tag{2.26}$$

3. Micromagnetics. In order to describe the theory of micromagnetics with a minimum of detail, we consider the case of a static, rigid, homogeneous ferromagnetic body. The basic goal of the model is to describe the magnetization induced in the body by an applied magnetic field $\mathbf{h}_0$. The mathematical formulation is as follows. In accordance with the Heisenberg-Weiss theory of magnetization, one assumes that the magnetization field $\mathbf{m}$ has constant magnitude $\mathcal{M}_0$ within the body,

$$|\mathbf{m}(\mathbf{x})| = \mathcal{M}_0. \tag{3.1}$$

Under this constraint, one seeks to minimize the energy functional

$$\overline{\mathcal{E}}(\mathbf{m}) = \frac{1}{2}\int_{\mathbf{R}^3} |\hat{\mathbf{h}}(\mathbf{m})|^2 + \int_{\mathcal{B}} \{\mathcal{W}(\mathbf{m}) - \mathbf{m}\cdot\mathbf{h}_0 + \varepsilon|\nabla\mathbf{m}|^2\}. \tag{3.2}$$

The various terms of the energy are described as follows.

- *Field energy.* The first term,

$$\mathcal{E}_F(\mathbf{m}) = \frac{1}{2}\int_{\mathbf{R}^3} |\hat{\mathbf{h}}(\mathbf{m})|^2, \tag{3.3}$$

represents the energy due to the resultant magnetic field. To reduce this term, we must minimize the sources of the field ($\operatorname{div}\mathbf{m}$ and discontinuities in $\mathbf{m}$ with jumps in the normal component if $\mathbf{m}$ is piecewise smooth (cf. (2.7))).

- *Anisotropy energy.* The second term,

$$\mathcal{E}_A(\mathbf{m}) = \int_{\mathcal{B}} \mathcal{W}(\mathbf{m}), \tag{3.4}$$

is intended to make the magnetization point in certain preferred directions. In a uniaxial crystal, the usual form considered is

$$\mathcal{W}(\mathbf{m}) = \alpha_1 m_1^2 + \alpha_2 m_2^2 + \alpha_3 m_3^2 \tag{3.5}$$

Here, $m_j = \mathbf{m}\cdot\mathbf{i}_j$. We assume that

$$\alpha_3 < \alpha_2 \leq \alpha_1, \tag{3.6}$$

and we say that $\mathbf{i}_3$ is the *easy direction of magnetization* since $\mathcal{W}$ is minimized over $|\mathbf{m}| = \mathcal{M}_0$ when $m_1 = m_2 = 0$. The constitutive equation (3.5) can, of course, be generalized, and there is a large body of literature suggesting various constitutive laws for various materials. But we concentrate on (3.5) throughout the paper.

- *Interaction energy.* This term

$$\mathcal{E}_I(\mathbf{m}) = -\int_{\mathcal{B}} \mathbf{m}\cdot\mathbf{h}_0 \tag{3.7}$$

tends to align the magnetization $\mathbf{m}$ with the applied field $\mathbf{h}_0$.

- *Exchange energy.*

$$\mathcal{E}_X(\mathbf{m}) = \int_{\mathcal{B}} \varepsilon |\nabla \mathbf{m}|^2. \tag{3.8}$$

While the first three terms (with appropriate modifications on the assumptions about the anisotropy energy) are to be found in the theories of other types of magnetism (e.g., paramagnetism), this term is designed, along with the constraint (3.1) on the magnitude of the magnetization, to produce the characteristic effects of ferromagnetic materials. In particular, its purpose is to keep the orientation of the magnetization locally constant and, thus, cause the piecewise constant minimizers that model magnetic domains.

Since this is the term I propose to replace, we note that it is usually derived (cf. [**3**, p. 35]) from a discrete model that penalizes the interaction of lattic spins through an energy term of the form

$$-C \sum \mathbf{S}_i \cdot \mathbf{S}_j. \tag{3.9}$$

Here, C is a constant, $\mathbf{S}_i$ is the spin angular momentum of the i^{th} lattice particle, and the sum is taken over nearest neighbors in the lattice.

There are a number of mathematical similarities between micromagnetics and the theories of liquid crystals (cf. [**11**]) and phase transitions in fluids (cf. [**4, 5**]). In particular, all use a gradient penalty like the exchange energy above and all employ some notion of nonconvexity (a nonconvex constraint like (3.1) in the case of liquid crystals and a nonconvex stored energy function—essentially a modification of the anisotropy energy into a function that is minimized at some nonzero modulus—in the case of phase transitions). However, there are important differences as well. Most important of these is the role of the field energy in micromagnetics. This is the term that encourages the rapid oscillations with which we seek to model the distinctive physical properties of ferromagnetic materials: domain structure, multiple equilibria, and hysteresis.

One can obtain the following abstract existence result for micromagnetism.

THEOREM 3.1. *There exists at least one $m \in H^1(\mathcal{B})$ such that*

$$|\mathbf{m}(\mathbf{x})| = \mathcal{M}_0 \quad \text{a.e. in } \mathcal{B} \tag{3.10}$$

and

$$\overline{\mathcal{E}}(\mathbf{m}) \leq \overline{\mathcal{E}}(\tilde{\mathbf{m}}) \tag{3.11}$$

for every $\tilde{\mathbf{m}} \in H^1(\mathcal{B})$ *satisfying* (3.10).

PROOF. The proof of this can be found in Visintin [**22**]. Essentially, the proof depends on the convexity of the exchange energy to take care of gradient terms and the compact imbedding from $H^1(\mathcal{B})$ to $L^2(\mathcal{B})$ to give us the strong convergence that allows us to handle the nonconvex constraint on $\mathbf{m}$. □

Because of the nonconvex constraint (3.10), there may be multiple solutions of the minimization problem, as well as additional relative minima. Since the abstract existence theory provides no information in this regard, we examine the Euler-Lagrange equations, which reduce to

$$\mathbf{m} \times [\varepsilon\Delta\mathbf{m} - (h_0 + \hat{\mathbf{h}}(\mathbf{m})) + \mathbf{A}\mathbf{m}] = 0 \tag{3.12}$$

with natural boundary conditions

$$\mathbf{m} \times [\nabla\mathbf{m} \cdot \mathbf{n}] = 0. \tag{3.13}$$

Here $\mathbf{A} = \sum_{j=1}^{3} \alpha_j \mathbf{i}_i \mathbf{i}_j$, where $\mathbf{i}_j \mathbf{i}_j$ are dyadic (tensor) products, i.e., $\mathbf{A}$ is a diagonal matrix with diagonal elements α_j. The only step in the derivation of these equations that is not completely elementary is the variation of the field energy.

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^3} |\hat{\mathbf{h}}(\mathbf{m} + t\mathbf{m}^\sharp)|^2 \bigg|_{t=0} &= \int_{R^3} \hat{\mathbf{h}}(\mathbf{m}) \cdot \hat{\mathbf{h}}(\mathbf{m}^\sharp) \\ &= -\int_{B_1(O)} \hat{\mathbf{h}}(\mathbf{m}) \cdot \mathbf{m}^\sharp. \end{aligned} \tag{3.14}$$

Here, we have used the linearity of $\hat{\mathbf{h}}$ in $\mathbf{m}$ and (2.2) applied to $\hat{\mathbf{h}}(\mathbf{m}^\sharp)$.

The Euler-Lagrange equations (3.12) are nonlinear and, accordingly, difficult to solve. Fortunately, for certain special geometries, the problem is tractable. In particular, since a uniformly magnetized ellipsoid induces a uniform resultant magnetic field (the case of the unit ball is worked out above (cf. (2.25)), the problem of an ellipsoid in a uniform applied field $\mathbf{h}_0$ can be shown to have uniformly magnetized solutions. In the case of the unit ball with applied field $\mathbf{h}_0 = H_0\mathbf{i}_3$ parallel to the easy direction of magnetization, there are two uniform solutions $\mathbf{m} = \pm M_0\mathbf{i}_3$ corresponding to the saturation of the specimen in the easy direction. There have been extensive studies of the stability of these branches of solutions and of the linearized equations of loss of stability ("nucleation equations," cf. [**3**]). However, there is very little information about other minimizers or relative minimizers of the energy that the experimental results of subloops and the Barkhausen effect suggest exist. Miranker and Willner [**14**] do show the existence of multiple solutions of the Euler-Lagrange equations in the case of an infinite slab of material, but they show the stability of only the two uniform solutions.

4. A nonlocal model of the exchange energy. I now propose a new version of the exchange energy

$$\mathcal{E}_{NX}(\mathbf{m}) = -\int_{\mathcal{B}}\left[\int_{\mathcal{B}} \mathbf{m}(\mathbf{x})\cdot\mathbf{m}(\mathbf{y})k(\mathbf{x}-\mathbf{y})\,d\mathbf{y}\right]d\mathbf{x}. \tag{4.1}$$

Here, k is an appropriate kernel. For definiteness, we examine below the consequences of using

$$k_\gamma(\mathbf{x}-\mathbf{y}) = C\frac{e^{-\gamma|\mathbf{x}-\mathbf{y}|}}{\kappa_\gamma|\mathbf{x}-\mathbf{y}|}, \tag{4.2}$$

where C and γ are material constants and

$$\kappa_\gamma = \int_{\mathcal{B}}\int_{\mathcal{B}} \frac{e^{-\gamma|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{y}\,d\mathbf{x}, \tag{4.3}$$

but my motivation for (and attachment to) this choice is rather weak. Mathematically, the main concern is that the operator

$$L_2(\mathcal{B}) \ni \mathbf{m} \mapsto \mathcal{K}(\mathbf{m}) \in L_2(\mathcal{B}) \tag{4.4}$$

be compact, where

$$\mathcal{K}(\mathbf{m})(\mathbf{x}) := \int_{\mathcal{B}} \mathbf{m}(\mathbf{y})k(\mathbf{x}-\mathbf{y})\,d\mathbf{y}. \tag{4.5}$$

Physically, we want our integral to correspond to the discrete energy defined in (3.9) (or perhaps some more accurate model than a nearest neighbor sum). The singular kernel suggested above seems to accomplish these goals and none of the results below depends too heavily on its particular form.

With the use of (4.1), the new total energy becomes

$$\begin{aligned}\mathcal{E}(\mathbf{m}) = \int_{\mathbf{R}^3} |\hat{\mathbf{h}}(\mathbf{m})|^2 + \int_{\mathcal{B}} \{\mathcal{W}(\mathbf{m}) - \mathbf{h}_0 \cdot \mathbf{m}\}& \\ - \int_{\mathcal{B}}\int_{\mathcal{B}} \mathbf{m}(\mathbf{x})\cdot\mathbf{m}(\mathbf{y})k(\mathbf{x}-\mathbf{y})\,d\mathbf{y}\,d\mathbf{x}.&\end{aligned} \tag{4.6}$$

The Euler-Lagrange equations for this energy subject to the constraint (3.10) are

$$\mathbf{m} \times [\mathbf{A}\mathbf{m} - (\hat{\mathbf{h}}(\mathbf{m}) + \mathbf{h}_0) - 2\mathcal{K}(\mathbf{m})]. \tag{4.7}$$

These are obtained by standard variational methods (cf. Edelen [**7**] for an extensive exposition on variational methods for nonlocal problems). Unfortunately, with the addition of the nonlocal operator $\mathcal{K}$, we no longer have uniform solutions for uniform applied fields on ellipsoids except for the case where the body occupies all of space.

A more important apparent difficulty is that we no longer have an abstract existence theorem comparable to Theorem 3.1 above. Instead, if we take an infimizing sequence $\{\mathbf{m}^j\}$ satisfying (5.13), we can still use the weak-star compactness of closed bounded sets in $L^\infty(\mathcal{B})$ to obtain a subsequence such that

$$\mathbf{m}^j \overset{*}{\rightharpoonup} \overline{\mathbf{m}}, \tag{4.8}$$

for some $\overline{\mathbf{m}} \in L^\infty(\mathcal{B})$, but we have no way of ensuring that the limit $\overline{\mathbf{m}}$ minimizes $\mathcal{E}$ or even that it satisfies the nonconvex constraint (3.10). However, even if the weak limit of the infimizing sequence is not a solution of the problem, we can identify the sequence itself (actually the "Young's measure" associated with the sequence) as a solution of the minimization problem. This practice is inspired by the work of L.C. Young, and, in the next section, we describe some of the mathematical tools necessary to make this idea concrete.

5. Weak convergence and minimizing sequences. The ideas presented in this section are based on the "generalized curves" of L.C. Young [**23**]. Our intention is to give an intuitive idea of these mathematical tools, so many theoretical details have been omitted.

Perhaps the easiest way to envision a Young's measure is as the representative of a weakly converging sequence. Recall that, while strong convergence (convergence in norm) is closely associated with pointwise convergence, weak convergence is, instead, associated with averaging. For example, suppose $f \in L^\infty(\mathbf{R})$ is a periodic function of period T. Let $u^n(x) = f(nx)$. Then it is a standard exercise to show that

$$u^n \overset{*}{\rightharpoonup} \alpha, \quad \text{in } L^\infty(\mathbf{R}), \tag{5.1}$$

where the constant α is the average of f:

$$\alpha = \frac{1}{T}\int_0^T f(x)\,dx. \tag{5.2}$$

Here, $\overset{*}{\rightharpoonup}$ indicates a weak-star convergence, i.e., $v^n \overset{*}{\rightharpoonup} \overline{v}$ in $L^\infty(\Omega)$ if and only if

$$\int_\Omega v^n\phi \to \int_\Omega \overline{v}\phi \tag{5.3}$$

for every $\phi \in L^1(\Omega)$.

Note that weak convergence is not continuous under composition, i.e., if $v^n \overset{*}{\rightharpoonup} \overline{v}$, it does not follow that $F(v^n) \overset{*}{\rightharpoonup} F(\overline{v})$ for every continuous function F. In terms of the example above, this is reflected in the fact that

$$F\left(\frac{1}{T}\int_0^T f(x)\,dx\right) \neq \frac{1}{T}\int_0^T F(f(x))\,dx$$

unless F is affine or the choice of f is fortuitous. In order to determine effect of composition on a weakly convergent sequence, we need information not only about its average value (the weak limit) but also about the oscillations about the average. The Young's measure is designed to give us this information. The following theorem defines the Young's measure in the specific case of a sequence of magnetizations.

THEOREM 5.1. *Let $\mathcal{B}$ be a body and let $\mathbf{m}^n : \mathcal{B} \mapsto \mathbf{R}^3$ be a sequence of magnetizations with*

$$|\mathbf{m}^n(\mathbf{x})| = \mathcal{M}_0 \quad a.e.\ in\ \mathcal{B} \tag{5.4}$$

and

$$\mathbf{m}^n \overset{*}{\rightharpoonup} \mathbf{m} \quad in\ L^\infty(\mathcal{B}). \tag{5.5}$$

Then,

$$|\mathbf{m}(\mathbf{x})| \leq \mathcal{M}_0 \quad a.e.\ in\ \mathcal{B}. \tag{5.6}$$

Furthermore, if at each $\mathbf{x} \in \mathcal{B}$ we define a probability measure $\nu_{\mathbf{X}}^{n,j}$ on $\mathbf{R}^3$ by

$$\langle \nu_{\mathbf{X}}^{n,j}, F \rangle = \frac{\int_{B_{1/j}(\mathbf{X}) \cap \mathcal{B}} F(\mathbf{m}^n(\mathbf{x}))\, d\mathbf{x}}{|B_{1/j}(\mathbf{x}) \cap \mathcal{B}|}, \tag{5.7}$$

where $F \in C(\mathbf{R}^3)$, $B_{1/j}(\mathbf{x})$ indicates the ball of radius $1/j$ about $\mathbf{x}$, and $|S|$ indicates the three-dimensional Lebesgue measure of a set S, then there is a set of probability measures $\nu_{\mathbf{X}}$ on $\mathbf{R}^3$ parameterized by $\mathbf{x} \in \mathcal{B}$ and with support on the sphere $|\mathbf{m}| = \mathbf{M}_0$ such that

$$\langle \nu_{\mathbf{X}}, F \rangle = \lim_{j \to \infty} \lim_{n \to \infty} \langle \nu_{\mathbf{X}}^{n,j}, F \rangle. \tag{5.8}$$

We call $\nu_{\mathbf{X}}$ the Young's measure of the sequence $\mathbf{m}^n$. It has the property that

$$\mathbf{m}(\mathbf{x}) = \langle \nu_{\mathbf{X}}, \mathrm{Id} \rangle \equiv \int_{\mathbf{R}^3} \lambda \, d\nu_{\mathbf{X}}(\lambda), \tag{5.9}$$

i.e., the weak limit of the sequence is the center of mass of the measure. Furthermore, if, for a continuous function $G : \mathbf{R}^3 \mapsto \mathbf{R}^k$, *we have*

$$G(\mathbf{m}^n) \overset{*}{\rightharpoonup} \overline{G}, \tag{5.10}$$

then

$$\overline{G}(\mathbf{x}) = \langle \nu_{\mathbf{X}}, G \rangle = \int_{\mathbf{R}^3} G(\lambda)\, d\nu_{\mathbf{X}}(\lambda). \tag{5.11}$$

Conversely, let $\nu_{\mathbf{X}}$ *be a family of probability measures parameterized by* $\mathbf{x} \in \mathcal{B}$ *with support on the sphere* $|\mathbf{m}| = \mathcal{M}_0$. *Then there exists a sequence of magnetizations* $\mathbf{m}^n$ *of which* $\nu_{\mathbf{X}}$ *is the Young's measure.*

The proof of this theorem follows directly from material given in [**18, 19**] and will not be repeated here. Instead, we note that the measure $\nu_{\mathbf{X}}^{n,j}(\lambda)$ gives the probability that the function $\mathbf{m}^n$ (n fixed) takes on the value λ in a ball of radius $1/j$ about the point $\mathbf{x}$. The double limit process in (5.8) defines the Young's measure by first taking the limit of the sequence $\mathbf{m}^n$ and then shrinking the radius of the ball to zero. Thus, a loose description of the Young's measure $\nu_x(\lambda)$ is that it "gives the probability that the oscillations of the sequence $\mathbf{m}^n$ hit the value λ at points near $\mathbf{x}$."

To demonstrate the use of Young's measures in a physical problem, we consider the problem of a unit ball of ferromagnetic material in a uniform applied field parallel to the easy direction of magnetization with the exchange energy assumed to be identically zero. This situation somewhat resembles the problem of twining in elastic crystals studied in [**8, 6, 10,** and **1**]. However, the ferromagnetism problem offers some simplifications (the unknown is the vector-valued magnetization rather than the tensor-valued deformation gradient) and some addition difficulties (there is no nonlocal term comparable to the field energy in the elasticity problem). The results for elasticity problems are well known. Our results are given both for clarity and to examine the effects of the new feature of the nonlocal field energy.

Mathematically, the problem reduces to minimizing

$$\tilde{\mathcal{E}}(\mathbf{m}) = \frac{1}{2}\int_{\mathbf{R}^3} |\hat{\mathbf{h}}(\mathbf{m})|^2 + \int_{\Omega} \{\mathcal{W}(\mathbf{m}) - \mathcal{H}_0 \mathbf{m} \cdot \mathbf{i}_3\} \tag{5.12}$$

over $\mathbf{m} \in L^2(B_1(O))$, subject to the constraint

$$|\mathbf{m}(\mathbf{x})| = \mathcal{M}_0 \quad \text{a.e. in } B_1(O). \tag{5.13}$$

Here $\mathcal{W}$ is as defined in (3.5) and $\mathcal{H}_0$ gives the strength and orientation of the applied field. The Euler-Lagrange equations for the problem are

$$\mathbf{m} \times (\hat{\mathbf{h}}(\mathbf{m}) + \mathcal{H}_0\mathbf{i}_3 - \mathbf{A}\mathbf{m}) = 0. \tag{5.14}$$

Note that, regardless of the value of $\mathcal{H}_0$, (5.14) and (5.13) have two solutions,

$$\mathbf{m} = \pm\mathcal{M}_0\mathbf{i}_3. \tag{5.15}$$

We refer to $\mathbf{m} = \operatorname{sgn}(\mathcal{H}_0)\mathcal{M}_0\mathbf{i}_3$ as the *saturated solution* and $\mathbf{m} = -\operatorname{sgn}(\mathcal{H}_0)\mathcal{M}_0\mathbf{i}_3$ as the *reverse saturated solution*.

The following theorem concerns the existence of minimizing sequences and the Young's measures that represent their limits. Essentially, the result is that, if the applied field is strong enough, the saturated solution is a classical minimizer of the energy. If the applied field is weak, the minimizing sequence is oscillatory and the weak limit is not a classical minimizer. In this case, we will interpret the Young's measure of the minimizing sequence as a "solution" of the problem.

THEOREM 5.2. *for $|\mathcal{H}_0| \geq \mathcal{M}_0/3$, the saturated solution*

$$\mathbf{m} = \operatorname{sgn}(\mathcal{H}_0)\mathcal{M}_0\mathbf{i}_3 \tag{5.16}$$

minimizes $\tilde{\mathcal{E}}$. However, for $|\mathcal{H}_0| < \mathcal{M}_0/3$, neither of the solutions of the Euler-Lagrange equation (5.14) *and the constraint* (5.13) *are strong relative minimizers of $\tilde{\mathcal{E}}$, i.e., if $|\mathcal{H}_0| < \mathcal{M}_0/3$ and $\mathbf{m}$ satisfies* (5.15)*, then, for every $\varepsilon > 0$, there exists $\mathbf{m}^\varepsilon \in L^2(B_1(O))$ with $\|\mathbf{m}^\varepsilon - \mathbf{m}\|_{L^2(B_1(O))} < \varepsilon$ and $\tilde{\mathcal{E}}(\mathbf{m}^\varepsilon) < \tilde{\mathcal{E}}(\mathbf{m})$. Furthermore, for $|\mathcal{H}_0| < \mathcal{M}_0/3$, there exists an oscillating sequence $\mathbf{m}^j_{\mathcal{H}_0}$ that minimizes the energy $\tilde{\mathcal{E}}$. This sequence converges weakly in $L^2(B_1(O))$ to $3\mathcal{H}_0\mathbf{i}_3$ and has Young's measure*

$$\mu = \left(\frac{1}{2} + \frac{3\mathcal{H}_0}{2}\right)\delta_{\mathcal{M}_0\mathbf{i}_3} + \left(\frac{1}{2} - \frac{3\mathcal{H}_0}{2}\right)\delta_{\mathcal{M}_0\mathbf{i}_3}. \tag{5.17}$$

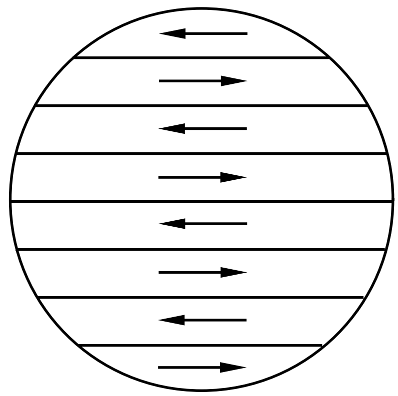

FIGURE 1. Oscillating magnetization $\mathbf{m}^{j,\theta}$.

PROOF. We begin by constructing an oscillating sequence of magnetizations that satisfies the constraint (5.13) and converges weakly to a uniform magnetization. Let $g(\theta, x) : [0,1] \times \mathbf{R} \mapsto \mathbf{R}$ be defined by

$$g(\theta, x) = \begin{cases} \mathcal{M}_0, & 0 \le x < 1-\theta, \\ -\mathcal{M}_0, & 1-\theta \le x < 1, \end{cases} \tag{5.18}$$

for $0 \le x < 1$, and extended by periodicity to the whole real line. For $\mathbf{x} \in B_1(O)$, we define

$$\mathbf{m}^{j,\theta}(\mathbf{x}) = g\left(\theta, \frac{|\mathbf{x}\cdot \mathbf{i}_1|}{j}\right)\mathbf{i}_3 \tag{5.19}$$

and note that

$$\mathbf{m}^{j,\theta} \overset{*}{\rightharpoonup} (1-2\theta)\mathcal{M}_0\mathbf{i}_3 \quad \text{in } L^\infty(B_1(O)) \quad \text{as } j \to \infty, \tag{5.20}$$

and, also,

$$m^{j,\theta} \overset{*}{\rightharpoonup} (1-2\theta)\mathcal{M}_0\mathbf{i}_3 \quad \text{in } L^2(B_1(O)) \quad \text{as } j \to \infty. \tag{5.21}$$

The Young's measure for the sequence is

$$\mu_\theta = (1-\theta)\delta_{\mathcal{M}_0\mathbf{i}_3} + \theta\delta_{-\mathcal{M}_0\mathbf{i}_3}. \tag{5.22}$$

We now wish to calculate
$$\lim_{j\to\infty} \tilde{\mathcal{E}}(\mathbf{m}^{j,\theta}).$$
(This limit can be interpreted as the energy of the Young's measure (5.22). We elaborate on this in the next section.) The field energy is the only interesting term. To calculate its limit, we first note that $\operatorname{div}\mathbf{m}^{j,\theta}$ is compact in $H^{-1}_{\text{loc}}(\mathbf{R}^3)$ (with $\mathbf{m}^{j,\theta}$ defined to be zero outside of $B_1(O)$). To see this, note that, since there are no jumps in the normal component of $\mathbf{m}^{j,\theta}$ at the planes of discontinuity inside the ball, we need only show that the normal component of the magnetization on the surface of the body lies in a compact set in $H^{-1}_{\text{loc}}(\mathbf{R}^3)$. But the normal component of the magnetization is bounded in L^∞ and, hence, in $L^2(\partial B_1(O))$, which imbeds compactly into $H^{-1/2}(\partial B_1(O))$. But, since the trace operator from $H^1(\mathbf{R}^3)$ to $H^{1/2}(\partial B_1(O))$ is bounded, a compact set in $H^{-1/2}(\partial B_1(O))$ can be identified with a compact set in $H^{-1}_{\text{loc}}(\mathbf{R}^3)$. Thus, we can use Theorem 2.3 to conclude
$$\hat{\mathbf{h}}(\mathbf{m}^{j,\delta}) \to \hat{\mathbf{h}}((1-2\theta)\mathcal{M}_0\mathbf{i}_3), \quad \text{(strongly) in } L^2(\mathbf{R}^3). \tag{5.23}$$
Using this, (3.5), and (5.21), we get
$$\lim_{j\to\infty} \tilde{\mathcal{E}}(\mathbf{m}^{j,\theta}) = \|\hat{\mathbf{h}}((1-2\theta)\mathcal{M}_0\mathbf{i}_3\|_{L^2(\mathbf{R}^3)} + \int_{B_1(O)} (\alpha_3\mathcal{M}_0^2 - \mathcal{H}_0\mathcal{M}_0) \tag{5.24}$$
$$= \frac{4\pi}{9}(1-2\theta)^2\mathcal{M}_0^2 - \frac{4\pi}{3}(1-2\theta) + \frac{4\pi}{3}\alpha_3\mathcal{M}_0^2. \tag{5.25}$$

Thus, the limiting energy is quadratic in θ and has an interior minimum on $\theta \in (0,1)$ whenever $|\mathcal{H}_0| < \mathcal{M}_0/3$. Since $\|\mathbf{m}^{j,\theta}\|_{L^2(B_1(O))}$ depends continuously on θ, this proves the claim that the saturated and reverse saturated solutions (which correspond to $\theta \in \{0,1\}$) are not relative minimizers of the energy when $|\mathcal{H}_0| < \mathcal{M}_0/3$.

We now show that, given $\mathcal{H}_0$, the sequence
$$\mathbf{m}^j_{\mathcal{H}_0} = \mathbf{m}^{j,\theta}, \tag{5.26}$$
with
$$\theta = \begin{cases} 0, & \mathcal{H}_0 \geq \frac{\mathcal{M}_0}{3}, \\ \frac{1}{2} + \frac{3\mathcal{H}_0}{2}, & |\mathcal{H}_0| < \frac{\mathcal{M}_0}{3}, \\ 1, & \mathcal{H}_0 \leq -\frac{\mathcal{M}_0}{3}, \end{cases} \tag{5.27}$$

is an infimizing sequence for the energy. Note that such a sequence minimizes the anisotropy energy absolutely at each of its elements. Thus, we need only show that

$$\overline{\mathbf{m}}_{\mathcal{H}_0} = \begin{cases} \mathcal{M}_0\mathbf{i}_3, & \mathcal{H}_0 \geq \frac{\mathcal{M}_0}{3}, \\ 3\mathcal{H}_0\mathbf{i}_3, & |\mathcal{H}_0| < \frac{\mathcal{M}_0}{3}, \\ -\mathcal{M}_0\mathbf{i}_3, & \mathcal{H}_0 \leq -\frac{\mathcal{M}_0}{3}, \end{cases} \tag{5.28}$$

(the weak limit of our candidate $\mathbf{m}^j_{\mathcal{H}_0}$) minimizes

$$\|\hat{\mathbf{h}}(\overline{\mathbf{m}})\|_{L^2(\mathbf{R}^3)} + \int_{B_1(O)} \mathcal{H}_0\mathbf{i}_3 \cdot \overline{\mathbf{m}}(\mathbf{x})\, dv_x \tag{5.29}$$

over all possible weak limits of admissible sequences. According to Theorem 5.1, we can do this by minimizing over functions $\overline{\mathbf{m}} \in L^\infty(B_1(O))$ with

$$|\overline{\mathbf{m}}(\mathbf{x})| \leq \mathcal{M}_0 \quad \text{a.e. in } B_1(O). \tag{5.30}$$

To show that our candidates minimize (5.29), we let $\{\mathbf{u}_i\}_{i=1}^\infty$ be an orthonormal basis for $L^2(\mathbf{R}^3)$ with

$$\mathbf{u}_1 = \hat{\mathbf{h}}\left(\frac{3}{2\sqrt{\pi}}\mathbf{i}_3\right).$$

Then, Parseval's equality and

$$\langle \hat{\mathbf{h}}(\mathbf{m}), \mathbf{u}_1 \rangle = \frac{1}{2\sqrt{\pi}} \int_{B_1(O)} \mathbf{m} \cdot \mathbf{i}_3\, dv \tag{5.31}$$

(which is a consequence of (2.2)) give us

$$\begin{aligned} &\frac{1}{2}\|\hat{\mathbf{h}}(\mathbf{m})\|_{L^2(\mathbf{R}^3)} - \int_{B_1(O)} \mathcal{H}_0\mathbf{i}_3 \cdot \mathbf{m} \\ &\quad = \frac{1}{2}\sum_{i=1}^\infty \langle \hat{\mathbf{h}}(\mathbf{m}), \mathbf{u}_i \rangle^2 - \int_{B_1(O)} \mathcal{H}_0\mathbf{i}_3 \cdot \mathbf{m} \\ &\quad = \frac{1}{2}\langle \hat{\mathbf{h}}(\mathbf{m}), \mathbf{u}_1 \rangle^2 - 2\sqrt{\pi}\mathcal{H}_0 \langle \hat{\mathbf{h}}(\mathbf{m}), \mathbf{u}_1 \rangle + \frac{1}{2}\sum_{i=2}^\infty \langle \hat{\mathbf{h}}(\mathbf{m}), \mathbf{u}_i \rangle^2. \end{aligned} \tag{5.32}$$

Thus, we can think of the energy as a functional on sequences $\{\langle\hat{\mathbf{h}}(\mathbf{m}), \mathbf{u}_i\rangle\} \in l^2$. The constraint (5.30) on $\mathbf{m}$ translates into a sequence of constraints on $\{\langle\hat{\mathbf{h}}(\mathbf{m}), \mathbf{u}_i\rangle\}$, one of which is

$$|\langle\hat{\mathbf{h}}(\mathbf{m}), \mathbf{u}_1\rangle| = \left|\frac{1}{2\sqrt{\pi}} \int_{B_1(O)} \mathbf{m} \cdot \mathbf{i}_3\right| \leq \frac{2\sqrt{\pi}\mathcal{M}_0}{3}. \tag{5.33}$$

The functional defined in (5.32) has an interior minimum when

$$\langle\hat{\mathbf{h}}(\mathbf{m}), \mathbf{u}_1\rangle = 2\sqrt{\pi}\mathcal{H}_0, \tag{5.34}$$

$$\langle\hat{\mathbf{h}}(\mathbf{m}), \mathbf{u}_i\rangle = 0, \quad i = 2, 3, \dots . \tag{5.35}$$

Equation (5.34) is satisfied when $\mathbf{m} = 3\mathcal{H}_0\mathbf{i}_3$, and, since

$$\begin{aligned} \langle\hat{\mathbf{h}}(3\mathcal{H}_0\mathbf{i}_3), \mathbf{u}_i\rangle &= -3\mathcal{H}_0 \int_{B_1(O)} \mathbf{u}_i \cdot \mathbf{i}_3 \, dv \\ &= 6\sqrt{\pi}\mathcal{H}_0 \int_{B_1(O)} -\frac{1}{2\sqrt{\pi}} \mathbf{u}_i \cdot \mathbf{i}_3 \, dv \\ &= 6\sqrt{\pi}\mathcal{H}_0 \langle\mathbf{u}_1, \mathbf{u}_i\rangle = 0, \quad i \neq 1, \end{aligned} \tag{5.36}$$

(5.35) is satisfied as well. When $|\mathcal{H}_0| \leq \mathcal{M}_0/3$, this corresponds to the weak limit of our sequence so that our sequence minimizes the energy. Finally, according to (5.33), the saturated solution represents a boundary minimizer when $|\mathcal{H}_0| > \mathcal{M}_0/3$, and the proof is complete. $\square$

In Figure 2 we have graphed the center of mass of the Young's measure for the minimizing sequences, i.e., the weak limit of the oscillating sequences and the classical values of the saturated solutions. The minimizing sequences obtained here are the result of two of the effects that govern ferromagnetic materials. The nonconvex constraint and the anisotropy energy combine to favor two states of magnetizations that lie in opposite direction, while the field energy favors a zero *average value* of the magnetization. Since there is no penalty on oscillations, both tendencies can be satisfied by a rapidly oscillating sequence of magnetizations. Of course, the results described here do not exhibit the multiple equilibrium solutions that we expect from a model of

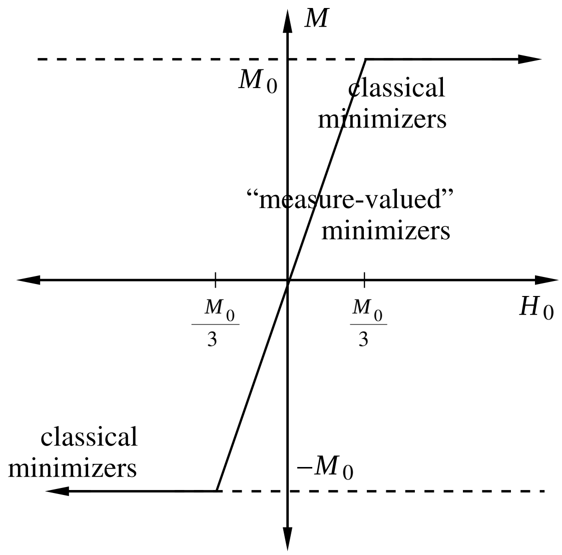

FIGURE 2. Solution curves.

ferromagnetic materials. We see below that the addition of a nonlocal exchange energy has the effect of penalizing oscillating sequences in a way that stabilizes the classical saturated solutions. But, before examining the effects of exchange energy, we examine the consequences of accepting the Young's measure of a minimizing sequence as a physical magnetization.

6. Measure-valued magnetizations. What do we mean by a physical solution of the minimization problems we are considering? Consider the problem of the sphere in a uniform applied field examined in the last section. In the saturated case $|\mathcal{H}_0| \geq \mathcal{M}_0/3$ we have no problem, the minimizing "sequence" converges strongly and its limit is a classical solution of both the constraint (5.13) and the Euler-Lagrange equations (5.14). But, below the level of saturation our minimizing sequences oscillate wildly and converge only weakly. Furthermore, the weak limit is not a classical solution of the problem, since it does not satisfy the constraint (5.13). As we have indicated before, in this case, we follow the lead of the recent studies of phase transitions mentioned above, and *we identify the Young's measure of the minimizing sequence as a solution of the problem.* In accepting a "measure-valued solution," we are saying (very loosely) that the solution of our problem is some sort of highly oscillatory domain configuration whose scale is too small to be recognized by our model. Thus, we choose not to carry along detailed pointwise information about the structure of the oscillations, but, instead, accept a type of solution that gives information only about the states observed in the oscillations (the support of the measure) and the amount of time spent in each state (the various weights of the measure). To be more specific, we make the following definition.

DEFINITION 6.1. A *measure-valued magnetization* on a body $\mathcal{B}$ is a parameterized probability measure $\nu_{\mathbf{X}}$ on $\mathbf{R}^3$ with support on the sphere $|\lambda| = \mathcal{M}_0$.

Recall that Theorem 5.1 says that such a measure represents a sequence $\mathbf{m}^n \in L^\infty(\mathcal{B})$ of classical magnetizations. Also, note that this definition subsumes the traditional definition of an admissible magnetization if we identify a classical magnetization $\mathbf{m}$ with a measure-valued magnetization represented by a single delta function,

$$\nu_{\mathbf{X}} = \delta_{\mathbf{m(x)}}. \tag{6.1}$$

Of course, the Young's measures of the minimizing sequences of the problem above are examples of measure-valued magnetizations.

At this point, we need to consider how we wish to use this new type of magnetization in more general minimization problems. One might proceed in the same fashion as above and construct minimizing sequences of classical magnetizations and identify the Young's measure of the sequence with the solution of the problem. (In fact, this is the general practice in phase transitions.) However, if measure-valued magnetizations are to be more than a theoretical device, we need to put them on the same footing as classical magnetizations: we need to define the magnetic field they produce and to evaluate their energy directly (without reference to the sequence of classical magnetizations from which they arose). The first goal is easily met. Since a measure-valued magnetization arises from a weakly convergent sequence of classical magnetizations, and since, by Theorem 2.1, the resultant magnetic field is weakly continuous, the obvious choice for the magnetic field of a measure-valued magnetization is the field generated by its center of mass, the weak limit of the classical sequence.

DEFINITION 6.2. Let $\mu_{\mathbf{X}}$ be a measure-valued magnetization and let

$$\overline{\mathbf{m}}(\mathbf{x}) = \int_{\mathbf{R}^3} \lambda \, d\mu_{\mathbf{X}}(\lambda) \tag{6.2}$$

be the center of mass of the measure. Then we define the *magnetic field generated by the measure-valued magnetization* to be

$$\hat{\mathbf{h}}(\mu) \equiv \hat{\mathbf{h}}(\overline{\mathbf{m}}). \tag{6.3}$$

The process of defining the energy of a measure-valued magnetization starts out easily enough. The anisotropy and interaction energies can be defined in a way that arises naturally from the definition of the Young's measure in terms of weakly convergent sequences:

$$\mathcal{E}_A(\mu) = \int_{\mathcal{B}} \int_{\mathbf{R}^3} \mathcal{W}(\lambda) \, d\mu_{\mathbf{X}}(\lambda) \, dx, \tag{6.4}$$

$$\mathcal{E}_I(\mu) = -\int_{\mathcal{B}} \overline{\mathbf{m}} \cdot \mathbf{h}_0. \tag{6.5}$$

Here, $\overline{\mathbf{m}}$ is the center of mass of the measure as defined in (6.2).

Similarly, if we assume that the operator $\mathcal{K}$ defined in (4.5) is compact and use the fact that the product of a weakly convergent and a strongly convergent sequence converges weakly, we can define

$$\mathcal{E}_{NL}(\mu) = -\int_{\mathcal{B}} \int_{\mathcal{B}} \overline{\mathbf{m}}(\mathbf{x}) \overline{\mathbf{m}}(\mathbf{y}) k(\mathbf{x} - \mathbf{y}) \, d\mathbf{y} \, d\mathbf{x}. \tag{6.6}$$

Finally, we define the field energy to be the energy of the magnetic field of the measure defined in (6.3):

$$\mathcal{E}_F(\mu) = \frac{1}{2} \int_{\mathbf{R}^3} |\hat{\mathbf{h}}(\mu)|^2 \, dv = \frac{1}{2} \int_{\mathbf{R}^3} |\hat{\mathbf{h}}(\overline{\mathbf{m}})|^2 \, dv. \tag{6.7}$$

Thus, the total energy of a measure-valued magnetization μ, with center of mass $\overline{m}$ given by (6.2), is given by

(6.8)
$$\begin{aligned}\mathcal{E}(\mu) =& \frac{1}{2}\int_{\mathbf{R}^3} |\hat{\mathbf{h}}(\overline{\mathbf{m}})|^2\,dv + \int_{\mathcal{B}}\left[\int_{\mathbf{R}^3} \mathcal{W}(\lambda)\,d\mu_{\mathbf{X}}(\lambda) - \int_{\mathcal{B}} \overline{\mathbf{m}}\cdot\mathbf{h}_0\right]dx \\ & - \int_{\mathcal{B}}\int_{\mathcal{B}} \overline{\mathbf{m}}(\mathbf{x})\overline{\mathbf{m}}(\mathbf{y})k(\mathbf{x}-\mathbf{y})\,d\mathbf{y}\,d\mathbf{x}.\end{aligned}$$

Unfortunately, the definition of the field energy presents a real problem. It is possible to have a sequence of magnetizations $\mathbf{m}^n$ that converge to a Young's measure μ, but for which

$$\lim_{n\to\infty} \mathcal{E}_F(\mathbf{m}^n) \neq \mathcal{E}_F(\mu). \tag{6.9}$$

Of course, Theorem 2.3 implies that such a sequence could not have $\operatorname{div}\mathbf{m}^n$ lying in a compact set in H^{-1}_{loc}. For example, consider the type of sequence of oscillating magnetizations represented by Figure 3. We could construct such a sequence that would converge weakly to zero, and, while the highly oscillatory sequence of magnetic fields generated by these magnetizations would converge weakly to zero, it would not converge strongly. Fortunately, Theorem 2.3 and Corollary 2.2 ensure that the field energy of a measure-value magnetization defined above gives the minimum possible limiting energy for any sequence of classical magnetizations converging to the measure-valued magnetization, i.e.,

THEOREM 6.1. *Let $\mu_{\mathbf{x}}$ be a measure-valued magnetization with center of mass $\overline{m}$ and let $\mathbf{m}^n$ be any sequence of classical magnetizations such that $\mathbf{m}^n \overset{*}{\rightharpoonup} \overline{\mathbf{m}}$. Then*

$$\mathcal{E}_F(\mu) \leq \liminf_{n\to\infty} \mathcal{E}_F(\mathbf{m}^n), \tag{6.10}$$

and, furthermore, if there is a sequence $\mathbf{m}^k$ of classical magnetizations such that

$$\operatorname{div}\mathbf{m}^k \subset \text{ a compact set in } H^{-1}_{\mathrm{loc}}(\mathbf{R}^3), \tag{6.11}$$

then

$$\mathcal{E}_F(\mu) = \lim_{k\to\infty} \mathcal{E}_F(\mathbf{m}^k). \tag{6.12}$$

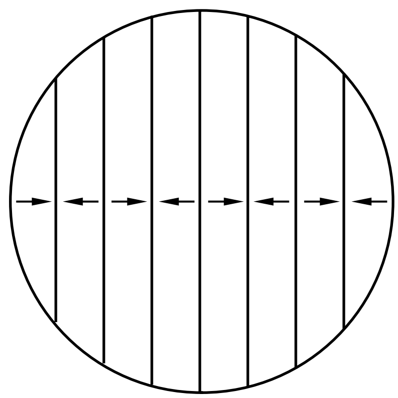

FIGURE 3. Oscillating magnetizations of this type produce large oscillating magnetic fields that converge weakly to zero but not strongly. Thus, the limit of their field energies is different from the field energy of their associated Young's measure.

However, this still leaves us with the following problem: We might have a measure-valued magnetization that minimized the total energy, but whose energy could not be attained in the limit by a sequence of classical magnetizations. Such a measure could not represent a sequence of classical magnetizations satisfying the differential constraint (6.11). This notion seems counter to my physical intuition (as the example in Figure 3 shows, such a sequence involves some sort of weird buildup of "magnetic charge"), but I would rather not rule out such sequences or their measures a priori on purely physical grounds. Instead, we should address the following question.

QUESTION 1. What conditions on a Young's measure $\nu_{\mathbf{x}}$ are sufficient to ensure that it is generated by a sequence of magnetizations satisfying the differential constraint (6.11)?

Tartar and Murat found necessary conditions on the measures derived from sequences satisfying differential constraints (cf. [**18**]), but, as far as I know, the problem of sufficient conditions has not been solved.

Fortunately, in the problems we have been studying, the energy terms do not depend on the entire Young's measure: the anisotropy energy depends only on second moments while the other terms depend only on the center of mass. The following theorem states that we can construct a Young's measure having arbitrary first and second moments of the type which concern us using a sequence satisfying the differential constraint (6.11).

THEOREM 6.2. *Given any* $\overline{\mathbf{m}} \in L^\infty(\mathcal{B})$, *with* $\|\overline{\mathbf{m}}\|_{L^\infty}(\mathcal{B}) \leq \mathcal{M}_0$, *and* $\mathbf{q} = (q_1, q_2, q_3) \in L^\infty(\mathcal{B})$, *with*

$$q_k \geq 0, \quad k = 1, 2, 3, \tag{6.13}$$

$$q_1 + q_2 + q_3 = \mathcal{M}_0^2 \quad \textit{a.e. in } \mathcal{B}, \tag{6.14}$$

$$q_k \geq (\overline{\mathbf{m}} \cdot \mathbf{i}_k)^2 \quad \textit{a.e. in } \mathcal{B}, \tag{6.15}$$

then there exists a sequence $\mathbf{m}^n$ *such that*

$$|\mathbf{m}^n| = \mathcal{M}_0 \quad \textit{a.e. in } \mathcal{B}, \tag{6.16}$$

$$\mathbf{m}^n \overset{*}{\rightharpoonup} \overline{\mathbf{m}} \quad \textit{in } L^\infty(\mathcal{B}), \tag{6.17}$$

$$\operatorname{div} \mathbf{m}^n \subset \textit{ a compact set in } H^{-1}_{\mathrm{loc}}(\mathbf{R}^3), \tag{6.18}$$

$$(\mathbf{m}^n \cdot \mathbf{i}_k)^2 \to q_k \ (\textit{strongly}) \ \textit{in } L^\infty(\mathcal{B}). \tag{6.19}$$

My proof of this theorem involves constructing a sequence of classical magnetizations with suitable domain structure. The sequence is not unique and, in many specific examples, a more physically reasonable sequence can be constructed. Thus, since the construction is rather long and technical, we postpone it until Section 8.

Theorem 6.2 above leads to the following abstract existence theory for measure-valued solutions.

THEOREM 6.3. *Let* $\mathbf{h}_0 \in L^2(\mathbf{R}^3)$ *be given. Then there exists a measure-valued magnetization* μ *which minimizes the energy* $\mathcal{E}$ *over* $\mathbf{M}$. *Furthermore, this measure* μ *represents a weakly convergent sequence of magnetizations* $\mathbf{m}^n$ *with* $\operatorname{div} \mathbf{m}^n$ *contained in a compact set in* $H^{-1}_{\mathrm{loc}}(\mathbf{R}^3)$ *and*

$$\lim_{n \to \infty} \mathcal{E}(\mathbf{m}^n) = \mathcal{E}(\mu). \tag{6.20}$$

PROOF. Since the energy $\mathcal{E}$ is bounded below on the set

$$\{\mathbf{m} \mid \|\mathbf{m}\|_{L^\infty(\mathcal{B})} = \mathcal{M}_0\},$$

there exists an infimizing sequence $\mathbf{m}^n$ in that set. Using weak-star compactness of the $\mathcal{M}_0$ ball in $L^\infty(\mathcal{B})$, we see that $\mathbf{m}^n$ has a weak-star convergent subsequence. We claim that the Young's measure μ associated with this sequence minimizes the energy over all measure-valued magnetizations. We first note that, since the energy depends only on the first and symmetric second moments of a measure, Theorems 6.2 and 6.1 imply that the energy of any measure can be achieved as the limit of energies of classical magnetizations. Thus, for any measure $\tilde{\mu}$ with smaller energy, there would exist a sequence $\tilde{m}^n$ such that

$$\lim_{n\to\infty} \mathcal{E}_F(\tilde{\mathbf{m}}^n) = \mathcal{E}_F(\tilde{\mu}) < \mathcal{E}_F(\mu) \leq \liminf_{n\to\infty} \mathcal{E}_F(\mathbf{m}^n), \tag{6.21}$$

contradicting the fact that $\mathbf{m}^n$ infimizes $\mathcal{E}$. □

7. Multiple minimizers of nonlocal energies. We now consider a model problem that indicates how multiple solutions of macroscopic problems arise. Of course, ideally, one would like to be able to solve the Euler-Lagrange equations for our energy functional explicitly and determine which solutions are relative minimizers of the energy; but, this was a difficult problem to solve analytically for the micromagnetic model and remains so for the new model. Our alternative is to minimize the energy over a class of magnetizations designed to give the energy a simple form. Specifically, we consider the case of a unit sphere of ferromagnetic material in a uniform applied field parallel to the easy direction of magnetization, and we seek to minimize the energy $\mathcal{E}$ over the class of minimizing sequences having uniform Young's measure.

While we do not expect this restricted set of measures to contain solutions of the full minimization problem for small applied fields (the nonlocal term introduces edge effects), such measures are solutions of the Euler-Lagrange equations if the body occupies all of space. Because of this, and because such functions represent relative minimizers in the case of convex, classical problems and nonconvex problems without exchange energy, this seems a reasonable class in which to search for approximate minimizers.

The use of uniform measure-valued magnetizations (and the fact that such magnetizations induce a uniform resultant magnetic field in a sphere) reduces the minimization problem to an elementary calculus problem on $\mathbf{R}^3$. A few basic calculations similar to those performed in the proof of Theorem 5.2 show that a minimizer over this space always has magnetization parallel to $\mathbf{i}_3$, the easy direction of magnetization and the direction of the applied field. Our candidates for solution are thus

$$\hat{\mathbf{M}} = \left\{ \mu \mathbf{i}_3 \mid \mu \in \mathcal{M}(\mathbf{R}), \int d\mu = 1 \right\}, \tag{7.1}$$

i.e., the set of probability measures on $\mathbf{R}^3$ with support on the ray generated by $\mathbf{i}_3$.

The energy of such a measure can be calculated from (6.8) and is given by

$$\mathcal{E}(\mu \mathbf{i}_3) = \frac{4\pi}{3} \left(\frac{a^2}{6} + \alpha_3 b - \mathcal{H}_0 a - C a^2 \right), \tag{7.2}$$

where

$$a = \int_{\mathbf{R}} \xi \, d\mu(\xi) \tag{7.3}$$

and

$$b = \int_{\mathbf{R}} \xi^2 \, d\mu(\xi) \tag{7.4}$$

are the first and second moments of the measure, respectively. These moments are subject to the constraints

$$a^2 \leq b, \tag{7.5}$$

$$b = \mathcal{M}_0^2. \tag{7.6}$$

Thus, we simply have to minimize

$$E(a) = \left[\frac{1}{6} - C \right] a^2 - \mathcal{H}_0 a \tag{7.7}$$

over $\alpha \in [-\mathcal{M}_0, \mathcal{M}_0]$. This elementary Calculus problem leads us to the following conclusions.

1. For $C < 1/6$, there are two types of solutions.

(a) For $|\mathcal{H}_0| \geq \mathcal{M}_0(1/3 - 2C)$, we get a saturated solution parallel to the direction of the applied field. These solutions are global minimizers of the energy over the set of uniform magnetizations.

(b) For $|\mathcal{H}_0| < \mathcal{M}_0(1/3 - 2C)$, we get an interior minimizer of E at $a = (3\mathcal{H}_0)/[\mathcal{M}_0(1 - 6C)]$. This corresponds to a measure-valued magnetization of the form

$$\mu = \theta\delta(-1) + (1 - \theta)\delta(1), \tag{7.8}$$

where $\theta \in [0, 1]$ solves

$$\theta(-1) + (1 - \theta)(1) = a, \tag{7.9}$$

i.e., the center of mass of μ is a. Note that this points the magnetization in the same direction as the applied field. Again, these solutions are global minimizers of the energy.

2. For $C > 1/6$, the situation changes.

(a) As before, for $|\mathcal{H}_0| \geq \mathcal{M}_0(1/3 - 2C)$, we get a saturated solution parallel to the direction of the applied field. These solutions are global minimizers of the energy over the set of uniform magnetizations.

(b) But now, for $|\mathcal{H}_0| < \mathcal{M}_0(1/3 - 2C)$, we no longer get an interior minimizer of E. Instead, the saturated solution parallel to the direction of the applied field is a global minimizer, while the reverse saturated solution is a relative local minimizer.

(c) This is essentially the situation in micromagnetics where the relative stability of branches of saturated solution has been one of the primary areas of investigation.

8. Proof of Theorem 6.2. My proof of Theorem 6.2 is a construction involving plane-wave oscillations. The following lemma describes some of the convergence properties of the type of oscillating sequences we use.

LEMMA 8.1. *Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be vectors in $\mathbf{R}^3$, and let*

$$\mathbf{v} = \theta\mathbf{v}_1 + (1 - \theta)\mathbf{v}_2, \tag{8.1}$$

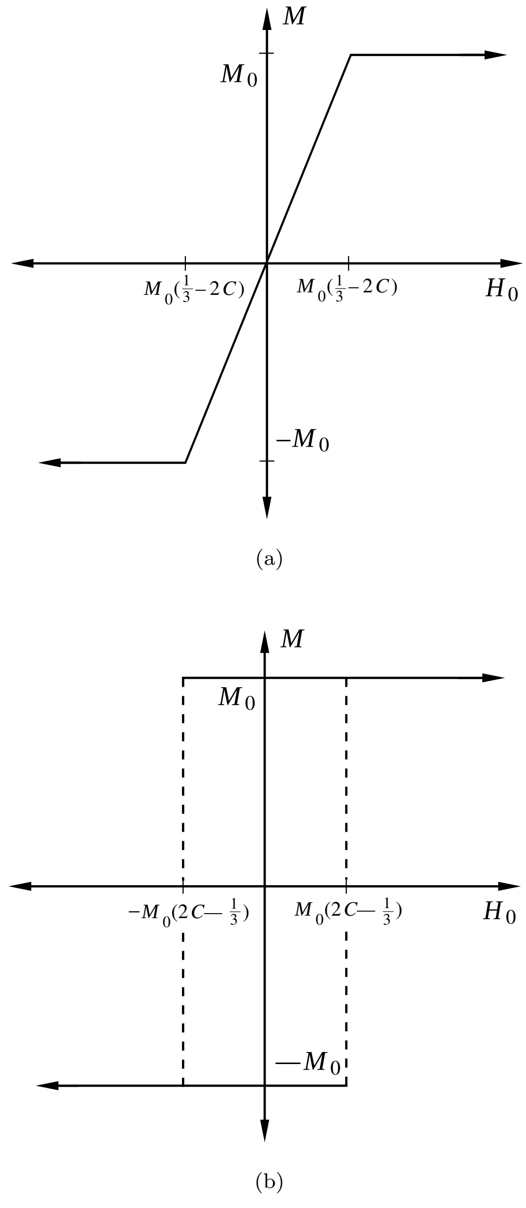

FIGURE 4. Solutions for restricted minimization problem with nonlocal exchange energy for (a) $0 \leq C < 1/6$ and (b) $C > 1/6$.

for some $\theta \in [0,1]$*. Let*

$$\mathbf{k} = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{|\mathbf{v}_1 \times \mathbf{v}_2|} \tag{8.2}$$

if $\mathbf{v}_1$ *is not parallel to* $\mathbf{v}_2$ *and any unit vector perpendicular to both otherwise. Define the sequence*

$$\mathbf{v}^j(\mathbf{x}) = \begin{cases} \mathbf{v}_1, & 0 \le j\mathbf{x}\cdot\mathbf{k} < \theta, \\ \mathbf{v}_2, & \theta \le j\mathbf{x}\cdot\mathbf{k} < 1, \end{cases} \tag{8.3}$$

and extend to all of space by periodicity. Then

$$\mathbf{v}^j \overset{*}{\rightharpoonup} \mathbf{v} \quad \text{in } L^\infty_{\text{loc}}(\mathbf{R}^3), \tag{8.4}$$

and, for any plane P *with unit normal* $\mathbf{n}$*, there is a subsequence (also labeled* $\mathbf{v}^j$*) such that*

$$\mathbf{v}^j \cdot \mathbf{n} \to \mathbf{v}\cdot\mathbf{n}, \quad (\text{strongly}) \text{ in } H^{-1/2}_{\text{loc}}(P). \tag{8.5}$$

PROOF. The proof of (8.4) is standard. To show that (8.5) holds, we note that, for any plane P with $|\mathbf{k}\times\mathbf{n}| \ne 0$, $\mathbf{v}^j\cdot\mathbf{n}$ is simply an oscillating sequence that is piecewise constant on strips of width $(\theta)/(j|\mathbf{k}\times\mathbf{n}|)$ and $(1-\theta)/(j|\mathbf{k}\times\mathbf{n}|)$ with values $\mathbf{v}_1\cdot\mathbf{n}$ and $\mathbf{v}_2\cdot\mathbf{n}$, respectively. Thus, using the same proof as that of (8.4), we see that

$$\mathbf{v}^j \cdot \mathbf{n} \overset{*}{\rightharpoonup} \mathbf{v}\cdot\mathbf{n}, \quad (\text{weakly}) \text{ in } L^\infty_{\text{loc}}(P), \tag{8.6}$$

and, therefore,

$$\mathbf{v}^j \cdot \mathbf{n} \rightharpoonup \mathbf{v}\cdot\mathbf{n}, \quad (\text{weakly}) \text{ in } L^2_{\text{loc}}(P). \tag{8.7}$$

We then use the compact imbedding of $L^2_{\text{loc}}(P)$ into $H^{-1/2}_{\text{loc}}(P)$ to obtain (8.5) for a subsequence. If $|\mathbf{k}\times\mathbf{n}| = 0$, we have $\mathbf{n} = \pm\mathbf{k}$ and use the fact that

$$\mathbf{k}\cdot\mathbf{v} = \mathbf{k}\cdot\mathbf{v}_1 = \mathbf{k}\cdot\mathbf{v}_2 = 0 \tag{8.8}$$

to get

$$\mathbf{v}\cdot\mathbf{n} \equiv \mathbf{v}^j\cdot\mathbf{n} = 0. \tag{8.9}$$

This complete the proof of the lemma. □

An immediate consequence of this is the following.

COROLLARY 8.2. *Let B be any bounded polyhedron. Define the sequence of functions*

$$\tilde{\mathbf{v}}^j(\mathbf{x}) = \begin{cases} \mathbf{v}^j(\mathbf{x}), & \mathbf{x}\in B, \\ 0, & \mathbf{x}\notin B, \end{cases} \tag{8.10}$$

and let

$$\tilde{\mathbf{v}}(\mathbf{x}) = \begin{cases} \mathbf{v}, & \mathbf{x}\in B, \\ 0, & \mathbf{x}\notin B. \end{cases} \tag{8.11}$$

Then

$$\tilde{\mathbf{v}}^j \overset{*}{\rightharpoonup} \tilde{\mathbf{v}}, \quad \text{in } L^\infty(B), \tag{8.12}$$

$$\operatorname{div}\tilde{\mathbf{v}}^j \to \operatorname{div}\tilde{\mathbf{v}}, \quad \text{(strongly) in } H^{-1}_{\text{loc}}(\mathbf{R}^3). \tag{8.13}$$

PROOF. The proof hinges on three facts: There are no jumps in the normal components of $\tilde{\mathbf{v}}^j$ in the interior of B. The normal components of $\tilde{\mathbf{v}}^j$ converge strongly in $H^{-1/2}$ on the boundary of B (by the previous lemma). The trace map from $H^1(B)$ to $H^{1/2}(\partial B)$ is bounded. These allow us to write

(8.14)

$$\|\operatorname{div}\tilde{\mathbf{v}}^j - \operatorname{div}\mathbf{v}\|_{H^{-1}_{\text{loc}}(\mathbf{R}^3)} = \sup_{\|\phi\|_{H^1(\mathbf{R}^3)=1}} \int_{\mathbf{R}^3} (\tilde{\mathbf{v}}^j - \mathbf{v})\cdot\nabla\phi$$

$$= \sup_{\|\phi\|_{H^1(\mathbf{R}^3)=1}} \int_{\partial B} \phi(\tilde{\mathbf{v}}^j - \mathbf{v})\cdot n \tag{8.15}$$

$$\leq \sup_{\|\phi\|_{H^1(\mathbf{R}^3)=1}} \|\tilde{\mathbf{v}}^j - \mathbf{v}\|_{H^{-1/2}(\partial B)}\|\phi\|_{H^{1/2}(\partial B)} \tag{8.16}$$

$$\leq C\|\tilde{\mathbf{v}}^j - \mathbf{v}\|_{H^{-1/2}(\partial B)} \tag{8.17}$$

$$\to 0. \tag{8.18}$$

This completes the proof of the corollary. □

With only slight modifications to the previous corollary, we get

Corollary 8.3. *Let B be any bounded polyhedron. Let*

$$\mathbf{v} = \theta_1\mathbf{v}_1 - \theta_2\mathbf{v}_1 + \theta_3\mathbf{v}_2 - \theta_4\mathbf{v}_2, \tag{8.19}$$

where

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 1, \tag{8.20}$$

and let $\mathbf{k}$ be defined as in Lemma 8.1. Define the sequence of functions $\tilde{\mathbf{v}}^j$ to be zero outside of B and, for $\mathbf{x}\in B$, let

$$\tilde{\mathbf{v}}^j(\mathbf{x}) = \begin{cases} \mathbf{v}_1, & 0 \leq j\mathbf{x}\cdot\mathbf{k} < \theta_1, \\ -\mathbf{v}_1, & \theta_1 \leq j\mathbf{x}\cdot\mathbf{k} < \theta_1 + \theta_2, \\ \mathbf{v}_2, & \theta_1+\theta_2 \leq j\mathbf{x}\cdot\mathbf{k} < \theta_1+\theta_2+\theta_3, \\ -\mathbf{v}_2, & \theta_1+\theta_2+\theta_3 \leq j\mathbf{x}\cdot\mathbf{k} < 1, \end{cases} \tag{8.21}$$

and extend to the rest of B by periodicity. Let

$$\tilde{\mathbf{v}}(\mathbf{x}) = \begin{cases} \mathbf{v}, & \mathbf{x}\in B \\ 0, & \mathbf{x}\notin B. \end{cases} \tag{8.22}$$

Then

$$\tilde{\mathbf{v}}^j \overset{*}{\rightharpoonup} \tilde{\mathbf{v}} \quad \textit{in } L^\infty(B) \tag{8.23}$$

and

$$\operatorname{div}\tilde{\mathbf{v}}^j \to \operatorname{div}\tilde{\mathbf{v}} \quad (\textit{strongly}) \textit{ in } H^{-1}_{\mathrm{loc}}(\mathbf{R}^3). \tag{8.24}$$

We now begin the actual construction of the sequence described in the Theorem (6.2). Let $\mathbf{R}^3$ be partitioned by a nested sequence of uniform rectangular grids with side lengths $1/2^n$ defined by dividing each cell of the grid into eight subcells for each successive grid. Since $\mathcal{B}$ is bounded, there is a constant N independent of n such that $\mathcal{B}$ is contained in $N2^{3n}$ of the grid cells. On each cell C that intersects with the interior of $\mathcal{B}$, define

$$\mathbf{m}^n = \left(\frac{1}{|C\cap\mathcal{B}|}\right)\int_{C\cap\mathcal{B}} \overline{\mathbf{m}}, \tag{8.25}$$

$$\mathbf{q}^n = \left(\frac{1}{|C\cap\mathcal{B}|}\right)^3\int_{C\cap\mathcal{B}} \mathbf{q}. \tag{8.26}$$

(Recall that $\mathbf{m}$ and $\mathbf{q}$ are defined to be zero outside of $\mathcal{B}$.) Note that, almost everywhere in $\mathcal{B}$, we have

$$q_1^n + q_2^n + q_3^n = \left(\frac{1}{|C\cap\mathcal{B}|}\right)\int_{C\cap\mathcal{B}} q_1+q_2+q_3 = \mathcal{M}_0^2, \tag{8.27}$$

$$\begin{aligned}(\mathbf{m}_k^n)^2 &= \left[\left(\frac{1}{|C\cap\mathcal{B}|}\right)\int_{C\cap\mathcal{B}} \overline{\mathbf{m}}_k\right]^2 \\ &\le \left(\frac{1}{|C\cap\mathcal{B}|}\right)\int_{C\cap\mathcal{B}} \overline{\mathbf{m}}_k^2 \le \left(\frac{1}{|C\cap\mathcal{B}|}\right)\int_{C\cap\mathcal{B}} q_k = q_k^n,\end{aligned} \tag{8.28}$$

and, further,

$$\mathbf{m}^n \to \overline{\mathbf{m}}, \quad \mathbf{q}^n \to \mathbf{q} \tag{8.29}$$

strongly in $L^p(\mathbf{R}^3)$ for any $1 \le p < \infty$. Note that, because of the strong convergence in L^2, we have

$$\operatorname{div}\mathbf{m} \to \operatorname{div}\overline{\mathbf{m}} \quad (\text{strongly}) \text{ in } H^{-1}_{\mathrm{loc}}(\mathbf{R}^3). \tag{8.30}$$

We now show that, for any fixed n, we can construct a function $\tilde{\mathbf{m}}^n$ such that, on each cell C, we have

$$|\tilde{\mathbf{m}}^n| = \mathcal{M}_0 \quad \text{almost everywhere} \tag{8.31}$$

$$\left| \int_C \tilde{\mathbf{m}}^n - \mathbf{m}^n \right| \le \frac{1}{Nn2^{3n}}; \tag{8.32}$$

$$\|\operatorname{div} \tilde{\mathbf{m}}^n|_C - \operatorname{div} \mathbf{m}^n|_C\|_{H^{-1}\mathrm{loc}\,(\mathbf{R}^3)} \le \frac{1}{Nn2^{3n}} \tag{8.33}$$

and

$$(\tilde{\mathbf{m}}^n_k)^2 = \mathbf{q}^n_k, \quad k = 1, 2, 3. \tag{8.34}$$

Here, the notation $f|_S$ indicates the restriction of a function to the set S.

Relations (8.31)–(8.34) imply that the sequence of functions $\tilde{\mathbf{m}}^n$ satisfies the requirements (6.16)–(6.19) of the Theorem (6.2). To see this, note the following.

1. Equation (8.31) is simply (6.16).

2. Observe that, for any $\varepsilon > 0$ and any $\phi \in L^1(\mathcal{B})$, we can choose $\overline{N}$ sufficiently large so that there is a step function ϕ^ε that is constant on the cells of side length $1/2^{\overline{N}}$ used in the construction of $\mathbf{m}^n$ and such that

$$\|\phi - \phi^\varepsilon\|_{L^1(\mathcal{B})} \le \frac{\varepsilon}{2\mathcal{M}_0}. \tag{8.35}$$

Also note that, for $n > \overline{N}$, the function ϕ^ε is constant on each of the grid cells (the sequence of cells is nested). Thus, using (8.25),

$$\begin{aligned} \int_{\mathcal{B}} \phi^\varepsilon(\mathbf{m}^n - \overline{\mathbf{m}}) &= \sum \int_C \phi^\varepsilon(\mathbf{m}^n - \overline{\mathbf{m}}) \\ &= \sum \phi^\varepsilon|_C \int_C (\mathbf{m}^n - \overline{\mathbf{m}}) \\ &= 0. \end{aligned} \tag{8.36}$$

Here, the sum is taken over the grid cells. A similar argument, this time based on (8.32), gives us

$$\left| \int_{\mathcal{B}} \phi^\varepsilon(\tilde{\mathbf{m}}^n - \mathbf{m}^n) \right| \le \frac{\|\phi^\varepsilon\|_{L^\infty(\mathcal{B})}}{n}. \tag{8.37}$$

Thus, we can use (8.36) and (8.37) to compute

(8.38)
$$\left|\int_{\mathcal{B}}(\tilde{\mathbf{m}}^n-\overline{\mathbf{m}})\phi\right|\leq\left|\int_{\mathcal{B}}\tilde{\mathbf{m}}^n(\phi-\phi^\varepsilon)\right|+\left|\int_{\mathcal{B}}\phi^\varepsilon(\tilde{\mathbf{m}}^n-\mathbf{m}^n)\right|+\left|\int_{\mathcal{B}}\phi^\varepsilon(\mathbf{m}^n-\overline{\mathbf{m}})\right|$$

(8.39)
$$\leq\|\tilde{\mathbf{m}}^n\|_{L^\infty(\mathcal{B})}\|\phi-\phi^\varepsilon\|_{L^1(\mathcal{B})}+\frac{\|\phi^\varepsilon\|_{L^\infty(\mathcal{B})}}{n}.$$

And we can use (8.35) to conclude that this can be made less than ε for n sufficiently large. Since ε was arbitrary, this verifies (6.17).

3. To see that (6.18) is satisfied, we simply use the triangle in equality in combination with (8.33) to obtain

(8.40)
$$\begin{aligned}\|\operatorname{div}\tilde{\mathbf{m}}^n-\operatorname{div}\overline{\mathbf{m}}\|_{H^{-1}\mathrm{loc}(\mathbf{R}^3)}&\leq\|\operatorname{div}\mathbf{m}^n-\operatorname{div}\overline{\mathbf{m}}\|_{H^{-1}_{\mathrm{loc}}(\mathbf{R}^3)}\\&\quad+\sum\|\operatorname{div}\tilde{\mathbf{m}}^n|_C-\operatorname{div}\mathbf{m}^n|_C\|_{H^{-1}_{\mathrm{loc}}}(\mathbf{R}^3)\\&\leq\|\operatorname{div}\mathbf{m}^n-\operatorname{div}\overline{\mathbf{m}}\|_{H^{-1}\mathrm{loc}(\mathbf{R}^3)}+\frac{1}{n}.\end{aligned}$$

Here, again, the sum is over grid cells. Using (8.30) we see that this goes to zero as n goes to infinity.

4. Relations (8.29) and (8.34) give us (6.19).

To construct the sequence $\tilde{\mathbf{m}}^n$, let C be any of the cells of side length $1/2^n$, and let (m_1^n,m_2^n,m_3^n) and (q_1^n,q_2^n,q_3^n) be the values calculated above that the step functions $\mathbf{m}^n$ and $\mathbf{q}^n$ take on in C. Note that, because of (8.28), we can write (m_1^n,m_2^n,m_3^n) as a convex combination of the square roots of (q_1^n,q_2^n,q_3^n), i.e.,

(8.41)
$$\begin{aligned}(m_1^n,m_2^n,m_3^n)&=\theta_1\left(\sqrt{q_1^n},\sqrt{q_2^n},\sqrt{q_3^n}\right)+\theta_2\left(-\sqrt{q_1^n},-\sqrt{q_2^n},-\sqrt{q_3^n}\right)\\&\quad+\theta_3\left(-\sqrt{q_1^n},\sqrt{q_2^n},\sqrt{q_3^n}\right)+\theta_4\left(\sqrt{q_1^n},-\sqrt{q_2^n},-\sqrt{q_3^n}\right)\\&\quad+\theta_5\left(\sqrt{q_1^n},-\sqrt{q_2^n},\sqrt{q_3^n}\right)+\theta_6\left(-\sqrt{q_1^n},\sqrt{q_2^n},-\sqrt{q_3^n}\right)\\&\quad+\theta_7\left(\sqrt{q_1^n},\sqrt{q_2^n},-\sqrt{q_3^n}\right)+\theta_8\left(-\sqrt{q_1^n},-\sqrt{q_2^n},\sqrt{q_3^n}\right),\end{aligned}$$

where

$$\sum_{i=1}^{8}\theta_i=1. \tag{8.42}$$

In order to use Lemma 8.1, we group the eight vectors in the convex combination into a pair of vectors and construct plane waves. Let

$$\bar{\theta} = \sum_{i=1}^{4} \theta_i, \tag{8.43}$$

$$\begin{aligned}\mathbf{v}_1 = \frac{1}{\bar{\theta}}\Big[&\theta_1\left(\sqrt{q_1^n}, \sqrt{q_2^n}, \sqrt{q_3^n}\right) + \theta_2\left(-\sqrt{q_1^n}, -\sqrt{q_2^n}, -\sqrt{q_3^n}\right)\\ &+ \theta_3\left(-\sqrt{q_1^n}, \sqrt{q_2^n}, \sqrt{q_3^n}\right) + \theta_4\left(\sqrt{q_1^n}, -\sqrt{q_2^n}, -\sqrt{q_3^n}\right)\Big],\end{aligned} \tag{8.44}$$

$$\begin{aligned}\mathbf{v}_2 = \frac{1}{(1-\theta)}\Big[&\theta_5\left(\sqrt{q_1^n}, -\sqrt{q_2^n}, \sqrt{q_3^n}\right) + \theta_6\left(-\sqrt{q_1^n}, \sqrt{q_2^n}, -\sqrt{q_3^n}\right)\\ &+ \theta_7\left(\sqrt{q_1^n}, \sqrt{q_2^n}, -\sqrt{q_3^n}\right) + \theta_8\left(-\sqrt{q_1^n}, -\sqrt{q_2^n}, \sqrt{q_3^n}\right)\Big].\end{aligned} \tag{8.45}$$

Note that

$$\bar{\theta}\mathbf{v}_1 + (1-\bar{\theta})\mathbf{v}_2 = \mathbf{m}^n. \tag{8.46}$$

As in Lemma 8.1 and Corollary 8.2, we let

$$\mathbf{k} = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{|\mathbf{v}_1 \times \mathbf{v}_2|}, \tag{8.47}$$

if $\mathbf{v}_1$ is not parallel to $\mathbf{v}_2$ and any unit vector perpendicular to both, otherwise define the sequence $\mathbf{v}^j$ to be zero outside of C, and, for $x \in C$, let

$$\mathbf{v}^j(\mathbf{x}) = \begin{cases} \mathbf{v}_1, & 0 \le j\mathbf{x}\cdot\mathbf{k} < \theta, \\ \mathbf{v}_2, & \theta \le j\mathbf{x}\cdot\mathbf{k} < 1, \end{cases} \tag{8.48}$$

extending to the rest of C by periodicity.

Now, by Corollary 8.2, we can choose $\bar{j}$ sufficiently large so that

$$\left|\int_C \mathbf{v}^{\bar{j}} - \mathbf{m}^n\right| \leq \frac{1}{2Nn2^{3n}}, \tag{8.49}$$

$$\left\|\operatorname{div}(\mathbf{v}^{\bar{j}} - \mathbf{m}^n)|_C\right\|_{H^{-1}_{\mathrm{loc}}(R^3)} \leq \frac{1}{2Nn2^{3n}}. \tag{8.50}$$

The function $\mathbf{v}^{\bar{j}}$ is a piecewise constant function taking one of the values $\mathbf{v}_1$ or $\mathbf{v}_2$ on "slabs" cut from the cell C. Since $\bar{j}$ is fixed, there is a finite number of the slabs, say K. Each of the values taken on is a convex combination of four vectors of the form specified in Corollary 8.3 (cf. (8.19), (8.44), and (8.45)). Thus, Corollary 8.3 implies that, on each of the slabs, we can construct an oscillating sequence, each element of which satisfies (8.31) and (8.34). Furthermore, on each of the slabs we can choose a fixed element of the sequence as we did in the construction of $\mathbf{v}^{\bar{j}}$ and thereby define a function $\tilde{\mathbf{m}}^n$ such that, on a slab S,

$$\left|\int_S \tilde{\mathbf{m}}^n - \mathbf{v}^{\bar{j}}\right| \leq \frac{1}{2NKn2^{3n}} \tag{8.51}$$

and

$$\left\|\operatorname{div}(\mathbf{v}^{\bar{j}} - \tilde{\mathbf{m}}^n)|_S\right\|_{H^{-1}_{\mathrm{loc}}(R^3)} \leq \frac{1}{2NKn2^{3n}}. \tag{8.52}$$

Summing (8.51) over all slabs in C and, using the triangle inequality gives (8.32), while a similar calculation using (8.52) gives (8.33) to complete the proof. □

9. Comments. I conclude with some comments on the ramifications and possible extensions of this work.

1. While nonconvexity, anisotropy, and exchange forces are usually the strongest effects in ferromagnetic materials, certain such materials exhibit a profound coupling with elastic effects. In particular, materials such as Metglass 2605SC being investigated by Savage and Spano [**17**] and Savage and Adler [**16**] change their elastic constants by a factor of ten when a magnetic field is applied. Two types of mathematical models of magnetoelastic interactions immediately suggest themselves.

(a) One could easily formulate a theory in which the ferromagnetic model presented in this work was coupled with an elasticity model satisfying some sort of convexity condition. For example, in Brown [**2**] and Rogers [**15**], the theory of micromagnetics is coupled with elasticity theories that assume strong ellipticity and polyconvexity, respectively. The existence theory of [**15**] could readily be extended to the present model.

(b) One could also couple the present ferromagnetic theory to a nonconvex elasticity theory of the type considered in [**1, 6, 10**]. This should have the effect of coupling magnetic domain structures to fine phase mixtures of elastic states. There seems to be some hint of this idea in the heuristic explanation of the profound magnetoelastic effects described in [**16**].

2. One of the purposes of this work is to point out the advantages of using measure-valued functions (or, rather, their first and second moments) to solve a minimization problem directly rather than appealing to a minimizing sequence of more classical solution candidates. However, we should not regard the choice made above for the class of measures to be considered as possible solutions as final. In particular, we must ask if the first and second moments of a measure-valued solution (or any finite collection of moments) are the only important physical quantities. If this is the case, then the appropriate class of measures are those that can be achieved as the limits of magnetizations with divergences compact in H^{-1}_{loc}. (One could argue that this is the correct class of measures on the grounds that there should be no blowup of magnetic charge, but I am not comfortable with such an argument.) I am unaware if it is known how to characterize this class of measures directly, without appealing to the sequences from which they arise.

3. In the case of elasticity, where measure-valued solutions are thought of as the limit of sequences of deformation gradients, we are on firmer ground in requiring that the admissible class of measure-valued solutions be obtainable as the limit of tensor valued functions whose curl is zero in H^{-1}_{loc} for physical reasons alone. But, again, I am unaware of an optimal way to characterize this class. (In particular, the choice of first and second moments is no longer arbitrary under the constraints on the curl.)

4. Young's measures are not the only available choice for a mathematical representative of oscillating sequences of magnetizations, nor do they contain all possible information about such sequences. Note that the choice of orientation of the planes of discontinuity in the minimizing sequences of Section 5 was somewhat arbitrary; any appropriate choice would yield the same Young's measure. While Young's measures suffice for the steady state problems studied here, information about the directions of oscillation may be crucial in dynamics problems. A new type of measure called an *H-measure* (H stands for homogenization) derived from weakly converging sequences and retaining some of this information has been developed by Tartar [**20**].

5. Versions of the micromagnetic exchange energy with different nonlinearity have been proposed (cf. Maugin [**13**]), i.e.,

$$\mathcal{E}_X = \int_{\mathcal{B}} \chi(\nabla \mathbf{m}), \tag{9.1}$$

but have not been extensively investigated in a setting closely connected to the physics of ferromagnetism. (There is, of course, an extensive literature on the "p-Laplacian," but I am unaware of any of this work that approaches such difficulties as field energy terms.) If we make similar modifications of the nonlocal exchange energy, we can see how higher order nonlinearity introduces new branches of solutions. Let

$$\tilde{\mathcal{E}}_{NX} = -\int_{\mathcal{B}} \mathcal{G}\left(\int_{\mathcal{B}} \mathbf{m}(\mathbf{x}) \cdot \mathbf{m}(\mathbf{y}) k(\mathbf{x}-\mathbf{y})\, d\mathbf{y}\right) d\mathbf{x}, \tag{9.2}$$

where $\mathcal{G} : \mathbf{R} \mapsto \mathbf{R}$ is monotone and $\mathcal{G}(0) = 0$. If we consider the case $\mathcal{G}(t) = t + k_1 t^3$, then the (Existence) Theorem 6.3 no longer applies because we now have to consider moments of the Young's measure higher than second order. However, if we restrict the class of measures to that considered in the all of space problem (7.1), we can use the restriction (7.6) to show that the energy of such measures is given by

$$\mathcal{E}(\mu \mathbf{i}_3) = \frac{4\pi}{3}\left(\frac{a^2}{6} + \alpha^3 b - \mathcal{H}_0 a - C(a^2 + \mathcal{M}_0^2 k_1 a^4)\right). \tag{9.3}$$

As in Section 7, the minimization problem reduces to the problem of minimizing

$$E(a,b) = -Ck_1\mathcal{M}_0^2 a^4 + \left[\frac{1}{6} - C\right] a^2 - \mathcal{H}_0 a \tag{9.4}$$

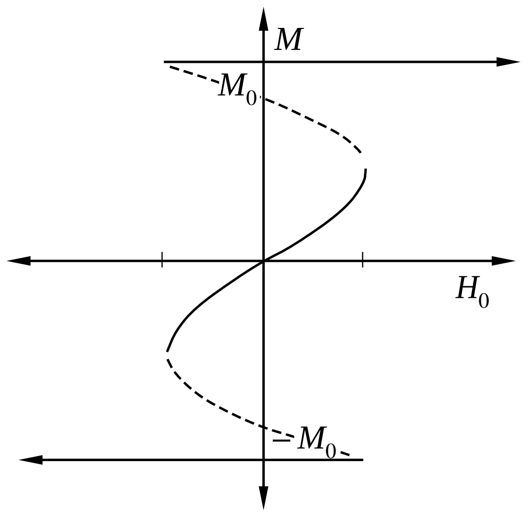

FIGURE 5. Solutions for restricted minimization problem with nonlinear form of the nonlocal exchange energy.

over $\alpha \in [-\mathcal{M}_0, \mathcal{M}_0]$. We omit the details here, but, for very small and very large values of C, the set of solutions remains much the same as in Section 7: a single family of solutions for small values of C compose of classical saturated solutions for large values of $\mathcal{H}_0$ and measure-valued solutions for small values of $\mathcal{H}_0$; and two branches of solutions for large values of C, composed of saturated and reverse saturated classical solutions (cf. Figure 4). However, when

$$\frac{1}{6(2k_1\mathcal{M}_0^4 + 1)} < C < \frac{1}{6},$$

there are three branches of solutions: saturated, reverse saturated and measure-valued (cf. Figure 5).

We can see in this illustration one possible way in which multiple equilibrium arise. Several others have been suggested, most importantly material imperfections.

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