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A NONSTANDARD PROOF OF THE JORDAN CURVE THEOREM

LOUIS NARENS

In this paper a proof of the Jordan curve theorem will be presented. Some familiarity with the basic notions of nonstandard analysis is assumed. The rest of the paper is self-contained except for some standard theorems about polygons.

The theorem will be proved in what ought to be a natural way: by approximation by polygons. This method is not usually found in the standard proofs since the approximating sequence of polygons is often unwieldly. But by using nonstandard analysis, one can approximate a Jordan curve by a single polygon that is infinitesimally close to the curve. This allows types of reasoning which are extremely difficult and unnatural on sequences of polygons.

Preliminaries. The basic concepts of nonstandard analysis and some acquaintance with polygons are assumed. Some basic definitions and theorems of point set topology are also assumed.

Throughout this paper the following notations and conventions will be used:

(1) All discussion, unless otherwise stated, is assumed to be about a nonstandard model of the Euclidean plane. "Otherwise stated" will often mean that the notion or concept will be prefaced by the word "standard".

(2) A standard concept and its extension will be denoted by the same symbol. If it is necessary to distinguish between them, reference to the model in which they are to be interpreted will be made.

(3) If A and B are sets of points and x is a point, then $|x, A|$ will denote the distance from x to A and $|B, A| = \inf_{x \in B} |x, A|$. (Thus if $A \cap B \neq \emptyset$ then $|A, B| = 0$.) $|x, y|$ will denote the distance from the point x to the point y.

(4) f will denote a fixed continuous function on $[0, 1]$ into the Euclidean plane with the property that $x < y$ and $f(x) = f(y)$ if and only if $x = 0$ and $y = 1$. C will denote the range of f.

(5) $x \cong y$ will mean that the distance from x to y is infinitesimal. If x is near-standard then ${}^\circ x$ will denote the standard y such that $x \cong y$.

(6) If x and y are points then xy will denote the *ordered*, closed line segment that begins at x and ends at y.

(7) If x and y are points then intv (x, y) is the set of all points z of xy such that $z \neq x$ and $z \neq y$.

2. The approximation of C. In this section the Jordan curve C will be approximated by a polygon P. Not all approximations of C by polygons are suitable for our purposes. A particular sort of approximation, called a *good* approximation, will be constructed. A good approximation not only approximates the point set C, but also has associated with it a function h, $h: P \to [0, 1]$, that approximates f^{-1} in the following sense: the point set C has the property that if $x, y \in C$ then $x \cong y$ if and only if $f^{-1}(x) \cong f^{-1}(y)$ or

$$|f^{-1}(x) - f^{-1}(y)| \cong 1 ;$$

h and P will have the property that if $x, y \in P$ then $x \cong y$ if and only if $h(x) \cong h(y)$ or $|h(x) - h(y)| \cong 1$.

DEFINITION 2.1. *A good approximation for C* is a simple closed polygon P and a function $h: P \to [0, 1]$ which assumes only (nonstandardly) finitely many values such that,

(1) if $x \in C$ then there is a $y \in P$ such that $x \cong y$,
(2) if $x \in P$ then there is a $y \in C$ such that $x \cong y$,
(3) if $x, y \in P$ and $h(x) \cong h(y)$ then $x \cong y$,
(4) if $x, y \in P$ and $x \cong y$ then $h(x) \cong h(y)$ or $|h(x) - h(y)| \cong 1$
(5) there are points $K = \{k_0, \cdots, k_s\}$ (where s is an infinite natural number) such that $P = (\bigcup_{i<s} k_i k_{i+1}) \cup k_s k_0$ and such that:
  (a) if $x = k_0$ then $h(x) = 0$,
  (b) if for $i < s$, $x \in k_i k_{i+1}$ and $x \neq k_i$ then $h(k_i) < h(x) = h(k_{i+1})$,
  (c) if $x \neq k_0$ and $x \in k_s k_0$ then $h(x) = 1$.

DEFINITION 2.2. Part (5) of Definition 2.1 gives a natural method of ordering P in terms of $\{k_0, \cdots, k_s\}$. For $x, y \in P$ and $x \neq y$ define $x < y$ if and only if (1) $x = k_0$, or (2) $x \in k_i k_{i+1}$ and $y \in k_j k_{j+1}$ and $i < j$, or (3) $x \in K_i K_{i+1}$ and $y \in K_s K_0$ and $y \neq K_0$, or (4) x and y belong to the same ordered segment ($k_i k_{i+1}$ or $k_s k_0$) and x comes before y in the ordering of that segment. (Also note that if $x, y \in P$ and $h(x) < h(y)$ then $x < y$.)

THEOREM 2.1. *A good approximation for C exists.*

Proof. Let M be an infinite natural number and

$$\beta_1 = \max_{|t'-t| \leq 1/M} |f(t), f(t')| \text{ and } \beta_2 = \max_{|1+t'-t| \leq 1/M} |f(t), f(t')|$$

and $\beta = \max\{\beta_1, \beta_2\}$. Since f is standard continuous, β is infinitesimal. Divide $[0, 1]$ into M equal intervals $[t_i, t_{i+1}]$ with $t_0 = 0$ and $t_M = 1$. For $i < M$ let $a_i = f(t_i)$. Then $|a_i, a_{i+1}| \leq \beta$. The points k_i will be defined inductively. $k_0 = a_0$, $k_{i+1} = a_p$ if and only if $|a_p, k_i| \leq \beta$ and for each $j > p$ either $|a_j, k_i| > \beta$ or $|f^{-1}(a_j) - f^{-1}(k_i)| \geq \frac{1}{2}$, where

$f^{-1}(x)$ is the least t such that $f(t) = x$. Since there are only "finitely" many a_i, there will be a last k_i defined. This last element is denoted by k_l. The points k_i will have the following properties:

(A1) $f^{-1}(k_l) \cong 1$,

(A2) if $a \in C$ then there is an $i \leqq l$ such that $k_i \cong a$,

(A3) for each $i \leqq l$ there is an $a \in C$ such that $k_i \cong a$,

(A4) if $f^{-1}(k_j)$ is not in the monad of 1 and $j > i + 1$ then $|k_i, k_j| > \beta$,

(A5) if $i + 1 < l$ then $k_i k_{i+1} \cap k_{i+1} k_{i+2} = \{k_{i+1}\}$,

(A6) if $0 \leqq p < i - 2$ and $f^{-1}(k_i)$ is not in the monad of 1 then $k_p k_{p+1} \cap k_i k_{i+1} = \varnothing$,

(A7) if $f^{-1}(k_p)$ is not in the monad of 0, $f^{-1}(k_q) \cong 1$, and $p \leqq q - 2$, then $k_p k_{p+1} \cap k_{q-1} k_q = \varnothing$.

Proof of (A1). Let $k_l = a_p$. $|a_p, a_M|$ must be infinitesimal or otherwise k_{l+1} would exist. This means that $f^{-1}(a_p) \cong 1$ or $f^{-1}(a_p) \cong 0$. If $f^{-1}(a_p) \cong 0$ then k_{l+1} would exist. Therefore $f^{-1}(a_p) = f^{-1}(k_l) \cong 1$.

Proof of (A2). By the method in which the a_i were defined, there is an index p such that $a_p \cong a$. Let j be the largest "natural number" $\leqq l$ such that $f^{-1}(k_j) \leqq f^{-1}(a_p)$. If $f^{-1}(a_p) - f^{-1}(k_j)$ were not infinitesimal then $f^{-1}(k_{j+1}) < f^{-1}(a_p)$. Therefore $f^{-1}(a_p) - f^{-1}(k_j)$ is infinitesimal and thus $k_j \cong a_p \cong a$.

Proof of (A3). Let $a = k_i$.

Proof of (A4). Immediately follows from the definition of k_{i+1}.

Proof of (A5). Suppose not. Let $b \in k_i k_{i+1} \cap k_{i+1} k_{i+2}$ and $b \neq k_{i+1}$. Then b, k_i, k_{i+1}, k_{i+2} are collinear. Since $|k_i, k_{i+2}| > \beta$, $k_{i+2} \notin k_i k_{i+1}$ and $k_i \notin k_{i+1} k_{i+2}$. But this can only happen to collinear points k_i, k_{i+1}, k_{i+2} if and only if $k_i k_{i+1} \cap k_{i+1} k_{i+2} = \{k_{i+1}\}$.

Proof of (A6). Assume that $k_p k_{p+1} \cap k_i k_{i+1} \neq \varnothing$. Then $k_p k_i, k_i k_{p+1}, k_{p+1} k_{i+1}, k_{i+1} k_p$ form the sides of a quadralateral. Without loss of generality assume that the angle at $k_p \geqq \pi/2$ radians. Then in the triangle with vertices at k_i, k_p, and k_{i+1}, the angle at $k_p \geqq \pi/2$ radians. This makes $k_i k_{i+1}$ the longest side. Therefore $|k_p, k_i| \leqq |k_i, k_{i+1}| \leqq \beta$. This contradicts (A4).

Proof of (A7). Similar to (A6).

If $i < l$, $x \in k_i k_{i+1}$, and $x \neq k_i$ define $h(x) = f^{-1}(k_{i+1})$.

$P_1 = \bigcup_{i<l} k_i k_{i+1}$ and h almost form a good approximation for C. Unfortunately P_1 is not a closed polygon and may intersect itself in the monad of k_0. To form a simple closed polygon from P_1 we let k_j be such that $h(k_j)$ is not in the monad of 0 or 1. Let

$$P_2 = \bigcup_{i \leqq j-1} k_i k_{i+1} \text{ and } P_3 = \bigcup_{j \leqq i < l} k_i k_{i+1} .$$

By (A5), (A6), and (A7) P_2 and P_3 are simple polygonal paths. P_2 and P_3 can be ordered in such a way that P_2 starts at k_0 and ends at k_j and P_3 starts at k_j and ends at k_l. Let $E = k_l k_0 \cap P_3$. Since $k_l k_0$ and P_3 are closed sets and $k_l \in P_3 \cap k_l k_0$, let e be the smallest member of E in the ordering induced by P_3. Then

$$P_4 = \{x \in P_3 \mid k_j \leqq x \leqq e\} \cup ek_0$$

is also a simple polygonal path. For $x \in ek_0$ define $h(x) = 1$. P_4 can be ordered in such a way that P_4 starts at k_j and ends at k_0. Since $k_0 \in P_2 \cap P_4$, let e_1 be the greatest member in the ordering of P_2 of $P_2 \cap P_4$. Then $P = \{x \in P_2 \mid e_1 \leqq x \leqq k_j\} \cup \{x \in P_4 \mid k_j \leqq x \leqq e_1\}$ is a simple closed polygon. Let $h_1(e_1) = 0$ and for $x \in P - \{e_1\}$ let $h_1(x) = h(x)$. Then P and h_1 will form a good approximation for C by (A2) and (A3) and this construction.

Notation. Throughout the rest of this paper, we let the polygon P and the function h be a fixed good approximation for C.

3. **The inside and outside of C.** The following theorem and other easily established facts about polygons will be used without proof in this paper. A proof of the following theorem can be found in [1].

THEOREM 3.1. (*The Jordan curve theorem for polygons.*) *A simple closed polygon Q divides the Euclidean plane into three non-empty disjoint sets, the polygon itself and two open components. One of the components is bounded and is called the inside of Q, and the other component is unbounded and called the outside of Q.*

DEFINITION 3.1. The *inside of* C is the set of all *standard* points x such that x is inside P and $|P, x|$ is not infinitesimal.

DEFINITION 3.2. The *outside of* C is the set of all *standard* points x such that x is outside P and $|P, x|$ is not infinitesimal.

THEOREM 3.2. *Each standard point is either in the inside of C, in the outside of C, or is on C.*

Proof. Suppose that x is a standard point and x is not in the inside of C and x is not in the outside of C. Then either x is inside P and $|P, x| \cong 0$, or x is outside P and $|P, x| \cong 0$, or x is on P. In any case, $|P, x| \cong 0$. Since P is a good approximation for C, $|C, x| \cong 0$. Thus there is a point y on C such that $x \cong y$. Let $t = f^{-1}(y)$. Let $^\circ t$ be the standard real number in $[0, 1]$ that is nearstandard to t. Since f is a standard continuous function, $f(^\circ t)$ is standard, and

$f(^\circ t)$ is on C, and $x \cong f(^\circ t)$. Since there is only one standard point in each monad, $x = f(^\circ t)$. Thus $x \in C$.

THEOREM 3.3. *The inside of C is bounded in the standard topology and the outside of C is unbounded in the standard topology.*

Proof. It is true in the standard model that there is a real number r such that for some fixed point x_0, $|x_0, f(t)| < r$ for all $t \in [0, 1]$. Therefore in the nonstandard model, $|x_0, C| < r$. Since P is a good approximation for C, $|x_0, P| < r + 1$. This implies that the inside of C is bounded. The outside of C is unbounded since there are standard points outside of P that are greater than any given standard distance.

THEOREM 3.4. *The inside of C and the outside of C are open.*

Proof. Let a be a point in the inside of C. Let $|a, C| = r$. Then $^\circ r > 0$. Let $E = \{x | x \text{ is standard and } |a, x| < {^\circ r}\}$. Then E is a standard open set containing a, and E is contained in the inside of C. Similarly for the outside of C.

DEFINITION 3.3. If $t, t' \in [0, 1]$ let

$$D(t, t') = \min \{|t', t|, |1 + t', t|, |1 + t, t'|\} .$$

Note that if $a, b \in P$ then $a \cong b$ if and only if $D(h(a), h(b)) \cong 0$.

LEMMA 3.1. *Let $a, b \in P$ and $a < b$, $P_1 = \{x | a \leqq x \leqq b\}$, $P_2 = P - P_1$. Assume that $D(h(a), h(b))$ is not infinitesimal, $x \in P_1$, $y \in P_2$, and $x \cong y$. Then $x \cong a$ or $x \cong b$.*

Proof. Suppose not. Since $x \in P_1$ and $a \leqq x \leqq b$, $h(a) \leqq h(x) \leqq h(b)$. By hypothesis, x is not infinitesimally close to a and x is not infinitely close to b. Therefore $h(x)$ is not infinitesimally close to $h(a)$ or $h(b)$. Since $y \in P_2$, $h(y) \leqq h(a)$ or $h(y) \geqq h(b)$.

Case 1. $h(y) \leqq h(a)$. Since $x \cong y$, either $h(x) - h(y) \cong 0$ or $h(x) - h(y) \cong 1$. But $h(x) - h(y)$ is not infinitesimal since $h(x) - h(y) \geqq h(x) - h(a)$ which is not infinitesimal. $h(x) - h(y)$ is not in the monad of 1. For if it were, $h(x) \cong 1$ and since $h(b) > h(x)$, $h(b) \cong 1$. But then $h(x) \cong h(b)$. A contradiction.

Case 2. $h(y) \geqq h(b)$. Similar to Case 1.

THEOREM 3.5. *The inside of C is nonempty.*

Proof. Let L be the set of line segments that have endpoints on P and are contained in (inside P) $\cup P$. Since the function h takes on only "finitely" many values, there is a fixed segment, ab, in L such that $D(h(a), h(b))$ is a maximum.

Case 1. $h(b) - h(a)$ is not infinitesimal. Without loss of generality, assume that $a < b$. Let c be the midpoint of ab, $P_1 = \{x \mid a \leqq x \leqq b\}$, and $P_2 = (P - P_1) \cup \{a, b\}$. Since c is not infinitesimally close to a or b, it follows from Lemma 3.1 that c is not infinitesimally close to both P_1 and P_2. Without loss of generality, suppose that c is not infinitesimally close to P_1. Let $r = |c, P_1|$ and $A = ab \cap P_1$. Since A is a closed set, let a_1 and b_1 be those members of A such that in the ordering of ab, for each $x, y \in A$ if $x < c$ and $y > c$ then $x \leqq a_1 < c < b_1 \leqq y$. (See Figure 3.1.) Then $a_1b_1 \cup \{x \in P_1 \mid a_1 \leqq x \leqq b_1\}$ can easily be shown to be a simple closed polygon, Q, with the property that the inside of Q is inside P. Further, $P_2 \cap$ (inside Q) $= \varnothing$. If d is inside Q and is the endpoint of the segment of length $r/3$, drawn perpendicular through c to the side a_1b_1 of Q, then it follows

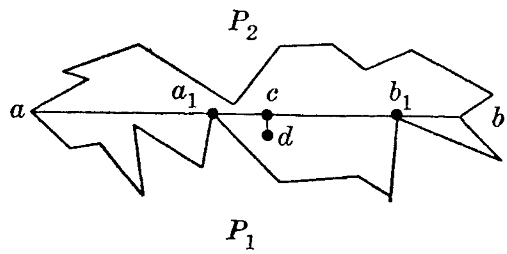

FIGURE 3.1.

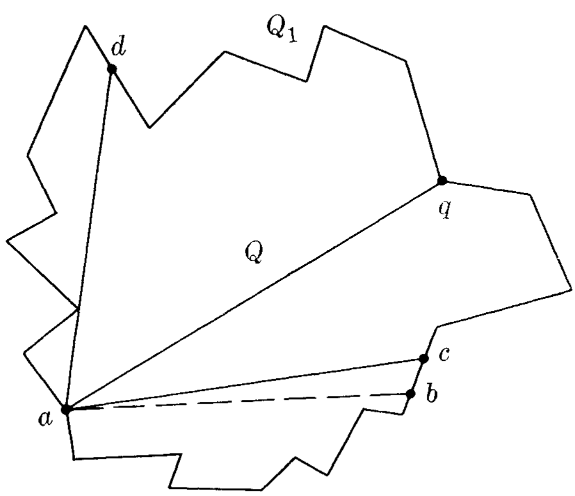

FIGURE 3.2.

that d is inside P and that $|d, P| \geqq r/3$. Since $r/3$ is not infinitesimal, d is inside C.

Case 2. $D(h(a), h(b))$ is infinitesimal. Without loss of generality, assume that $a < b$ and $h(a)$ is not in the monad of 0. Let

$$B = \{x \in P \mid ax \in L \text{ and } h(x) = h(b)\} .$$

In the ordering "$<$" of P let $c = \sup B$. By continuity, $ac \in L$. Since $c \geqq b$, $h(c) \geqq h(b)$. Since $h(a)$ is not in the monad of 0, and $D(h(a), h(c))$ is infinitesimal, and $h(c) \geqq h(b)$, and $h(b) - h(a)$ is a maximum it follows that $h(c) = h(b)$. If $e \in P$ and $e > c$ then $ae \notin L$. For if $e \in P$ and $e > c$ and $ae \in L$, then either $h(e) = h(c)$, contradicting the definition of c, or $h(e) > h(c)$, contradicting the maximality of $D(h(a), h(b))$ together with $h(a)$ not being in the monad of 0. Similarly a point $d \in P$ can be found such that for all $e < d, ea \notin L$ and $da \in L$. (See Figure 3.2.) Let $Q_1 = \{x \in P \mid x \leqq d \text{ or } x \geqq c\}$. Let $Q = da \cup ac \cup Q_1$. Then Q is a simple closed polygon with inside $Q \subset$ inside P. Since a is a vertex of Q, it follows that a segment aq can be drawn such that intv (a, q) is inside Q and q is on Q. It then follows that $q \in Q_1$. Thus $aq \in L$. But by the method in which c and d were chosen, this is impossible.

4. Connectivity of the inside and outside of C.

DEFINITION 4.1. If $A \subset B$ and B is a standard compact set, let $^\circ A = \{^\circ x \mid x \in A\}$.

LEMMA 4.1. *If A is a nonempty connected set and $A \subset B$ where B is a standard compact set, then $^\circ A$ is connected in the standard model.*

Proof. Suppose $^\circ A$ is not connected in the standard model. Then in the standard model, there are open sets W and V such that $W \cap {^\circ A} \neq \emptyset$, $V \cap {^\circ A} \neq \emptyset$, $W \cap V = \emptyset$, and $^\circ A \subset W \cup V$. But $x \in W$ in the standard model implies that the monad of x is contained in W in the nonstandard model. Therefore in the nonstandard model, W and V are open sets, $W \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, $W \cap V = \emptyset$, and $A \subset W \cup V$. Hence A is not connected. But this contradicts the hypothesis that A is connected. Therefore $^\circ A$ is connected in the standard model.

LEMMA 4.2. *Let e, g be points on P such that* intv (e, g) *is inside P. Let a, b be points on eg such that $|a, P|, |b, P|$, and $|a, b|$ are not infinitesimal. Then there is a connected set, A, such that A is inside $P, a, b \in A$, and $|A, P|$ is not infinitesimal.*

Proof. Assume the hypotheses. Without loss of generality assume that e and g are on the Y-axis. Let w be an arbitrary point of ab. Let c and d be points of P such that cd is parallel to the X-axis, $w \in cd$, and $\operatorname{intv}(c, d)$ is inside P. It will be shown that (c, d) is not infinitesimal. There are three cases to consider: (i) $w \cong a$, (ii) $w \cong b$, (iii) w is not infinitesimally closed to either a or b. Cases (i) and (ii) immediately follow from the assumption that $|a_1 P|$ and $|b, P|$ are not infinitesimal. (iii) will be shown by contradiction. Assume that $c \cong$ d. Without loss of generality we may assume that c is in the left half-plane d is in the right half-plane, and that in the ordering of $P, c < d$. For convenience we assume that $h(d)$ is not in the monad of 1. Let $P_1 = \{x \mid x \in P \text{ and } c \leqq x \leqq d\}$. Then for each $x \in P_1$, $h(d) - h(x) \cong 0$. However, since c is in the left-plane and d is in the right half-plane, P_1 intersects the Y-axis at some point p. From (iii) it follows that $\min(|w, a|, |w, b|)$ is not infinitesimal. Since

$$\min(|w, a|, |w, b|) \leqq |w, p| \leqq |d, p| ,$$

it follows that $|d, p|$ is not infinitesimal. Hence $h(d) - h(p)$ is not infinitesimal—a contradiction.

One again, let $w = (0, w_1)$ be an arbitrary point of abc, $= (c_1, w_1)$ and $d = (d_1, w_1)$ be points on P such that cd is parallel to the X-axis and $\operatorname{intv}(c, d)$ is inside P. Let $u(w) = (1/2(d_1 - c_1), w_1)$. Then, by construction, $u(w)$ is inside P and $|u(w), P|$ is not infinitesimal. Let

$$A = au(a) \cup \{u(w) \mid w \in ab\} \cup bu(b) .$$

Then A is a connected set that is inside P and is such that $a \in A$, $b \in A$, and $|A, P|$ is not infinitesimal.

LEMMA 4.3. *Let a_1, a_2 be distinct points inside C. Then there is a connected set A such that $a_1 \in A, a_2 \in A$, A is inside P, and $|A, P|$ is not infinitesimal.*

Proof. Let a_1, a_2 be distinct points inside C. Let $c_1 d_1$ be a segment through a_1 such that $|a_2, c_1 d_1|$ is not infinitesimal, c_1 and d_1 are on P, and $\operatorname{intv}(c_1, d_1)$ is inside P. Similarly, let $c_2 d_2$ be a segment through a_2 that is parallel to $c_1 d_1$, and such that c_2 and d_2 are on P and $\operatorname{intv}(c_2, d_2)$ is inside P. Without loss of generality, it may be assumed that $c_1 < c_2 < d_2 < d_1$.

Let $P_1 = \{x \in P \mid c_1 \leqq x \leqq c_2\}$ and $P_2 = \{x \in P \mid d_2 \leqq x \leqq d_1\}$. (See Figure 4.1.) $|P_1, P_2|$ is not infinitesimal. For if x_1 and x_2 are arbitrary members of P_1 and P_2 respectively, then $c_1 \leqq x_1 \leqq c_2 \leqq d_2 \leqq x_2 \leqq d_1$, and hence $h(c_1) \leqq h(x_1) \leqq h(c_2) \leqq h(d_2) \leqq h(x_2) \leqq h(d_1)$. Since $c_1 d_1$ passes through a_1 and $|a_1, P|$ is not infinitesimal, $h(d_1) - h(c_1)$ is not in the monad of 0 or 1. Similarly $h(d_2) - h(c_2)$ is not in the monad of 0 or 1. But, $h(d_1) - h(c_1) \geqq h(x_2) - h(x_1) \geqq h(d_2) - h(c_2)$. Therefore $h(x_2) - h(x_1)$ is not in the monad of 0 or 1. Thus $|x_1, x_2|$ is not infinitesimal. Therefore $|P_1, P_2|$ is not infinitesimal.

Let $r = \min\{|P_1, P_2|, |c_1d_1, c_2d_2|, |a_1, P|, |a_2, P|\}$. Without loss of generality, it may be assumed that c_1d_1 is on the Y-axis. Since each line segment l that is parallel to c_1d_1 has the same X-coordinate for each of its points, we shall say that the *coordinate of l* is the X-coordinate of its points. Let v be the X-coordinate of c_2d_2. Without loss of generality we may assume that v is positive. Let

$$L = \{cd \mid c \in P_1,\, d \in P_2,\ \text{intv}\,(c, d) \text{ is inside } P, \text{ and } cd \text{ is parallel to } c_1d_1\}\,.$$

Let $B = \{x \mid \text{there is an } l \in L \text{ such that } x \in l \text{ and } |x, P| > r/4\}$. Let E be the largest connected subset of B that contains a_1. Let

$$T = \{t \mid \text{there is a } z \in E \text{ such that } t \text{ is an } X\text{-coordinate of } z\}\,.$$

Let s be the supremum of T. $s = v$. For if s were less than v, let $cd \in L$ and such that $s = X$-coordinate of cd. Let p be the midpoint of cd. If $|p, P| \geqq r/4$ then the disk D about p of radius $r/4$ intersects E. Then $(E \cup D) \cap B$ would be a connected subset of B containing a_1 that is larger than E. This is impossible since E is the largest connected subset of B containing a_1. Thus $|p, P| < r/4$. Since $|p, P_1| \geqq r/2$ and $|p, P_2| \geqq r/2$, there is a point $q \in P - (P_1 \cup P_2)$ such that $|p, q| \leqq r/4$. Since $Q = c_1d_1 \cup P_1 \cup c_2d_2 \cup P_2$ is a simple closed polygon, and p is inside Q and q is outside Q, $pq \cap Q \neq \varnothing$. This can only happen if $pq \cap c_1d_1 \neq \varnothing$ or $pq \cap c_2d_2 \neq \varnothing$. Assume that $pq \cap c_1d_1 \neq \varnothing$. (The case of $pq \cap c_2d_2 \neq \varnothing$ follows similarly.) Then $|p, c_1d_1| < r/4$. Since $c \in P_1$ and $d \in P_2$ and cd is parallel to c_1d_1, there is a point $w \in cd$ such that $|w, a_1| < r/4$. Since $|a_1, P| \geqq r$, the disk D_1 of radius $r/4$ about the point w has the property that $|D_1, P| > r/4$. Also, $D_1 \cap E \neq \varnothing$ and $(B \cap D_1) - E \neq \varnothing$. Therefore $(B \cap D_1) \cup E$ is a larger, connected subset of B than E. A contradiction.

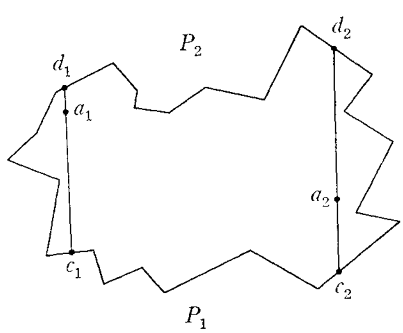

FIGURE 4.1.

Since $s = v$, $E \cap c_2d_2 \neq \varnothing$. Let $e \in E \cap c_2d_2$. Since $|E, P| \geqq r/4$, e is not infinitesimally close to P. By Lemma 4.2, there is a connected set F such that $e \in F$, $a_2 \in F$, $|F, P|$ is not infinitesimal, and F is inside P. Let $A = E \cup F$. Then $a_1, a_2 \in A$, A is connected, $|A, P|$ is not infinitesimal, and A is inside P.

THEOREM 4.1. *The inside of C is connected (in the standard model).*

Proof. Suppose not. Let $X =$ inside C. Then in the standard model, there are open sets V and W such that $V \cap X \neq \varnothing$, $W \cap X \neq \varnothing$, $W \cap V = \varnothing$, and $X \subset W \cup V$. Let $a \in V \cap X$ and $b \in W \cap X$. Then in the nonstandard model, by Lemma 4.3 there is a connected set A such that $a, b \in A$, A is inside P, and $|A, P|$ is not infinitesimal. Note that since $|A, P|$ is not infinitesimal, ${}^\circ A$ is inside C. That is, ${}^\circ A \subseteqq X$. Also, by Lemma 4.1, ${}^\circ A$ is connected in the standard model. Therefore in the standard model,

$$V \cap {}^\circ A \neq \varnothing,\ W \cap {}^\circ A \neq \varnothing,\ V \cap W = \varnothing\ ,$$

and ${}^\circ A \subset V \cup W$. Hence ${}^\circ A$ is not connected. A contradiction.

LEMMA 4.4. *Let a, b be points in the outside of C. Then there is a line l that intersects P and such that*

(1) *$|a, l|$ and $|b, l|$ are not infinitesimal, and*

(2) *a and b are in the same half-plane determined by l.*

Proof. Let a, b be points in the outside of P and c a point in the inside of C. Then a, b, and c are standard and $|a, P|$, $|b, P|$, and $|c, P|$ are not infinitesimal. Let k be the line through a and c. Let H_1 and H_2 be the closed half-planes determined by k. Without loss of generality, suppose that $b \in H_1$. Since $|c, P|$ is not infinitesimal, there is a point d that is inside P and such that $|d, H_1|$ is not infinitesimal. Let l be the line through d that is parallel to k. Then $|a, l|$ and $|b, l|$ are not infinitesimal and a and b are in the same half-plane determined by l.

THEOREM 4.2. *The outside of C is connected (in the standard model).*

Proof. Let a and b be points in the outside of C. It only needs to be shown that there is a connected set in the outside of C that contains a and b. By Lemma 4.4, let l be a line through P such that a and b are in the same closed half-plane, H, that is determined by l. Then a square S can be found with the following four properties:

(1) one side of S is on l,

(2) $l \cap P$ is contained on a side of S,

(3) a and b are inside S and $|a, S|$ and $|b, S|$ are not infinitesimal,

(4) $S \subset H$.

Let $T =$ (inside of S) $\cap$ (outside of P). Then it is easy to show that T is an open set that has a simple closed polygon, Q, as a boundary. In other words, inside of $Q = T$, a and b are in the inside of Q, $|a, Q|$ and $|b, Q|$ are not infinitesimal. Now, ${}^\circ Q = \{{}^\circ x \mid x \in Q\}$ can easily be seen to be a Jordan curve in the standard model (composed partly of the square ${}^\circ S = \{{}^\circ x \mid x \in S\}$ and partly of C) with a and b in the inside of ${}^\circ Q$. Thus by Theorem 4.1, in the standard model there is a polygonal path, A, which is inside ${}^\circ Q$ and such that $a, b \in A$. But since the inside of ${}^\circ Q \subset$ the outside of C, it follows that a and b belong to a connected set that is in the outside of C.

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UNIVERSITY OF CALIFORNIA, LOS ANGELES