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A NONUNIFORM BOUND ON CONVERGENCE TO NORMALITY

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Various asymptotically correct bounds on the uniform metric for distance between distribution functions in the central limit theorem for sums of independent and identically distributed random variables have previously been given. It is shown in the present paper that corresponding nonuniform bounds can be given for the difference between distribution functions. These results have much wider applicability, such as for obtaining probabilities of moderate deviation or for dealing with L_p metrics, $1 \leqq p \leqq \infty$.

1. Introduction and results. Let X_i, $i = 1, 2, \cdots$ be independent and identically distributed random variables with $EX_1 = 0$, $EX_1^2 = 1$, and distribution function F. Write $S_n = \sum_{i=1}^n X_i$, $n \geqq 1$, $G(x) = P(|X_1| \leqq x)$, $F_n(x) = P(S_n \leqq xn^{\frac{1}{2}})$, and denote by Φ the distribution function of the unit normal law. We shall establish the following theorem.

THEOREM 1. *Suppose $E|X_1|^{2+\alpha} < \infty$ for some $0 \leqq \alpha \leqq 1$. Then for all x,*

$$|F_n(x) - \Phi(x)| < (1 + |x|^{2+\alpha})^{-1} c_n(\alpha), \tag{1}$$

where

$$c_n(\alpha) = C\{n^{-\frac{1}{2}} \int_0^{n^{\frac{1}{2}}} y^3 \, dG(y) + n^{-\alpha/2} \int_{n^{\frac{1}{2}}}^\infty y^{2+\alpha} \, dG(y)\}, \tag{2}$$

C being a universal constant. Furthermore, writing $\Delta_n = \sup_x |F_n(x) - \Phi(x)|$, the following results hold.

(i) *The conditions $\sum n^{-1}\Delta_n < \infty$ and $\sum n^{-1} c_n(0) < \infty$ are equivalent and hold if and only if $EX_1^2 \log(1 + |X_1|) < \infty$.*

(ii) *For $0 < \delta < 1$, the conditions $\sum n^{-1+\delta/2}\Delta_n < \infty$ and $\sum n^{-1+\delta/2} c_n(0) < \infty$ are equivalent and hold if and only if $E|X_1|^{2+\delta} < \infty$.*

(iii) *For $0 < \delta < 1$, the conditions $\Delta_n = O(n^{-\delta/2})$ and $c_n(0) = O(n^{-\delta/2})$ as $n \to \infty$ are equivalent and hold if and only if $\int_z^\infty x^2 \, dG(x) = O(z^{-\delta})$ as $z \to \infty$.*

This theorem extends results of Heyde [2] and of Ibragimov [5] which deal with the uniform metric in (i), (ii) and (iii). The result is useful for many purposes, such as for obtaining probabilities of moderate deviation (e.g. to generalize Theorem 4 of Davis [1]), for obtaining results on L_p metrics, $1 \leqq p < \infty$, for departure from normality or for estimating differences of the kind $Eb(S_n) - Eb(n^{\frac{1}{2}}Y)$ for large n and suitable functions b, Y having a unit normal distribution. A sample application, establishing a similarity of behaviour of the L_p metrics,

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$1 \leqq p \leqq \infty$, is given in the following corollary which extends Theorem 2 of Heyde [3].

COROLLARY 2. *Put*

$$||F_n - \Phi||_p = (\int_{-\infty}^{\infty} |F_n(x) - \Phi(x)|^p \, dx)^{1/p}, \qquad 1 \leqq p < \infty,$$
$$||F_n - \Phi||_\infty = \sup_x |F_n(x) - \Phi(x)| .$$

Then, for $0 \leqq \delta < 1$ *and* $1 \leqq p \leqq \infty$, *the following two conditions are equivalent*:

(a) $EX_1^2 \log(1 + |X_1|) < \infty$ *if* $\delta = 0$, $E|X_1|^{2+\delta} < \infty$ *if* $0 < \delta < 1$,

(b) $\sum n^{-1+\delta/2} ||F_n - \Phi||_p < \infty$.

It should be remarked that the complementary result to this corollary which deals with the case of $||F_n - \Phi||_p = O(n^{-\delta/2})$, $0 < \delta < 1$, has been obtained, by quite different methods, in Theorem 4.3 of Ibragimov [5].

2. Proofs. To prove Theorem 1 we start by taking $x \geqq 2$ for definiteness. The case $x \leqq -2$ can then be dealt with by replacing the X_i's by $-X_i$'s and the case $|x| < 2$ will follow from a bound on the uniform metric of Osipov and Petrov [7].

Put

$$X_k^{(n)}(x) = X_k \qquad \text{if} \quad |X_k| \leqq n^{\frac{1}{2}} x ,$$
$$= 0 \qquad \text{otherwise},$$

and write $S_n^{(n)}(x) = \sum_{k=1}^n X_k^{(n)}(x)$. We have

$$F_n(x) = P(S_n \leqq n^{\frac{1}{2}}x, S_n^{(n)}(x) \leqq n^{\frac{1}{2}}x) + P(S_n \leqq n^{\frac{1}{2}}x, S_n^{(n)}(x) > n^{\frac{1}{2}}x)$$
$$\leqq P(S_n^{(n)}(x) \leqq n^{\frac{1}{2}}x) + nP(|X_1| > n^{\frac{1}{2}}x)$$

and similarly

$$1 - F_n(x) \leqq P(S_n^{(n)}(x) > n^{\frac{1}{2}}x) + nP(|X_1| > n^{\frac{1}{2}}x)$$

so that

$$F_n(x) \geqq P(S_n^{(n)}(x) \leqq n^{\frac{1}{2}}x) - nP(|X_1| > n^{\frac{1}{2}}x)$$

and hence

$$|F_n(x) - \Phi(x)| \leqq |P(S_n^{(n)}(x) \leqq n^{\frac{1}{2}}x) - \Phi(x)| + nP(|X_1| > n^{\frac{1}{2}}x) . \tag{3}$$

Now put $\mu_n(x) = EX_1^{(n)}(x)$, $\sigma_n^2(x) = \operatorname{Var} X_1^{(n)}(x)$. We have, writing

$$y_n = [\sigma_n(x)]^{-1} n^{-\frac{1}{2}} [n^{\frac{1}{2}}x - n\mu_n(x)] ,$$
$$|P(S_n^{(n)}(x) \leqq n^{\frac{1}{2}}x) - \Phi(x)|$$
$$\leqq |P([\sigma_n(x)]^{-1}(S_n^{(n)}(x) - n\mu_n(x)) \leqq n^{\frac{1}{2}}y_n) - \Phi(y_n)| \tag{4}$$
$$+ |\Phi(y_n) - \Phi([\sigma_n(x)]^{-1}x)| + |\Phi([\sigma_n(x)]^{-1}x) - \Phi(x)| .$$

But, using Theorem 2 of Nagaev [6],

$$|P([\sigma_n(x)]^{-1}(S_n^{(n)}(x) - n\mu_n(x)) \leqq n^{\frac{1}{2}}y_n) - \Phi(y_n)|$$
$$\leqq CE|X_1^{(n)}(x) - \mu_n(x)|^3 n^{-\frac{1}{2}} [\sigma_n(x)]^{-3} (1 + |y_n|^3)^{-1} \tag{5}$$
$$\leqq 8CE|X_1^{(n)}(x)|^3 n^{-\frac{1}{2}} |x - n^{\frac{1}{2}}\mu_n(x)|^{-3} ,$$

C denoting a universal constant.

Now, for $x \geqq 2$ we have

$$|\mu_n(x)| = |\int_{|y|>n^{\frac{1}{2}}x} y\, dF(x)| \leqq \int_{n^{\frac{1}{2}}x}^{\infty} y\, dG(x) \leqq n^{-\frac{1}{2}}x^{-1} \int_{n^{\frac{1}{2}}x}^{\infty} y^2\, dG(y) \leqq n^{-\frac{1}{2}}x^{-1},$$

so that

$$n^{\frac{1}{2}}|\mu_n(x)| \leqq x^{-1} \leqq \tfrac{1}{2}x, \qquad |x - n^{\frac{1}{2}}\mu_n(x)|^3 \geqq \tfrac{1}{8}x^3, \tag{6}$$

and hence, from (5),

$$\begin{aligned} |P([\sigma_n(x)]^{-1}(S_n^{(n)}(x) - n\mu_n(x)) \leqq n^{\frac{1}{2}}y_n) - \Phi(y_n)| &\leqq 64CE|X_1^{(n)}(x)|^3 n^{-\frac{1}{2}}x^{-3} \\ &= 64Cn^{-\frac{1}{2}}x^{-3}\{\int_0^{n^{\frac{1}{2}}} y^3\, dG(y) + \int_{n^{\frac{1}{2}}}^{n^{\frac{1}{2}}x} y^3\, dG(y)\} \\ &\leqq 64Cn^{-\frac{1}{2}}x^{-3}\{\int_0^{n^{\frac{1}{2}}} y^3\, dG(y) + (n^{\frac{1}{2}}x)^{1-\alpha} \int_{n^{\frac{1}{2}}}^{\infty} y^{2+\alpha}\, dG(y)\} \\ &\leqq 64Cx^{-(2+\alpha)}\{n^{-\frac{1}{2}} \int_0^{n^{\frac{1}{2}}} y^3\, dG(y) + n^{-\alpha/2} \int_{n^{\frac{1}{2}}}^{\infty} y^{2+\alpha}\, dG(y)\}. \end{aligned} \tag{7}$$

Further, for $x \geqq 1$,

$$\begin{aligned} nP(|X_1| > n^{\frac{1}{2}}x) &\leqq x^{-(2+\alpha)}n^{-\alpha/2} \sup_{u \geqq n^{\frac{1}{2}}} u^{2+\alpha}P(|X_1| > u) \\ &\leqq x^{-(2+\alpha)}n^{-\alpha/2} \sup_{u \geqq n^{\frac{1}{2}}} \int_u^{\infty} y^{2+\alpha}\, dG(y) \\ &\leqq x^{-(2+\alpha)}n^{-\alpha/2} \int_{n^{\frac{1}{2}}}^{\infty} y^{2+\alpha}\, dG(y). \end{aligned} \tag{8}$$

It now remains to bound the terms $|\Phi(y_n) - \Phi([\sigma_n(x)]^{-1}x)|$ and $|\Phi([\sigma_n(x)]^{-1}x) - \Phi(x)|$ from (4). Since $e^{\frac{1}{2}u^2} > \frac{1}{2}u^2$ for $u > 0$ we have

$$\begin{aligned} |\Phi(y_n) - \Phi([\sigma_n(x)]^{-1}x)| &\leqq (2\pi)^{-\frac{1}{2}} \int_{(x-n^{\frac{1}{2}}|\mu_n(x)|)[\sigma_n(x)]^{-1}}^{x[\sigma_n(x)]^{-1}} e^{-\frac{1}{2}u^2}\, du \\ &\leqq 2(2\pi)^{-\frac{1}{2}} \int_{(x-n^{\frac{1}{2}}|\mu_n(x)|)[\sigma_n(x)]^{-1}}^{x[\sigma_n(x)]^{-1}} u^{-2}\, du \\ &\leqq 4(2\pi)^{-\frac{1}{2}}n^{\frac{1}{2}}x^{-2}|\mu_n(x)| \end{aligned} \tag{9}$$

using (6). Also,

$$n^{\frac{1}{2}}|\mu_n(x)| \leqq n^{\frac{1}{2}} \int_{n^{\frac{1}{2}}x}^{\infty} y\, dG(y) \leqq x^{-1} \int_{n^{\frac{1}{2}}x}^{\infty} y^2\, dG(y) \leqq x^{-1} \int_{2n^{\frac{1}{2}}}^{\infty} y^2\, dG(y) \tag{10}$$

and hence, from (9) and (10),

$$\begin{aligned} |\Phi(y_n) - \Phi([\sigma_n(x)]^{-1}x)| &\leqq 4(2\pi)^{-\frac{1}{2}}x^{-3} \int_{2n^{\frac{1}{2}}}^{\infty} y^2\, dG(y) \\ &\leqq 4(2\pi)^{-\frac{1}{2}}x^{-(2+\alpha)} \int_{2n^{\frac{1}{2}}}^{\infty} y^2\, dG(y) \\ &\leqq 4(2\pi)^{-\frac{1}{2}}x^{-(2+\alpha)}n^{-\alpha/2} \int_{n^{\frac{1}{2}}}^{\infty} y^{2+\alpha}\, dG(y). \end{aligned} \tag{11}$$

Next, since $e^{\frac{1}{2}u^2} > \frac{1}{8}u^4 \geqq \frac{1}{4}u^3$ for $u \geqq 2$, we have for $x \geqq 2$,

$$\begin{aligned} |\Phi([\sigma_n(x)]^{-1}x) - \Phi(x)| &\leqq 4(2\pi)^{-\frac{1}{2}} \int_x^{x[\sigma_n(x)]^{-1}} u^{-3}\, du \\ &= 2(2\pi)^{-\frac{1}{2}}x^{-2}[1 - \sigma_n^2(x)] \\ &\leqq 4(2\pi)^{-\frac{1}{2}}x^{-2} \int_{n^{\frac{1}{2}}x}^{\infty} y^2\, dG(y) \\ &\leqq 4(2\pi)^{-\frac{1}{2}}x^{-(2+\alpha)}n^{-\alpha/2} \int_{n^{\frac{1}{2}}x}^{\infty} y^{2+\alpha}\, dG(y) \\ &\leqq 4(2\pi)^{-\frac{1}{2}}x^{-(2+\alpha)}n^{-\alpha/2} \int_{2n^{\frac{1}{2}}}^{\infty} y^{2+\alpha}\, dG(y). \end{aligned} \tag{12}$$

Hence, using (3), (4), (7), (8), (11) and (12), we have for $|x| \geqq 2$ that there exists a universal constant C such that, a fortiori,

$$|F_n(x) - \Phi(x)| \leqq C(1 + |x|^{2+\alpha})^{-1}\{n^{-\frac{1}{2}} \int_0^{n^{\frac{1}{2}}} y^3\, dG(y) + n^{-\alpha/2} \int_{n^{\frac{1}{2}}}^{\infty} y^{2+\alpha}\, dG(y)\}. \tag{13}$$

Furthermore, the inequality of Osipov and Petrov [7] can be specialized to give (see (2) of Heyde [4] wherein we take $\tau_n = C_n = n^{\frac{1}{2}}$; K_0 is a universal constant)

$$\begin{aligned} |F_n(x) - \Phi(x)| &\leqq nP(|X_1| > n^{\frac{1}{2}}) + K_0 n^{-\frac{1}{2}}[\sigma_n(1)]^{-3} \int_0^{n^{\frac{1}{2}}} y^3\, dG(y) \\ &\quad + n^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}}[\sigma_n(1)]^{-1}\left|\int_{|y| \leqq n^{\frac{1}{2}}} y\, dF(y)\right| \\ &\quad + (2\pi e)^{-\frac{1}{2}} \int_{n^{\frac{1}{2}}}^{\infty} y^2\, dG(y) , \end{aligned} \tag{14}$$

and noting (8), (10) and $\sigma_n(1) \to 1$ as $n \to \infty$, we observe that the right hand side of (14) is upper bounded by the right hand side of (13) for suitable universal C and $|x| < 2$. This ensures that (1) also holds for $|x| < 2$ and hence completes the proof of (1). In order to complete the remainder of the proof we first obtain a simple bound for $c_n(0)$.

Write $L(y) = \int_y^\infty x^2\, dG(x)$. Then

$$\begin{aligned} n^{-\frac{1}{2}} \int_0^{n^{\frac{1}{2}}} y^3\, dG(y) &= -n^{-\frac{1}{2}} \int_0^{n^{\frac{1}{2}}} y\, dL(y) \\ &\leqq n^{-\frac{1}{2}} \int_0^{n^{\frac{1}{2}}} L(y)\, dy , \end{aligned}$$

and, since $L(y) \downarrow$ as y increases,

$$c_n(0) \leqq 2Cn^{-\frac{1}{2}} \int_0^{n^{\frac{1}{2}}} L(y)\, dy . \tag{15}$$

Now suppose $0 \leqq \delta < 1$ and $EX_1^2 \log(1 + |X_1|) < \infty$ if $\delta = 0$, $E|X_1|^{2+\delta} < \infty$ if $0 < \delta < 1$. Here and below we use C to denote a positive universal constant which may differ from one expression to the next. Using (15), we have

$$\begin{aligned} \sum_{n=1}^\infty n^{-1+\delta/2} c_n(0) &\leqq C \sum_{n=1}^\infty n^{-(3-\delta)/2} \sum_{k=1}^n \int_{(k-1)^{\frac{1}{2}}}^{k^{\frac{1}{2}}} L(y)\, dy \\ &\leqq C \sum_{n=1}^\infty n^{-(3-\delta)/2} \sum_{k=1}^n L((k-1)^{\frac{1}{2}})(k^{\frac{1}{2}} - (k-1)^{\frac{1}{2}}) \\ &\leqq C \sum_{k=1}^\infty k^{-\frac{1}{2}} L((k-1)^{\frac{1}{2}}) \sum_{n=k}^\infty n^{-(3-\delta)/2} \\ &\leqq C \sum_{k=1}^\infty k^{-1+\delta/2} L((k-1)^{\frac{1}{2}}) \\ &= C \sum_{k=1}^\infty k^{-1+\delta/2} \sum_{n=k}^\infty E\{X_1^2 I((n-1)^{\frac{1}{2}} < |X_1| \leqq n^{\frac{1}{2}})\} \\ &\leqq C \sum_{n=1}^\infty n^{\delta/2} E\{X_1^2 I((n-1)^{\frac{1}{2}} < |X_1| \leqq n^{\frac{1}{2}})\} \\ &\leqq CE|X_1|^{2+\delta} < \infty , \qquad \text{if} \quad 0 < \delta < 1 , \quad \text{or,} \\ &\leqq C \sum_{n=1}^\infty \log(n+1) E\{X_1^2 I((n-1)^{\frac{1}{2}} < |X_1| \leqq n^{\frac{1}{2}})\} \\ &\leqq CEX_1^2 \log(1 + |X_1|) < \infty , \qquad \text{if} \quad \delta = 0 . \end{aligned}$$

Conversely, if $0 \leqq \delta < 1$ and $\sum n^{-1+\delta/2} c_n(0) < \infty$, then (2) gives $\sum n^{-1+\delta/2} L(n^{\frac{1}{2}}) < \infty$ and

$$\begin{aligned} \sum_{n=1}^\infty n^{-1+\delta/2} L(n^{\frac{1}{2}}) &= \sum_{n=1}^\infty n^{-1+\delta/2} \sum_{k=n}^\infty E\{X_1^2 I(k^{\frac{1}{2}} < |X_1| \leqq (k+1)^{\frac{1}{2}})\} \\ &= \sum_{k=1}^\infty E\{X_1^2 I(k^{\frac{1}{2}} < |X_1| \leqq (k+1)^{\frac{1}{2}})\} \sum_{n=1}^k n^{-1+\delta/2} \\ &\geqq C \sum_{k=1}^\infty k^{\delta/2} E\{X_1^2 I(k^{\frac{1}{2}} < |X_1| \leqq (k+1)^{\frac{1}{2}})\} \\ &\geqq CE|X_1|^{2+\delta} , \qquad \text{if} \quad 0 < \delta < 1 , \quad \text{or} \\ &\geqq C \sum_{k=1}^\infty \log k E\{X_1^2 I(k^{\frac{1}{2}} < |X_1| \leqq (k+1)^{\frac{1}{2}})\} \\ &\geqq CEX_1^2 \log(1 + |X_1|) , \qquad \text{if} \quad \delta = 0 . \end{aligned}$$

The equivalence of $\sum n^{-1+\delta/2} \Delta_n < \infty$ and $EX_1^2 \log(1 + |X_1|) < \infty$ if $\delta = 0$,

$E|X_1|^{2+\delta} < \infty$ if $0 < \delta < 1$, follows from Heyde [2] and hence the proof of (i) and (ii) is complete.

Finally, to prove (iii) we first suppose that $L(y) = O(y^{-\delta})$ as $y \to \infty$. Then, using (15), $c_n(0) = O(n^{-\delta/2})$. On the other hand, if $c_n(0) = O(n^{-\delta/2})$ then certainly, from (2),

$$L(n^{\frac{1}{2}}) = \int_{n^{\frac{1}{2}}}^{\infty} y^2 \, dG(y) = O(n^{-\delta/2})$$

which yields $L(y) = O(y^{-\delta})$ as $y \to \infty$ as required. The equivalence of $\Delta_n = O(n^{-\delta/2})$ and $L(z) = O(z^{-\delta})$ as $z \to \infty$ was obtained by Ibragimov [5]. This completes the proof of Theorem 1.

To establish Corollary 2 we first observe that the equivalence of (a) and (b) for $p = \infty$ follows from Theorem 1 as does the result that (a) implies (b) for $1 \leqq p < \infty$ since (a) gives $\sum n^{-1+\delta/2} c_n(0) < \infty$ and $||F_n - \Phi||_p = O(c_n(0))$. To see that (b) implies (a) note that the proof of the necessity part of the theorem of [2] is still applicable with minor modification (involving applications of Hölder's and Minkowski's inequalities if $p > 1$).

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