A nonparametric exponentially weighted moving average signed-rank chart for monitoring location

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ABSTRACT

Nonparametric control charts can provide a robust alternative in practice to the data analyst when there is a lack of knowledge about the underlying distribution. A nonparametric exponentially weighted moving average (NPEWMA) control chart combines the advantages of a nonparametric control chart with the better shift detection properties of a traditional EWMA chart. A NPEWMA chart for the median of a symmetric continuous distribution was introduced by Amin and Searcy (1991) using the Wilcoxon signed-rank statistic (see Gibbons and Chakraborti, 2003). This is called the nonparametric exponentially weighted moving average Signed-Rank (NPEWMA-SR) chart. However, important questions remained unanswered regarding the practical implementation as well as the performance of this chart. In this paper we address these issues with a more in-depth study of the two-sided NPEWMA-SR chart. A Markov chain approach is used to compute the run-length distribution and the associated performance characteristics. Detailed guidelines and recommendations for selecting the chart's design parameters for practical implementation are provided along with illustrative examples. An extensive simulation study is done on the performance of the chart including a detailed comparison with a number of existing control charts, including the traditional EWMA chart for subgroup averages and some nonparametric charts i.e. runs-rules enhanced Shewhart-type SR charts and the NPEWMA chart based on signs. Results show that the NPEWMA-SR chart performs just as well as and in some cases better than the competitors. A summary and some concluding remarks are given.

Keywords: Contaminated normal, Distribution-free, Markov chain, Median, Outlier, Quality control, Robust, Run-length, Search Algorithm, Simulation.

1. Introduction

The cumulative sum (CUSUM) and the exponentially weighted moving average (EWMA) control charts enjoy widespread popularity in practice. They are particularly effective in detecting small sustained shifts quickly (see e.g. Montgomery, 2005 pages 386 and 411). The superiority of these charts over the Shewhart chart stems from the fact that they use information in the data from start-up and not the most recent time point only. The performance of CUSUM and EWMA charts are similar (see e.g. Montgomery, 2005 page 405), but from a practical standpoint the EWMA chart is often preferred because of its relative ease of use. Traditional EWMA charts for the mean were introduced by Roberts (1959) and they contain Shewhart-type charts as a special case. The literature on EWMA charts is enormous and continues to grow at a substantial pace (see e.g. the overview by Ruggeri et al. (2007) and the references therein). In typical applications of the EWMA chart it is usually assumed that the underlying process distribution is normal (or, at least, approximately so); see e.g. Huwang et al. (2010). Such an assumption(s) should ideally be verified which would typically involve some preliminary work such as exploratory (e.g. graphical) and confirmatory (e.g. testing hypotheses) data analysis. If normality is in doubt or can not be justified for lack of information or data, a nonparametric (NP) chart is more desirable. These charts are attractive because their run-length distribution is the same for all continuous distributions so that they can be applied without any knowledge of the form of the underlying distribution. For comprehensive overviews of the literature on nonparametric control charts see Chakraborti et al. (2001), (2007) and (2010). A control chart that combines the shift detection properties of the EWMA with the robustness of a NP chart is thus clearly desirable.

Amin and Searcy (1991) considered such a chart based on the Wilcoxon signed-rank (SR) statistic for monitoring the known or the specified or the target value of the median of a process; we label this the NPEWMA-SR chart. However, much work remained to be done. Chakraborti and Graham (2007), noted that "…more work is necessary on the practical implementation of the (NPEWMA-SR) charts…". Given the potential practical benefits of this control chart, in this article we perform an in-depth study to gain insight into its design, implementation and performance. More precisely:

i. We use a Markov-chain approach to calculate the in-control (IC) run-length distribution and the associated performance characteristics;
ii. We examine the average run-length (*ARL*) as a performance measure and, for a more thorough assessment of the chart's performance, we also calculate and study the standard deviation (*SDRL*), the median (*MDRL*), the 1st and 3rd quartiles as well as the 5th and 95th percentiles for an overall assessment of the run-length distribution;
iii. We provide easy to use tables for the chart's design parameters to aid practical implementation; and
iv. We do an extensive simulation-based performance study comparison with competing traditional and nonparametric charts.

The rest of the article is organized as follows: In Section 2 some statistical background information is given and the NPEWMA-SR chart is defined. In Section 3 the computational aspects of the run-length distribution plus the design and implementation of the chart are discussed. Section 4 provides two illustrative examples. In Section 5, the IC and out-of-control (OOC) chart performance are compared to those of the traditional EWMA chart for the mean (denoted EWMA-$\overline{X}$ hereafter), the runs-rules enhanced Shewhart-type SR charts, i.e. the basic (or original) *1-of-1* chart, the *2-of-2* DR and the *2-of-2* KL Shewhart-type SR charts and the

NPEWMA chart based on signs (denoted NPEWMA-SN). We conclude with a summary and some recommendations in Section 6.

2. Background and definition of the NPEWMA-SR chart

2.1 Statistical Background

The Wilcoxon signed-rank (SR) test is a popular nonparametric alternative to the one-sample *t*-test for testing hypotheses (or setting-up confidence intervals) about the location parameter (mean/median) of a symmetric continuous distribution. Note that for a *t*-test to be valid the assumption of normality is needed, but that is not necessary for the SR test. The SR test is quite efficient, the asymptotic relative efficiency (ARE) of the SR test relative to the *t*-test is 0.955, 1, 1.097 and 1.5 for the Normal, Uniform, Logistic and Laplace distribution, respectively (see e.g. Gibbons and Chakraborti, 2003 page 508). This indicates that the SR test is more powerful for some heavier tailed distributions. In fact, it can be shown that the ARE of the SR test to the *t*-test is at least 0.864 for any symmetric continuous distribution. So, very little seems to be lost and much to be gained in terms of efficiency when the SR test is used instead of the *t*-test. Graham et al. (2009) proposed a NPEWMA chart based on the sign (SN) statistic, the so-called NPEWMA-SN chart. Although both the sign and the signed-rank charts are nonparametric, the SR chart is expected to be more efficient since the SR test is more efficient than the SN test for a number of light to moderately heavy-tailed normal-like distributions (see e.g. Gibbons and Chakraborti (2003)). Thus the NPEWMA-SR chart is an exceptionally viable alternative to the traditional EWMA and the NPEWMA-SN charts. In this paper the EWMA chart based on the SR statistic, the NPEWMA-SR chart, is considered, which can be used to monitor the median of a symmetric continuous distribution (for a discussion of some tests of symmetry see the review article by Konijin (2006)). Also, as a referee pointed out, because many practitioners in the quality field may have a better intuitive understanding of a median (half of the output from a process is below a certain level) than a mean, the application of the SR charts facilitates a simple switch over from the well entrenched traditional methods used in the quality field.

Suppose that X_{ij}, $i = 1,2,3, \ldots$ and $j = 1,2, \ldots, n$ denote the j^{th} observation in the i^{th} rational subgroup of size $n > 1$. Let R_{ij}^{+} denote the rank of the absolute values of the differences $|X_{ij} - \theta_0|, j = 1,2, \ldots, n$, within the i^{th} subgroup. Define

$$SR_i = \sum_{j=1}^{n} sign(X_{ij} - \theta_0) R_{ij}^{+} \qquad i = 1,2,3, \ldots \tag{1}$$

where $sign(A) = -1, 0, 1$ if $A < 0, = 0, > 0$ and θ_0 is the known or the specified or the target value of the median, θ, that is monitored. Thus SR_i is the difference between the sum of the ranks of the absolute differences corresponding to the positive and the negative differences, respectively. Note that the statistic SR is linearly related to the better-known signed-rank statistic T_n^+ through the relationship $SR = 2T_n^+ - n(n+1)/2$ (the reader is referred to Gibbons and Chakraborti (2003) page 197 for more details on the T_n^+ statistic).

Bakir (2004) proposed a nonparametric Shewhart-type control chart based on the SR statistic. Chakraborti and Eryilmaz (2007) extended this idea and proposed various nonparametric charts based on runs-rules of the SR statistic and showed that their charts are more sensitive in detecting small shifts. Other nonparametric charts based on runs-type signalling rules have also been proposed in the literature (see e.g. Chakraborti et al. (2009)).

2.2 The NPEWMA-SR chart

The NPEWMA-SR chart is constructed by accumulating the statistics $SR_1, SR_2, SR_3, \ldots$ sequentially from each subgroup. The plotting statistic is

$$Z_i = \lambda\, SR_i + (1-\lambda) Z_{i-1} \quad \text{for} \quad i = 1,2,3,\ldots \tag{2}$$

where the starting value is taken as $Z_0 = 0$ and $0 < \lambda \leq 1$ is the smoothing constant. Note that λ = 1 yields the Shewhart-type SR chart of Bakir (2004).

To calculate the control limits of the NPEWMA-SR chart the IC mean and variance of the plotting statistic Z_i are necessary; these can conveniently be obtained applying a recursive substitution and using the relationship between SR and T_n^+. The IC mean and standard deviation of Z_i are given by $\mathrm{E}(\mathrm{Z_i}) = 0$ and $\sigma_{Z_i} = \sqrt{\left(\frac{n(n+1)(2n+1)}{6}\right)\left(\frac{\lambda}{2-\lambda}\right)(1-(1-\lambda)^{2i})}$, respectively, and follows directly from the expressions of the null expectation and variance of the well-known signed-rank statistic (see e.g. Gibbons and Chakraborti, 2003 page 198) coupled with the properties of the plotting statistic of the EWMA chart (see e.g. Montgomery, 2005 page 406). Hence, the exact time varying upper control limit (UCL), lower control limit (LCL) and centerline (CL) of the NPEWMA-SR chart for the median are given by

$$UCL/LCL = \pm L\sqrt{\left(\frac{n(n+1)(2n+1)}{6}\right)\left(\frac{\lambda}{2-\lambda}\right)(1-(1-\lambda)^{2i})} \text{ and } CL = 0. \tag{3}$$

The "steady-state" control limits and the CL are given by

$$UCL/LCL = \pm L\sqrt{\left(\frac{n(n+1)(2n+1)}{6}\right)\left(\frac{\lambda}{2-\lambda}\right)} \text{ and } CL = 0. \tag{4}$$

These are typically used when the NPEWMA-SR chart has been running for several time periods and are obtained from (3) as $i \to \infty$ so that $\left(1-(1-\lambda)^{2i}\right) \to 1$. If any Z_i plots on or outside either of the control limits, the process is declared OOC and a search for assignable causes is started. Otherwise, the process is considered IC and the charting procedure continues.

It should be noted that because T_n^+ is known to be distribution-free for all symmetric continuous distributions (see e.g. Gibbons and Chakraborti, 2003) so is the statistic SR and hence the NPEWMA-SR chart.

In the developments that follow:

i. We study two-sided charts with symmetrically placed control limits i.e. equidistant from the *CL*. This is the typical application of the traditional EWMA-$\overline{X}$ chart. The methodology can be easily modified where a one-sided chart is more meaningful.

ii. We use the steady-state control limits which significantly simplifies the calculation of the IC run-length distribution via the Markov chain approach.

iii. We investigate the entire run-length distribution in terms of the mean (*ARL*), the standard deviation (*SDRL*), the median run-length (*MDRL*), the 1st and the 3rd quartiles as well as the 5th and the 95th percentiles (Amin and Searcy (1991) only evaluated the *ARL*). It's a well-known fact that important information about the performance of a control chart may be missed by focusing only on the *ARL*, because the run-length distribution is highly right-skewed (see e.g. Radson and Boyd (2005) and Chakraborti (2007)).

Note that λ and L are the two design parameters of the chart which directly influence the chart's performance; this implies that suitable combinations need to be used. The choice of λ and L is discussed in more detail in Section 3.2. Next we discuss the computational aspects of the run-length distribution.

3. The Run-length distribution and Implementation of the chart

3.1 Computation of the Run-Length distribution

For the calculation of the run-length distribution and associated characteristics computer simulation experiments and the Markov chain approach have proven to be useful. While each of these methods has their own advantages and/or disadvantages, the most important benefit with using the Markov chain approach is that one can find explicit expressions (formulas) for the characteristics of interest. For a detailed discussion on how to implement the Markov chain approach for a NPEWMA control chart, see Graham et al. (2009); here we summarize the key results only. Given the Markov chain representation of the IC run-length distribution, the probability mass function (pmf), the expected value (*ARL*), the standard deviation (*SDRL*) and the cumulative distribution function (cdf[1]) of the run-length variable *N* can all be calculated as

$$P(N=t;\lambda,L,r,\theta)=\underline{\xi}Q^{t-1}(I-Q)\underline{1} \quad \text{for} \quad t=1,2,3,\ldots \tag{5}$$

$$ARL(\lambda,L,r,\theta)=\underline{\xi}(\mathrm{I}-Q)^{-1}\underline{1}, \tag{6}$$

$$SDRL(\lambda,L,r,\theta)=\sqrt{\underline{\xi}(I+Q)(I-Q)^{-2}\underline{1}-(ARL)^2}, \text{ and} \tag{7}$$

$$P(N\leq t;\lambda,L,r,\theta)=1-\underline{\xi}Q^{t}\underline{1} \quad \text{for} \quad t=1,2,3,\ldots \tag{8}$$

respectively (see Fu and Lou (2003); Theorems 5.2 and 7.4 pages 68 and 143) where $r+1$ denotes the total number of states (i.e. there are *r* non-absorbing states and one absorbing state which is entered when the chart signals), $I=I_{r\times r}$ is the identity matrix, $Q=Q_{r\times r}$ is called the essential transition probability sub-matrix which contains all the probabilities of going from one non-absorbing state to another, $\underline{1}=\underline{1}_{r\times 1}$ is a column vector with all elements equal to one and $\underline{\xi}=\underline{\xi}_{1\times r}$ is a row vector called the initial probability vector which contains the probabilities that the Markov chain starts in a given state. The vector $\underline{\xi}=(\xi_{-m},\ldots,\xi_m)$ with $m=(r-1)/2$, is typically chosen such that $\sum_{i=-m}^{m}\xi_i=1$. We set $\xi_0=1$ and let $\xi_i=0$ for all $i\neq 0$; this implies that $Z_0=0$ with probability one as mentioned earlier in Section 2.2. Note that the key component in using the Markov chain approach is to obtain the essential transition probability sub-matrix $Q_{r\times r}$. The elements of the latter are called the one-step transition probabilities; $Q_{r\times r}=[p_{ij}]$ for $i,j=-m,-m+1,\ldots,m-1,m$. The transition probability, p_{ij}, is the conditional probability that the plotting statistic at time k, Z_k, lies within state *j*, given that the plotting statistic at time $k-1$, Z_{k-1}, lies within state *i* (an approximation to the latter probability is obtained by setting Z_{k-1} equal to S_i which denotes the midpoint of state *i*) and we obtain

$$\begin{aligned} p_{ij} &= P(Z_k \text{ lies within state } j|Z_{k-1} \text{ lies within state } i) \\ &= P(S_j-\gamma<Z_k\leq S_j+\gamma|Z_{k-1}=S_i) \end{aligned}. \tag{9}$$

It should be noted that the midpoints can be calculated using the expression $S_j=LCL+(2(m+j)+1)\gamma$ for $j=-m,-m+1,\ldots,m-1,m$ and $S_0=0$ because of the symmetrically positioned control limits i.e. – *LCL* = *UCL*.

By substituting the definition of the plotting statistic (see equation (2)) into (9) and using the relationship between the statistic SR and usual signed-rank statistic T_n^+ we get that p_{ij} equals

[1] Using the cdf in (8) we can calculate any IC percentile of the run-length distribution.

$$P\left(S_j-\gamma<\lambda SR_k+(1-\lambda)Z_{k-1}\le S_j+\gamma \mid Z_{k-1}=S_i\right)$$
$$=P\left(S_j-\gamma<\lambda SR_k+(1-\lambda)S_i\le S_j+\gamma\right)$$
$$=P\left(\frac{(S_j-\gamma)-(1-\lambda)S_i}{\lambda}<SR_k\le\frac{(S_j+\gamma)-(1-\lambda)S_i}{\lambda}\right)$$
$$=P\left(\left(\frac{(S_j-\gamma)-(1-\lambda)S_i}{\lambda}+\frac{n(n+1)}{2}\right)/2<T_n^+\ \le\left(\frac{(S_j+\gamma)-(1-\lambda)S_i}{\lambda}+\frac{n(n+1)}{2}\right)/2\right).$$

Note that the accuracy of the Markov chain approach increases as r (the number of non-absorbing states) increases (see also e.g. Knoth (2006)). Verification of the Markov chain approach using 100,000 Monte Carlo simulations suggests that the discrepancies are within 1% of the simulated values when r = 1001. Taking larger values of r would result in more accurate answers, but in doing so, some run-length characteristics could not be computed within a practical time. In addition, it is recommended that r be chosen to be an odd positive integer ($r = 2m + 1$) so that there is a unique middle entry which simplifies the calculations.

3.2 Choice of Design Parameters

The choice of the design parameters (λ, L) generally entails two steps: First, one has to (use a search algorithm to) find the (λ, L) combinations that yield the desired in-control *ARL* (denoted ARL_0). Second, one has to choose, among these (λ, L) combinations, the one that provides the best performance i.e. the smallest out-of-control *ARL* (ARL_δ) for the shift (δ) that is to be detected. Note that, the smoothing parameter $0<\lambda\le 1$ is typically selected first (which depends on the magnitude of the shift to be detected) and then the constant $L > 0$ is selected (which determines the width of the control limits i.e. the larger the value of L, the wider the control limits and vice versa).

The above-mentioned procedure was used in the design of the NPEWMA-SR chart and the run-length distribution was calculated for various values of λ and L for subgroup sizes n = 5 and 10 (for a detailed discussion on the choice of n see Bakir and Reynolds (1979) wherein they concluded that the best subgroup size is somewhere between 5 and 10 depending on the desired ARL_0 and the size of the shift (δ) to be detected). Using a search algorithm with five values of λ (i.e. 0.01, 0.025, 0.05, 0.1 and 0.2) along with values of L ranging from 2 to 3 in increments of 0.1, the (λ, L) combinations were identified which lead to an ARL_0 close to the industry standard of 370 and 500; these results are shown in Tables 1 and 2. Note that, the first row of each of the cells in Tables 1 and 2 shows the ARL_0 and $SDRL_0$ values whereas the second row shows the IC 5th, 25th, 50th, 75th and 95th percentiles (in this order).

From Tables 1 and 2 we observe that for a specified or fixed value of λ, all the characteristics of the IC run-length distribution increase as L increases. Also, we observe that the IC run-length distribution is positively skewed (as is expected) because the $ARL_0 > MDRL_0$ in all cases. Tables 1 and 2 were used to find those combinations of λ and L values that give the desired IC performance. These are useful for a practical implementation of the control chart. For example, from Table 1 for n = 5, we observe that for (λ = 0.025, L = 2.2) the ARL_0 = 347.83 and for (λ = 0.025, L = 2.3) the ARL_0 = 431.13, which implies that the value of L that leads to an ARL_0 of 370 is between 2.2 and 2.3. Refining the search algorithm leads to (λ = 0.025, L = 2.230) with an ARL_0 of 370.35 (see Table 3); more details are given below.

< Insert Table 1 >

< Insert Table 2 >

3.3 Implementation of the NPEWMA-SR chart

To implement the chart, a practitioner needs values of the design parameters (λ, L). The first step is to choose λ. If small shifts (roughly 0.5 standard deviations or less) are of primary concern the typical recommendation is to choose a small λ say equal to 0.01, 0.025 or 0.05; if moderate shifts (roughly between 0.5 and 1.5 standard deviations) are of greater concern choose λ = 0.10, whereas if larger shifts (roughly 1.5 standard deviations or more) are of concern choose λ = 0.20 (see e.g. Montgomery (2005), page 411). Next we choose L, in conjunction with the chosen λ, so that a desired nominal ARL_0 is attained.

Table 3 lists some (λ, L)-combinations for the popular ARL_0 values of 370 and 500 and for subgroups of size n = 5 and n = 10, respectively. In each case, the ARL_0 values were calculated using the Markov chain approach and are called the attained ARL_0 values. Note that because of the discreteness of the SR statistic, the desired nominal ARL values are not attained exactly.

Table 3. (λ, L)-combinations for the NPEWMA-SR chart for nominal ARL_0 = 370 and 500.[1]

	Nominal ARL_0 = 370		**Nominal ARL_0 = 500**	
Shift to be detected	**(λ, L)**	**Attained ARL_0**	**(λ, L)**	**Attained ARL_0**
	***n* = 5**			
Small	(0.01, 1.822)	370.14	(0.01, 1.975)	499.45
	(0.025, 2.230)	370.35	(0.025, 2.368)	499.04
	(0.05, 2.481)	370.29	(0.05, 2.602)	499.83
Moderate	(0.10, 2.668)	370.13	(0.10, 2.775)	500.11
Large	(0.20, 2.764)	369.91	(0.20, 2.852)	499.27
	***n* = 10**			
Small	(0.01, 1.821)	370.05	(0.01, 1.975)	500.51
	(0.025, 2.230)	370.85	(0.025, 2.367)	500.06
	(0.05, 2.486)	370.49	(0.05, 2.610)	500.67
Moderate	(0.10, 2.684)	370.09	(0.10, 2.794)	500.13
Large	(0.20, 2.810)	370.19	(0.20, 2.905)	498.92

[1]Table 3 is more extensive and unlike in Amin and Searcy (1991) who give some (λ, UCL)-values.

So, for example, suppose n = 5 and one is interested in detecting a small shift in the location with a NPEWMA-SR with an ARL_0 of 370. Then one can use the (λ, L)-combination: (0.05, 2.481) which yields an attained ARL_0 of 370.29. Table 3 should be very useful for implementing the NPEWMA-SR chart in practice.

4. Examples

To illustrate the effectiveness and the application of the NPEWMA-SR control chart we provide two illustrative examples where the proposed chart is compared to the (i) EWMA-$\overline{X}$ chart, (ii) *1-of-1*, *2-of-2* DR and *2-of-2* KL Shewhart-type SR charts (see Chakraborti and Eryilmaz (2007) for a detailed description of *2-of-2* DR and KL charts, respectively) and the (iii) NPEWMA-SN chart, suitably adapted for $n > 1$. For the three EWMA charts we choose the design parameters (λ, L) so that $ARL_0 \approx 370$ and 500 for Examples 1 and 2, respectively. It should be noted that the industry standard ARL_0 values of 370 and 500 are far from being attainable when using the *1-of-1* Shewhart-type SR chart, because the highest ARL_0 that it can attain for subgroups of size 5 is 16 (see Bakir (2004), page 616). In addition, the *2-of-2* SR charts under the DR and KL schemes also can't attain the industry standard ARL_0 values; see Chakraborti and Eryilmaz (2007) Table 11, where it is shown that the highest ARL_0 value that the *2-of-2* DR scheme can attain for n = 5 is 271.15 when UCL = 15, whereas the *2-of-2* KL scheme can attain ARL_0 values of 136.00 and 526.34 for UCL = 13 and 15, respectively, for n = 5. Although the ARL_0 values of the Shewhart-type SR charts for UCL = 15 when n = 5 are far from the desired nominal ARL values, we include these charts for illustrative purposes.

Example 1

We first illustrate the NPEWMA-SR chart using a well-known dataset from Montgomery (2001; Table 5.2) on the inside diameters of piston rings manufactured by a forging process. Table 5.2 contains fifteen prospective samples each of five observations (n = 5). We assume that the underlying process distribution is symmetric with a known median of 74mm. The values of the SR statistics and the NPEWMA-SR plotting statistics were calculated using (1) and (2), respectively, and are presented in Table 4. The control charts are shown in panels (a) – (d) of Figure 1 along with the values of the control limits.

Table 4. The SR_i statistics and the NPEWMA-SR plotting statistics, Z_i

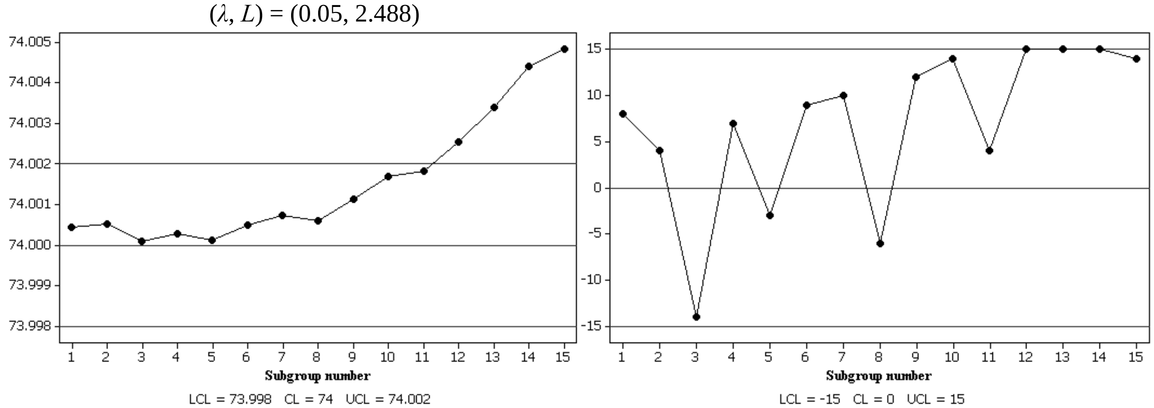
Subgroup number	SR_i	Z_i
1	8	0.400
2	4	0.580
3	-14	-0.149
4	7	0.208
5	-3	0.048
6	9	0.496
7	10	0.971

8	-6	0.622
9	12	1.191
10	14	1.832
11	4	1.940
12	15	2.593
13	15	3.213
14	15	3.803
15	14	4.313

From panels (a), (c) and (d) in Figure 1 we see that the EWMA-$\overline{X}$ control chart is the first to signal at subgroup number 12, whereas the NPEWMA-SN and the NPEWMA-SR charts both signal later at subgroup number 13. This is not surprising, as normal theory counterparts typically outperform nonparametric methods when the assumptions are met and a goodness-of-fit test does not reject normality for these data. The *1-of-1* SR chart signals on subgroup number 12, whereas the *2-of-2* SR charts using the DR and KL signalling rules only signals later on sample number 13. In this example the EWMA-$\overline{X}$ slightly outperformed the nonparametric charts, but it should be noted that the assumptions necessary for the parametric chart seemed to be met. Typically in practice, however, normality can be in doubt or may not be justified for lack of information or data and a nonparametric method may be more desirable. The next example illustrates this.

(a) EWMA-$\overline{X}$
(λ, L) = (0.05, 2.488)

(b) *1-of-1*, *2-of-2* DR and *2-of-2* KL Shewhart-type



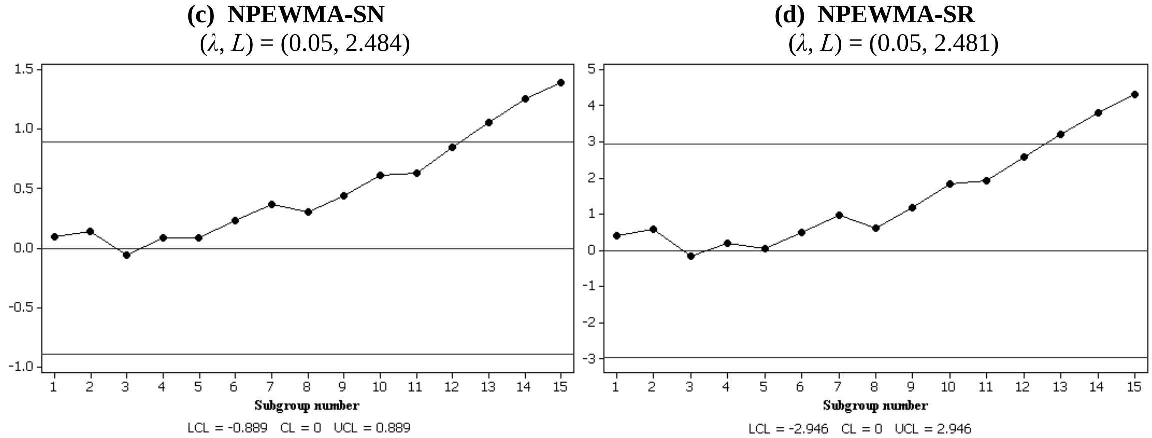


Figure 1. EWMA-$\overline{\boldsymbol{X}}$, *1-of-1, 2-of-2* DR and *2-of-2* KL Shewhart-type signed-rank, NPEWMA-SN and NPEWMA-SR control charts for Example 1.

Example 2

The second example is to illustrate the effectiveness and the application of the nonparametric chart when normality is in doubt use some simulated data from a Logistic distribution with location parameter 0 and scale parameter $\sqrt{3}/\pi$: $LG(0, \sqrt{3}/\pi)$, so that the observations come from a symmetric distribution with a median of zero and a standard deviation of 1. Suppose that the median increases or has sustained an upward step shift of 0.5. Accordingly, subgroups each of size 5 (n = 5) were generated from the Logistic distribution with the same scale parameter but with the location parameter equal to 0.5, resulting in observations that have a median of 0.5 and a standard deviation of 1.

The control charts are shown in panels (a) – (d) of Figure 2 and we observe that the nonparametric EWMA control charts are the first to signal at subgroup number 7, whereas the EWMA-$\overline{X}$ chart signals later at subgroup number 9. The *1-of-1* SR chart signals on subgroup number 7, whereas the *2-of-2* SR charts using the DR and KL signalling rules didn't signal. Although this is an example using simulated data, it shows that there are situations in practice where the NPEWMA-SR chart offers an effective alternative over available parametric and nonparametric control charts.

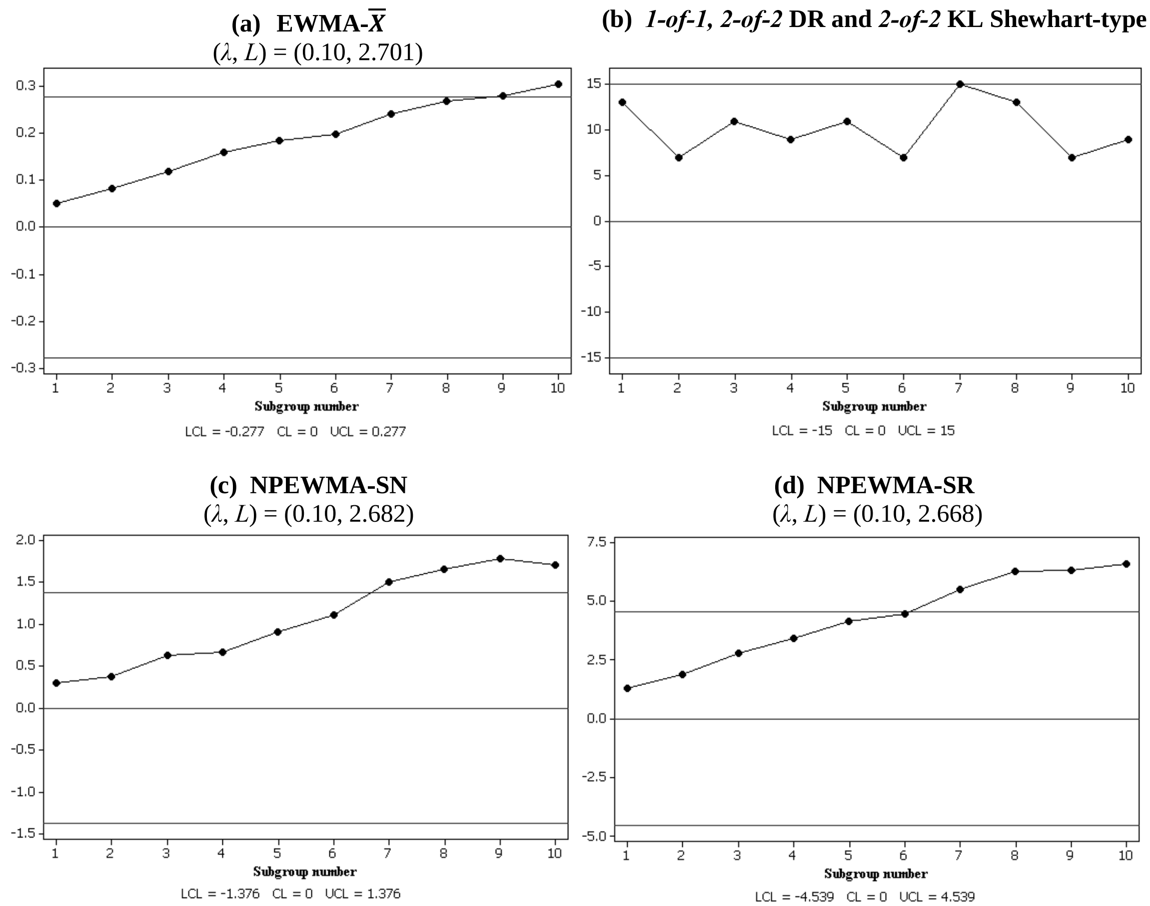


Figure 2. EWMA-$\overline{\boldsymbol{X}}$, *1-of-1, 2-of-2* DR and *2-of-2* KL Shewhart-type signed-rank, NPEWMA-SN and NPEWMA-SR control charts for Example 2.

5. Performance Comparison

The IC performance of a chart shows how robust a chart is whereas the OOC performance needs to be examined to assess the chart's efficacy, that is its effectiveness in detecting a shift. From a practical standpoint, it is also of interest to compare the OOC performance of the NPEWMA-SR chart with existing charts. We first compare the EWMA-type charts, i.e. the NPEWMA-SR chart to the traditional EWMA-$\overline{X}$ and the NPEWMA-SN charts. Following this, we compare the NPEWMA-SR chart to the *1-of-1*, the *2-of-2* DR and the *2-of-2* KL Shewhart-type SR charts.

Our study includes a wide collection of symmetric distributions including the normal and normal-like non-normal distributions: (a) the standard normal distribution, $N(0,1)$; (b) the scaled Student's t-distribution, $t(\nu)/\sqrt{\frac{\nu}{\nu-2}}$, with degrees of freedom $\nu = 4$ and 8, respectively; (c) the Laplace (or double exponential) distribution, $DE(0,1/\sqrt{2})$; (d) the logistic distribution,

$LG(0,\sqrt{3}/\pi)$; (e) the contaminated normal (*CN*) distribution: a mixture of $N(0,\sigma_1^2)$ and $N(0,\sigma_2^2)$, represented by $(1-\alpha)N(0,\sigma_1^2)+\alpha N(0,\sigma_2^2)$.

The *CN* distribution is often used to study the effects of outliers. Note that all distributions in the study have mean/median 0 and are scaled such that they have a standard deviation of 1 so that the results are easily comparable across distributions. Thus, for example, the scale parameters of the Laplace and the Logistic distributions were set equal to $1/\sqrt{2}$ and $\sqrt{3}/\pi$, respectively. For the CN distribution the σ_i's are chosen so that the standard deviation of the mixture distribution equals 1, that is, $(1-\alpha)\sigma_1^2+\alpha\sigma_2^2=1$. We take $\sigma_1/\sigma_2=2$ and the level of contamination $\alpha=0.05$.

5.1 In-control Robustness

Because the NPEWMA-SR and the NPEWMA-SN charts are nonparametric, the IC run-length distribution and the associated characteristics should remain the same for all symmetric continuous distributions. A Markov chain approach was used in the calculations for the two NPEWMA charts whereas for the traditional EWMA-$\bar{X}$ chart, the values of the IC run-length characteristics were estimated using 100,000 simulations as the exact closed-form expressions for the run-length distribution is not available for all the distributions considered in the study; the main stumbling block being the exact distribution of the mean (i.e $\bar{X}$) for small subgroup sizes. The results are shown in Table 5 for λ = 0.01, 0.025, 0.05, 0.10 and 0.20, respectively. Note that, the values of L were chosen such that in each case $ARL_0\approx 500$ and, in case of the EWMA-$\bar{X}$ chart, the values of L were chosen such that the $ARL_0\approx 500$ for the $N(0,1)$ distribution.

The first row of each cell in Table 5 shows the ARL_0 and $SDRL_0$ values, respectively, whereas the second row shows the values of the 5th, 25th, 50th, 75th and 95th percentiles (in this order).

< Insert Table 5 >

For a better understanding of the IC run-length distributions, the values of Table 5 were used to construct boxplot-like graphs (see Radson and Boyd (2005)) for λ = 0.05, 0.10 and 0.20; these graphs are shown in panels (a), (b) and (c), of Figure 3, respectively. Each boxplot shows the mean of the run-length distribution as a square and the median as a circle inside the box and the “whiskers” are extended to the 5th and the 95th percentiles instead of the usual minimum and maximum. Note that only one boxplot is shown for each of the two NPEWMA charts (the first two boxplots on the left), because their IC run-length characteristics are the same for all symmetric continuous distributions and that a reference line was inserted on the vertical axis at 500, which is the desired nominal ARL_0 value in this case.

Several interesting observations can be made from an examination of Table 5 and Figure 3:

i. As expected, both NPEWMA charts are IC robust for all λ and for all distributions under consideration, including the *CN* distribution, indicating that the nonparametric charts are more resistant to outliers. Also, the IC run-length distributions of the NPEWMA-SN and the NPEWMA-SR charts look almost identical. As an aside, comparing the two NPEWMA charts to the *1-of-1*, the *2-of-2* DR and the *2-of-2* KL Shewhart-type SR charts, we find that the two NPEWMA charts are better options, because it offers a more attractive (larger) set of attainable ARL_0 values for use in routine practice; see Tables 1, 2 and 3 of this paper for the NPEWMA-SR chart and Tables 1, 2 and 3 of Graham et al. (2009) for the NPEWMA-SN chart for individuals data (the latter chart was suitably adapted for $n > 1$ and similar tables were constructed, but these are omitted here to conserve space). In Section 4 we pointed out that the highest ARL_0 value of the *1-of-1* and the *2-of-2* DR charts are 16 and 271.15, respectively, while the two highest ARL_0 values of the *2-of-2* KL chart are 136.00 and 526.34, respectively. However, from Table 3 we can see that the NPEWMA-SR chart can attain the industry standard ARL_0 values of 370 and 500 almost exactly; this is also true for the NPEWMA-SN chart (see Graham et al. (2009) Tables 2 and 3).

ii. The EWMA-$\overline{X}$ chart is not IC robust and its run-length distribution has a higher variance as seen from the interquartile ranges. Its IC characteristics vary (sometimes dramatically) as the underlying distribution changes. For example, focussing on the ARL_0 as a measure of location, for $\lambda = 0.20$ (see Figure 3 (c) and Table 5) the ARL_0 of the EWMA-$\overline{X}$ chart varies from 497.31 (when the underlying distribution is $N(0,1)$) to 367.65 (when the underlying distribution is $t(4)$). In addition, for $\lambda = 0.2$, the ARL_0 values of the EWMA-$\overline{X}$ chart are much smaller than 500 (farther below the reference line) for all distributions other than the normal. This is problematic as there will be many more false alarms than what is nominally expected.

iii. The EWMA$-\overline{X}$ chart appears to be less IC robust for larger values of λ, especially for the *CN* distribution. Thus, this chart may be problematic when outliers are likely to be present.

<Insert Figure 3>

5.2 Out-of-control chart Performance Comparison

For the OOC chart performance comparison it is customary to ensure that the ARL_0 values of the competing charts are fixed at (or very close to) an acceptably high value, such as 500 in this case, and then compare their out-of-control *ARL*'s i.e. their ARL_δ values, for specific values of the shift δ; the chart with the smaller ARL_δ value is generally preferred.

Table 6 shows the OOC performance characteristics of the run-length distribution for various distributions and shifts of size δ = 0.5(0.5)2.5 in the mean/median, expressed in terms of the population standard deviation (which, in our case, equals one), for λ = 0.05 and n = 10. It may be noted that in order for the NPEWMA-SR chart to be able to signal after one subgroup (i.e. to obtain an ARL_δ of 1), the maximum allowable value for the *UCL* is $\lambda n(n+1)/2$ and, in general, in order for the chart to be able to signal after the i^{th} subgroup, the maximum allowable *UCL* is $(1-(1-\lambda)^i)n(n+1)/2$. This result can be established by substituting the maximum value of SR_i (equal to $n(n+1)/2$) into equation (2) and rewriting the plotting statistic as $Z_i = \lambda\sum_{j=0}^{i-1}(1-\lambda)^j SR_{i-j} + (1-\lambda)^i Z_0$ by recursive substitution. Thus, the first time the chart can signal is on the subgroup number

$$i \geq \frac{ln(1-2UCL/(n(n+1))}{ln(1-\lambda)}\,. \qquad (10)$$

For example, for n = 10, λ = 0.05 and L = 2.610 (this (λ, L)-combination can be used for ARL_0 is 500 (see Table 5)) we get UCL = 8.200 from (4) and then the right-hand side of (10) equals 3.148. Thus the NPEWMA-SR chart can only signal for the first time on or beyond subgroup number 4, which is confirmed from Table 6. Similar conditions apply to the performance of the NPEWMA-SN chart.

The results of Table 6 can again be displayed as boxplot-like graphs as in Figure 3 for easier understanding but these are omitted here to conserve space. It should be noted that the Markov-chain approach could not be used to obtain the run-length characteristics of the NPEWMA-SR chart for the OOC performance comparisons, because the distribution of the SR statistic is not available for most non-normal distributions and/or when a shift occurred in the process. Consequently, extensive computer simulation was used to estimate these quantities. The simulation algorithm is described below.

Simulation algorithm

Step 1: After specifying the subgroup size and the size of the shift to be detected, we generate random subgroups from a standard normal, Student's *t*, Laplace, Logistic or contaminated normal distribution, respectively.

Step 2: Select the two design parameters, λ and L (see Section 3.2) for a given ARL_0 and shift size.

Step 3: Calculate the SR_i and the plotting statistic Z_i statistics (see equations (1) and (2), respectively) for each subgroup.

Step 4: Calculate the steady-state control limits using equation (4) and compare Z_i to the control limits.

Step 5: The number of subgroups needed until Z_i plots on or outside the control limits is recorded as an observation from the run-length distribution.

Step 6: Repeat steps 1 to 5 a total of 100,000 times.

Step 7: Once we have obtained a "dataset" with 100,000 observations from the run-length distribution, proc univariate of SAS®v 9.1.3 was used to obtain the run-length characteristics.

< Insert Table 6 >

A summary of our observations from the OOC performance characteristics shown in Table 6 is as follows:

i. The NPEWMA-SR chart outperforms the NPEWMA-SN chart for all distributions under consideration except for the Laplace distribution, for which the performances of the charts are very similar (which is not surprising in view of the ARE values mentioned in Section 1). Both nonparametric charts perform significantly better than the EWMA-$\overline{X}$ chart for all distributions except the normal with (δ < 1.5) and even then the performances of the charts are very comparable. Similar conclusions can be drawn for λ = 0.01, 0.025, 0.10 and 0.20 where the run-length characteristics of the NPEWMA-SR chart tends to 6, 4, 3 and 2, respectively, as the shift increases.

ii. For larger shifts in location ($\delta \geq 1.5$), all the values of the run-length characteristics of the NPEWMA-SR chart become smaller and ultimately converge to 4 as the shift increases (due to the restriction given in (10)) and those of the NPEWMA-SN chart also become smaller and ultimately converge to 3 as this shift increases (due to a similar type of restriction) and those of the EWMA-$\overline{X}$ can (and do) get smaller.

Next we compare the OOC performance of the NPEWMA-SR chart to that of the Shewhart-type SR charts. Table 14 of Chakraborti and Eryilmaz (2007) give the ARL values for n = 10 for the *1-of-1*, the *2-of-2* DR and the *2-of-2* KL Shewhart-type SR charts, respectively. Note that the control limits were chosen such that the $ARL_0 \approx 480$ for each chart.

Table 7. *ARL* values under the *N*(0,1) distribution when *n* = 10.

Shift	*1-of-1* *UCL*/*LCL* = ± 55	*2-of-2* DR *UCL*/*LCL* = ± 39	*2-of-2* KL *UCL*/*LCL* = ± 37	NPEWMA-SR (λ = 0.05, *L* = 2.595) *UCL*/*LCL* = ± 8.153
0.0	± 480.00	± 480.00	± 480.00	± 480.00
0.2	208.76	147.19	113.17	22.25
0.4	66.93	30.37	22.52	9.56
0.6	25.22	9.60	7.51	6.43
0.8	10.72	4.49	3.89	5.11
1.0	5.64	2.90	2.66	4.44
1.2	3.37	2.31	2.22	4.11

From Table 7 we find that:

i. The NPEWMA-SR chart far outperforms all charts for shifts in location of 0.6 standard deviations or less.

ii. For shifts in the location of 0.8 standard deviations and larger, the performances of the charts are similar, particularly that of the runs-rule enhanced charts and the NPEWMA-SR charts.

iii. The *ARL* of the NPEWMA-SR charts tends to 4 as the shift increases. This is due to the restriction (10) as explained before.

The first row of each cell in Table 8 shows the ARL_0 and $SDRL_0$ values, respectively, whereas the second row shows the values of the 5th, 25th, 50th, 75th and 95th percentiles (in this order) for the traditional and the nonparametric EWMA charts, for the normal distribution when the standard deviation increases from 1 to 10. We see that while the NPEWMA-SR chart is insensitive to misspecification or changes in the variance, the traditional EWMA-$\overline{X}$ is clearly not. In fact, a two fold increase of the standard deviation can have a very significant effect on the ARL_0 of the EWMA-$\overline{X}$ chart. Thus while for the traditional EWMA-$\overline{X}$ chart a shift in the variance can easily lead to a signal on the location chart that is not the case with the NPEWMA-SR chart.

Table 8. Performance characteristics of the IC run-length distribution for the NPEWMA-SR and the EWMA-$\overline{X}$ chart with *n* = 10 for *N*(0, σ^2) data.

σ	**NPEWMA-SR** (λ = 0.05, *L* = 2.595)	**EWMA-$\overline{X}$** (λ = 0.05, *L* = 2.602)
1	482.28 (467.86) 38, 149, 339, 663, 1416	481.82 (465.87) 38, 150, 340, 662, 1413
2		32.69 (28.48) 5, 13, 24, 44, 89
3		13.44 (11.22) 3, 6, 10, 18, 36
4		7.99 (6.50) 2, 3, 6, 11, 21
10		2.33 (1.69) 1, 1, 2, 3, 6

6. Concluding Remarks

EWMA charts take advantage of the sequentially (time ordered) accumulating nature of the data arising in a typical SPC environment and are known to be more efficient in detecting smaller shifts. The traditional parametric EWMA-$\overline{X}$ chart can lack in-control robustness and as such the corresponding false alarm rates can be a practical concern. Nonparametric EWMA charts offer an attractive alternative in such situations as they combine the inherent advantages of nonparametric charts (IC robustness) with the better small shift detection capability of EWMA-type charts. We study the nonparametric EWMA control chart based on the signed-rank statistic and its properties via the in-control and out-of-control run-length distribution using a Markov chain approach and simulation, respectively. A performance comparison of the NPEWMA-SR chart is done with its competitors: the EWMA-$\overline{X}$ chart, the *1-of-1*, the *2-of-2* DR and the *2-of-2* KL Shewhart-type signed-rank charts and the NPEWMA chart based on signs, and it is seen that the NPEWMA-SR chart performs as well as and, in many cases, better than its competitors. Thus, on the basis of minimal required assumptions, robustness of the in-control run-length distribution and out-of-control performance, the NPEWMA-SR chart is a strong contender in practical SPC applications. Note that, the focus in this article has been the situation where the process median is known or specified in advance. Adaptations to the case where the median is unknown or unspecified are currently being investigated and will be reported in a separate paper.

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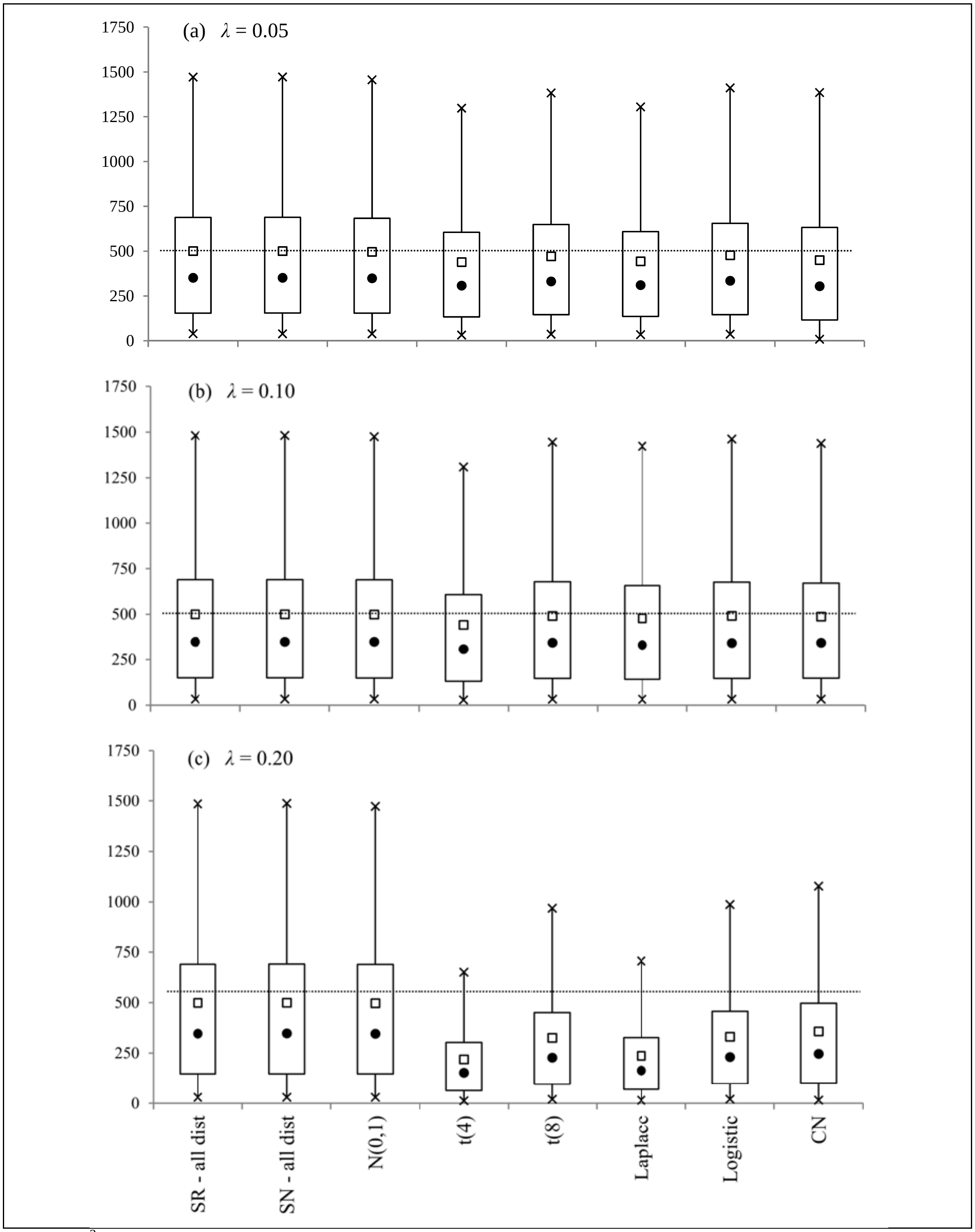

Figure 3[2]. Boxplot-like graphs of the IC run-length distributions of the NPEWMA-SR chart (first boxplot on the left), the NPEWMA-SN chart (second boxplot to the left) and the EWMA- $\overline{X}$ chart (remaining 6 boxplots on the right)

[2]Panel (a): NPEWMA-SR (λ=0.05, L=2.610); NPEWMA-SN (λ=0.05, L=2.612); EWMA- $\overline{X}$ (λ=0.05, L=2.613)
Panel (b): NPEWMA-SR (λ=0.10, L=2.794); NPEWMA-SN (λ=0.10, L=2.797); EWMA- $\overline{X}$ (λ=0.10, L=2.815)
Panel (c): NPEWMA-SR (λ=0.20, L=2.905); NPEWMA-SN (λ=0.20, L=2.933); EWMA- $\overline{X}$ (λ=0.20, L=2.962)

Table 1. Performance characteristics of the IC run-length distribution for the NPEWMA-SR chart with n = 5.

L	***λ***				
	Small shifts			**Moderate shifts**	**Large shifts**
	0.01	**0.025**	**0.05**	**0.10**	**0.20**
2.0	525.37 (483.82) 64, 182, 378, 713, 1490	229.47 (211.92) 28, 79, 165, 311, 652	127.18 (117.83) 15, 43, 91, 173, 362	73.72 (68.60) 9, 25, 53, 100, 211	46.05 (43.21) 5, 15, 33, 63, 132
2.1	642.12 (596.56) 74, 218, 460, 873, 1832	281.79 (262.60) 32, 95, 201, 383, 806	156.62 (146.42) 17, 52, 112, 213, 449	91.51 (85.95) 10, 30, 65, 125, 263	58.07 (54.94) 6, 19, 41, 79, 168
2.2	788.31 (738.60) 86, 263, 562, 1074, 2262	347.83 (326.92) 37, 115, 248, 474, 1000	194.21 (183.14) 20, 64, 138, 265, 560	114.41 (108.39) 12, 37, 81, 156, 331	73.92 (70.50) 7, 24, 52, 101, 215
2.3	974.71 (920.71) 100, 320, 693, 1331, 2812	431.13 (408.49) 43, 140, 306, 589, 1246	242.64 (230.66) 24, 78, 172, 332, 703	144.31 (137.82) 14, 46, 102, 198, 419	95.16 (91.52) 8, 30, 67, 131, 278
2.4	1214.47 (1156.06) 117, 392, 860, 1661, 3521	539.08 (514.64) 51, 173, 321, 738, 1566	305.68 (292.78) 28, 97, 216, 419, 890	183.97 (177.00) 16, 58, 130, 252, 537	123.83 (119.93) 10, 38, 87, 170, 363
2.5	1517.63 (1454.79) 137, 482, 1072, 2080, 4421	677.62 (651.38) 60, 214, 478, 929, 1977	386.96 (373.15) 33, 121, 273, 531, 1132	236.12 (228.68) 19, 73, 166, 324, 692	163.43 (159.27) 12, 50, 115, 225, 481
2.6	1918.28 (1850.91) 162, 600, 1351, 2633, 5612	860.65 (832.58) 71, 268, 605, 1182, 2522	496.96 (481.21) 39, 153, 348, 682, 1456	307.15 (299.22) 23, 94, 215, 423, 904	220.15 (215.72) 16, 66, 154, 303, 651
2.7	2436.64 (2364.77) 193, 753, 1711, 3350, 7156	1102.44 (1072.54) 85, 339, 773, 1517, 3243	640.44 (624.75) 48, 195, 449, 882, 1887	404.57 (396.15) 29, 122, 283, 558, 1195	300.03 (295.35) 20, 90, 209, 414, 889
2.8	3128.26 (3051.86) 233, 955, 2192, 4307, 9219	1417.73 (1386.01) 103, 431, 993, 1953, 4184	838.61 (821.99) 59, 253, 586, 1156, 2479	541.06 (532.15) 36, 162, 378, 747, 1603	417.77 (412.83) 26, 124, 291, 577, 1242
2.9	4053.52 (3972.60) 285, 1224, 2835, 5588, 11982	1860.88 (1827.32) 127, 559, 1300, 2567, 5508	1108.26 (1090.69) 74, 331, 774, 1530, 3285	730.87 (721.46) 46, 217, 510, 1010, 2171	590.31 (585.08) 35, 174, 411, 816, 1758
3.0	5309.20 (5223,82) 354, 1588, 3706, 7327, 15734	2456.38 (2421.01) 160, 732, 1714, 3392, 7288	1471.46 (1452.99) 93, 437, 1026, 2033, 4371	997.49 (987.60) 61, 294, 694, 1379, 2968	856.39 (850.86) 49, 250, 595, 1185, 2554

Table 2. Performance characteristics of the IC run-length distribution for the NPEWMA-SR chart with n = 10.

L	***λ***				
	Small shifts			**Moderate shifts**	**Large shifts**
	0.01	**0.025**	**0.05**	**0.10**	**0.20**
2.0	526.24 (484.78) 64, 182, 378, 714, 1493	230.19 (212.76) 28, 79, 165, 312, 655	127.34 (118.12) 15, 43, 91, 173, 363	73.50 (68.52) 8, 25, 53, 100, 210	45.18 (42.44) 5, 15, 32, 62, 130
2.1	643.37 (597.91) 74, 218, 461, 875, 1836	282.21 (263.15) 32, 95, 202, 384, 807	156.79 (146.74) 17, 52, 112, 214, 450	90.91 (85.52) 10, 30, 65, 124, 262	56.44 (53.49) 6, 18, 40, 77, 163
2.2	790.58 (740.97) 86, 264, 564, 1077, 2269	347.75 (327.01) 37, 115, 248, 474, 1000	193.83 (182.93) 20, 64, 138, 265, 559	113.21 (107.38) 11, 37, 80, 155, 327	71.25 (68.09) 7, 23, 50, 98, 207
2.3	976.99 (923.11) 100, 320, 694, 1334, 2819	431.42 (408.95) 43, 140, 306, 590, 1247	241.48 (229.70) 23, 78, 171, 330, 700	142.18 (135.91) 13, 45, 101, 195, 413	90.73 (87.34) 8, 29, 64, 124, 265
2.4	1211.01 (1152.78) 117, 391, 858, 1657, 3511	538.45 (514.21) 50, 172, 381, 737, 1565	302.73 (290.07) 27, 96, 214, 415, 882	179.97 (173.24) 16, 57, 127, 247, 526	116.62 (113.01) 9, 36, 82, 160, 342
2.5	1520.23 (1457.55) 137, 483, 1073, 2083, 4429	676.74 (650.72) 59, 213, 477, 928, 1975	383.02 (369.46) 33, 120, 270, 526, 1120	229.79 (222.61) 19, 71, 162, 316, 674	151.71 (147.86) 11, 46, 106, 209, 447
2.6	1916.65 (1849.51) 162, 600, 1350, 2631, 5607	857.99 (830.16) 70, 267, 603, 1179, 2515	488.46 (473.99) 39, 151, 343, 672, 1434	296.15 (288.51) 22, 91, 208, 408, 872	199.65 (195.58) 14, 60, 140, 275, 590
2.7	2438.25 (2366.61) 193, 753, 1712, 3353, 7161	1096.87 (1067.23) 84, 337, 770, 1509, 3227	630.57 (615.18) 47, 192, 442, 868, 1858	386.19 (378.10) 28, 117, 270, 532, 1141	265.79 (261.48) 18, 80, 186, 367, 788
2.8	3131.18 (3055.01) 233, 955, 2194, 4311, 9228	1415.89 (1384.45) 103, 430, 991, 1951, 4179	817.76 (801.47) 57, 247, 572, 1127, 2417	508.59 (500.05) 34, 152, 355, 702, 1507	358.54 (354.00) 23, 106, 250, 495, 1065
2.9	4056.22 (3975.58) 285, 1225, 2837, 5592, 11990	1853.14 (1819.89) 127, 557, 1295, 2556, 5485	1076.12 (1058.94) 72, 322, 751, 1485, 3189	678.68 (669.69) 43, 202, 473, 937, 2015	490.96 (486.19) 30, 145, 342, 679, 1461
3.0	5298.98 (5213.92) 353, 1585, 3699, 7313, 15704	2430.95 (2395.95) 158, 724, 1696, 3357, 7213	1427.59 (1409.53) 90, 424, 995, 1972, 4241	913.59 (904.16) 56, 270, 636, 1263, 2718	678.75 (673.76) 40, 199, 472, 939, 2023

Table 5. Performance characteristics of the IC run-length distribution for the NPEWMA-SR chart, the NPEWMA-SN chart and the EWMA-$\overline{X}$ chart for selected (λ, L)-combinations and n = 10.

NPEWMA-SR chart					
(λ, L)	**(0.01, 1.975)**	**(0.025, 2.367)**	**(0.05, 2.610)**	**(0.10, 2.794)**	**(0.20, 2.905)**
For all symmetric continuous distributions	500.51 (460.04) 62, 174, 360, 679, 1418	500.06 (476.41) 48, 161, 354, 684, 1451	500.67 (486.10) 40, 154, 352, 688, 1471	500.13 (491.61) 34, 150, 349, 690, 1481	498.92 (494.15) 30, 147, 347, 690, 1485
NPEWMA-SN chart					
(λ, L)	**(0.01, 1.973)**	**(0.025, 2.369)**	**(0.05, 2.612)**	**(0.10, 2.797)**	**(0.20, 2.933)**
For all continuous distributions	498.08 (457.78) 62, 173, 358, 675, 1411	499.21 (475.65) 48, 161, 353, 683, 1448	501.04 (486.58) 39, 155, 352, 689, 1472	500.25 (491.88) 34, 150, 349, 690, 1482	499.64 (495.00) 30, 147, 348, 691, 1488
EWMA-$\overline{X}$ chart					
Dist / **(λ, L)**	**(0.01, 1.975)**	**(0.025, 2.368)**	**(0.05, 2.613)**	**(0.10, 2.815)**	**(0.20, 2.962)**
***N*(0,1)**	500.73 (460.49) 61, 173, 360, 678, 1424	499.25 (476.72) 47, 161, 353, 682, 1447	496.37 (482.62) 39, 152, 350, 681, 1462	498.96 (490.01) 34, 149, 349, 689, 1475	497.31 (492.20) 30, 147, 346, 688, 1479
***t*(4)**	524.98 (485.57) 61, 180, 376, 712, 1500	497.84 (479.58) 44, 158, 352, 678, 1447	480.84 (470.36) 38, 148, 337, 661, 1421	441.57 (436.35) 29, 131, 308, 608, 1309	367.65 (365.04) 22, 108, 255, 509, 1094
***t*(8)**	508.37 (469.60) 61, 175, 366, 688, 1437	497.66 (474.17) 46, 160, 353, 682, 1437	494.13 (478.31) 39, 153, 349, 682, 1445	490.80 (479.81) 33, 147, 344, 678, 1445	471.10 (466.43) 28, 137, 329, 653, 1407
Laplace	512.94 (471.37) 62, 176, 369, 698, 1457	493.12 (470.21) 45, 158, 350, 677, 1431	491.87 (479.56) 39, 150, 345, 675, 1450	477.52 (473.51) 32, 142, 331, 657, 1423	438.70 (434.15) 26, 129, 305, 607, 1300
Logistic	506.92 (467.73) 62, 175, 364, 687, 1443	498.93 (475.23) 47, 159, 353, 684, 1446	491.81 (479.10) 39, 152, 345, 677, 1452	491.58 (485.19) 33, 147, 342, 676, 1462	473.63 (471.09) 28, 138, 328, 654, 1416
CN	332.72 (436.21) 2, 22, 163, 481, 1221	431.71 (475.43) 4, 89, 281, 611, 1379	494.67 (479.24) 39, 152, 349, 683, 1448	487.51 (477.50) 33, 148, 343, 671, 1438	476.14 (473.16) 29 ,140, 331, 662, 1411

Table 6[3]. The OOC performance characteristics of the run-length distribution for the EWMA-$\overline{X}$, the NPEWMA-SN and the NPEWMA-SR charts for $\lambda = 0.05$, $n = 10$ and number of simulations = 100,000.

	EWMA-$\overline{X}$ chart with $\lambda = 0.05$ and L such that $ARL_0 \approx 500$						**NPEWMA-SR chart with $\lambda = 0.05$ and L such that $ARL_0 \approx 500$**					
	L	**Shift (number of standard deviations)**						**Shift (number of standard deviations)**				
		0.5	**1**	**1.5**	**2**	**2.5**		**0.5**	**1**	**1.5**	**2**	**2.5**
***N*(0,1)**	***L*=2.613**	6.71 (1.89) 4, 5, 6, 8, 10	3.33 (0.64) 2, 3, 3, 4, 4	2.26 (0.44) 2, 2, 2, 3, 3	1.98 (0.15) 2, 2, 2, 2, 2	1.68 (0.47) 1, 1, 2, 2, 2	***L*=2.610**	7.65 (1.97) 5, 6, 7, 9, 11	4.46 (0.58) 4, 4, 4, 5, 5	4.00 (0.07) 4, 4, 4, 4, 4	4.00 (0.00) 4, 4, 4, 4, 4	4.00 (0.00) 4, 4, 4, 4, 4
***t*(4)**	***L*=2.682**	30.94 (17.73) 11, 18, 27, 39, 65	11.76 (4.21) 6, 9, 11, 14, 20	7.29 (2.01) 5, 6, 7, 8, 11	5.34 (1.25) 4, 5, 5, 6, 8	4.26 (0.89) 3, 4, 4, 5, 6	***L*=2.610**	6.51 (1.47) 5, 5, 6, 7, 9	4.27 (0.47) 4, 4, 4, 5, 5	4.01 (0.11) 4, 4, 4, 4, 4	4.00 (0.02) 4, 4, 4, 4, 4	4.00 (0.01) 4, 4, 4, 4, 4
***t*(8)**	***L*=2.640**	29.53 (16.99) 10, 18, 25, 37, 62	11.50 (4.22) 6, 9, 11, 14, 19	7.18 (2.05) 4, 6, 7, 8, 11	5.27 (1.27) 4, 4, 5, 6, 8	4.20 (0.90) 3, 4, 4, 5, 6	***L*=2.610**	7.21 (1.77) 5, 6, 7, 8, 10	4.39 (0.55) 4, 4, 4, 5, 5	4.01 (0.09) 4, 4, 4, 4, 4	4.00 (0.01) 4, 4, 4, 4, 4	4.00 (0.00) 4, 4, 4, 4, 4
Laplace	***L*=2.666**	30.48 (17.58) 11, 18, 26, 38, 65	11.68 (4.27) 6, 9, 11, 14, 20	7.24 (2.05) 4, 6, 7, 8, 11	5.32 (1.27) 4, 4, 5, 6, 8	4.23 (0.89) 3, 4, 4, 5, 6	***L*=2.610**	6.54 (1.51) 5, 5, 6, 7, 9	4.34 (0.52) 4, 4, 4, 5, 5	4.02 (0.13) 4, 4, 4, 4, 4	4.00 (0.02) 4, 4, 4, 4, 4	4.00 (0.00) 4, 4, 4, 4, 4
Logistic	***L*=2.635**	29.46 (17.00) 10, 17, 25, 37, 62	11.47 (4.22) 6, 8, 11, 14, 19	7.17 (2.05) 4, 6, 7, 8, 11	5.26 (1.27) 4, 4, 5, 6, 8	4.20 (0.90) 3, 4, 4, 5, 6	***L*=2.610**	7.20 (1.77) 5, 6, 7, 8, 10	4.39 (0.55) 4, 4, 4, 5, 5	4.01 (0.10) 4, 4, 4, 4, 4	4.00 (0.01) 4, 4, 4, 4, 4	4.00 (0.00) 4, 4, 4, 4, 4
CN	***L*=2.656**	24.49 (18.26) 3, 11, 20, 33, 59	7.42 (4.73) 2, 4, 6, 10, 16	3.82 (2.20) 1, 2, 3, 5, 8	2.45 (1.28) 1, 2, 2, 3, 5	1.78 (0.85) 1, 1, 2, 2, 3	***L*=2.610**	7.42 (1.87) 5, 6, 7, 8, 11	4.41 (0.56) 4, 4, 4, 5, 5	4.01 (0.08) 4, 4, 4, 4, 4	4.00 (0.01) 4, 4, 4, 4, 4	4.00 (0.00) 4, 4, 4, 4, 4
	NPEWMA-SN chart with $\lambda = 0.05$ and L such that $ARL_0 \approx 500$											
***N*(0,1)**	***L*=2.612**	9.01 (2.76) 5, 7, 9, 11, 14	4.78 (0.85) 4, 4, 5, 5, 6	3.65 (0.57) 3, 3, 4, 4, 4	3.15 (0.35) 3, 3, 3, 3, 4	3.01 (0.12) 3, 3, 3, 3, 3						
***t*(4)**	***L*=2.612**	6.94 (1.76) 5, 6, 7, 8, 10	4.21 (0.69) 3, 4, 4, 5, 5	3.47 (0.53) 3, 3, 3, 4, 4	3.16 (0.37) 3, 3, 3, 3, 4	3.05 (0.22) 3, 3, 3, 3, 4						
***t*(8)**	***L*=2.612**	8.08 (2.31) 5, 6, 8, 9, 12	4.53 (0.77) 3, 4, 4, 5, 6	3.58 (0.56) 3, 3, 4, 4, 4	3.17 (0.38) 3, 3, 3, 3, 4	3.04 (0.19) 3, 3, 3, 3, 3						
Laplace	***L*=2.612**	6.56 (1.59) 5, 5, 6, 7, 9	4.29 (0.71) 3, 4, 4, 5, 5	3.57 (0.55) 3, 3, 4, 4, 4	3.22 (0.42) 3, 3, 3, 3, 4	3.07 (0.25) 3, 3, 3, 3, 4						
Logistic	***L*=2.612**	8.00 (2.26) 5, 6, 8, 9, 12	4.53 (0.77) 3, 4, 4, 5, 6	3.59 (0.56) 3, 3, 4, 4, 4	3.18 (0.39) 3, 3, 3, 3, 4	3.04 (0.20) 3, 3, 3, 3, 3						
CN	***L*=2.612**	8.61 (2.57) 5, 7, 8, 10, 13	4.65 (0.81) 4, 4, 5, 5, 6	3.59 (0.56) 3, 3, 4, 4, 4	3.14 (0.35) 3, 3, 3, 3, 4	3.02 (0.15) 3, 3, 3, 3, 3						

[3] The values of the run-length characteristics of the NPEWMA-SR chart become smaller and ultimately converge to 4 as the shift increases (due to the restriction given in (10)), those of the NPEWMA-SN chart also become smaller and ultimately converge to 3 as this shift increases (due to a similar type of restriction) and those of the EWMA-$\overline{X}$ can (and do) get smaller.