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A normal distribution or a Weibull distribution for fatigue lives

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Abstract: The results of six series of 18 to 30 similar fatigue tests on three types of specimens are used to check the applicability of the normal distribution function and the three-parameter Weibull distribution function. A least-square procedure is adopted to obtain the three parameters of the Weibull function. Comments are made on the results obtained.

INTRODUCTION

The results of a number of similar fatigue tests can be described by fitting a statistical distribution function to the data. Usually, the statistical variable adopted is the 10-logarithm of the fatigue life N.

$$v = \log N \qquad (1)$$

Two statistical distribution functions frequently considered in the literature for fatigue lives are:

(i) normal distribution function (Gauss)

(ii) 3-parameter Weibull distribution function.

The normal distribution function is fully described by two characteristic parameters, which can be estimated by a simple calculation. For the 3-parameter Weibull distribution a more elaborate procedure is necessary. In this document the two distribution functions are applied to six test series of 18 to 30 similar fatigue tests. A survey of the test series is given in the table below. The results of the test series are compiled in Appendix A.

specimen	test series	2024-T3 Alclad, thickness (mm)	stress (MPa)	Number of specimens
			S_{max} (R = 0)	
Unnotched K_t = 1.15	1 2	2.0	225 157	20 18
			S_{max} (R = 0)	
Edge notched K_t= 2.85	3 4	5.0	103 64	20 30
			S_m S_a	
Riveted lap joint[1]	5 6	0.8	88 71 88 31	20 20

[1]Two rows with 8 rivets each, diameter 3.1 mm, specimen width 160 mm.

The purpose of the present document is to examine whether the test results can be fitted by the two distribution functions. A least square procedure is described to obtain the three parameters of the Weibull distribution function. Some comments are made on the comparison of the two distribution functions and on the statistical evaluation of laboratory test series.

THE NORMAL DISTRIBUTION FUNCTION

The equation of the normal distribution function of a variable v is:

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{v-\mu}{\sigma}\right)^2} dv \qquad (2)$$

where μ is the mean value and σ the standard deviation. P(x) is the probability to find values $v < x$.

The data sets to be studied consist of n values $x_i = \log N_i$ (i = 1 to n). The estimates for μ and σ are obtained as:

$$\mu = (\Sigma x_i)/n \qquad (3)$$

$$\sigma = \sqrt{\frac{\Sigma (x_i-\mu)^2}{n-1}} \qquad (4)$$

The probability P_i to be associated with each value x_i is approximated by:

$$P_i = \frac{i-0.5}{n} \qquad (5)$$

The results of six test series have been plotted in Figures 1a to 1f. The vertical scale of this figure corresponds to the normal probability scale. It implies that a normal distribution will become a straight line, see Appendix B.

WEIBULL DISTRIBUTION FUNCTION

The 3-parameter Weibull distribution function is written as:

$$P(x) = 1 - e^{-\left(\frac{x-x_0}{a}\right)^b} \qquad (6)$$

In this equation x_0 is the location parameter (lower values of x are impossible, it is a lower limit), 'a' is the scale parameter, and b is the shape parameter. Values of $b \approx 3.23$ give a shape which is somewhat similar to the normal distribution function. Sometimes Eq.(6) is written as:

$$P(x) = 1 - e^{-\left(\frac{x - x_o}{a - x_o}\right)^b} \tag{7}$$

That does not change the equation, it only changes the value of a with an amount x_o. Here we will use Equation (6). The equation can be rewritten as:

$$x = x_o + a\{-\ln(1 - P)\}^{1/b} \tag{8}$$

For the P-values:

$$P_i = \frac{i - ½}{n} \qquad (i = 1 \text{ to } n) \tag{9}$$

we can calculate x and the deviation between x_i and x.

$$\text{dev.} = x_o + a\{-\ln(1 - P_i)\}^{1/b} - x_i \tag{10}$$

The sum of squared deviations is:

$$S = \Sigma(\text{dev})^2 \tag{11}$$

The criterion selected to find a, b and x_o is to minimize the sum S, i.e. *a least squares criterion* is adopted. It implies:

$$\frac{dS}{dx_o} = 0, \ \frac{dS}{da} = 0, \ \frac{dS}{db} = 0 \tag{12}$$

Combining equations (10) to (12) leads to three equations:

$$n x_o - \Sigma(x_i) + a\Sigma\{-\ln(1 - P_i)\}^{1/b} = 0 \tag{13}$$

$$x_o\Sigma\{-\ln(1 - P_i)\}^{1/b} + a\Sigma\{-\ln(1 - P_i)\}^{2/b} - \Sigma x_i\{-\ln(1 - P_i)\}^{1/b} = 0 \tag{14}$$

$$x_o\Sigma\{-\ln(1 - P_i)\}^{1/b}\ln\{-\ln(1 - P_i)\} + a\Sigma\{-\ln(1 - P_i)\}^{2/b}\ln\{-\ln(1 - P_i)\} -$$

$$- \Sigma x_i\{-\ln(1 - P_i)\}^{1/b}\ln\{-\ln(1 - P_i)\} = 0 \tag{15}$$

The equations do not allow an explicit solution of x_o, a and b. However, with (13) and (14) 'a' and x_o can be expressed as a function of b. Substitution in (15) then leads to an equation, from which b can be obtained by a simple iteration procedure, see Appendix C.

The results of five of the six test series have been plotted in Figures 2a to 2f. The vertical scale is calculated in such a way that the Weibull distribution function of the particular test series becomes a straight line. The three constants a, b, x_o have to be used in that calculation, see Appendix C. For test series 3 the calculation did not lead to a minimum of $\Sigma(\text{dev.})^2$. For this test series the Weibull distribution function could not be fitted to the test data.

The root mean square (r.m.s.) value of the deviations in Eq.(11) is obtained as:

$$\text{r.m.s.dev.} = \sqrt{\frac{\Sigma(\text{dev.})^2}{n-1}} \qquad (16)$$

It is a kind of a standard deviation for the plots in Figure 2. A smaller r.m.s.dev. indicates a better fit of the distribution function to the test data. Similarly, the r.m.s.dev. has been calculated for the normal distribution functions in Figure 1, in order to examine the fit between this function and the data.

DISCUSSION

The results are recapitulated in the table below. It shows the well-known trend of higher $\sigma_{\log N}$ for a longer life. The trend is further illustrated by Figure 3. Here, a more interesting factor is the r.m.s.dev., which is much smaller (3 to 10 times) than $\sigma_{\log N}$. This should obviously be expected, because the r.m.s.dev. indicates scatter around a fitted function, whereas $\sigma_{\log N}$ measures scatter around the mean value. The r.m.s.dev. is primarily adopted here to indicate how well the data can be fitted by a distribution function. Also for the r.m.s.dev., the table shows higher values for higher N-values. It suggests that a satisfactory fit is more easily obtained for a shorter life.

Some noteworthy observations can be made when considering r.m.s.dev. values for the two distribution functions in relation to the graphs in figures 1 and 2. On the average, the r.m.s.dev. for the Weibull distribution function is similar or smaller than for the normal distribution function. It suggests that the Weibull distribution function on the average gives a better fit than the normal distribution function. However, a noteworthy exception is found for test series 3, where a Weibull distribution function could not be fitted to the data with the least square approach. This is not a surprise if the data of test series 3 in Fig.1c are considered. The dotted curve in this graph indicates a trend which neither suggests a lower life limit, nor an infinite upper limit, which both are characteristic for the Weibull distribution. It should be noted that the normal distribution function in Fig.1c also disagrees with the trend of the data. In this respect, test series 4 is another noteworthy data set. As shown by Fig.1d, the data exhibit the characteristic behaviour of a Weibull distribution, and significant deviations from the normal distribution occur. The agreement of this test series with the Weibull distribution function is confirmed in Fig.2d, where the results line up very well with that function.

specimen	test series	normal distribution			Weibull distribution			
		mean life (kc)	$\sigma_{\log N}$	r.m.s.dev.	r.m.s.dev.	a	b	X_o (kc)
unnotched	1	196	0.114	0.0187	0.0149	0.708	6.841	42.7
	2	1290	0.163	0.0335	0.0325	0.680	4.388	309.8
edge-notched	3	185	0.069	0.0242	-	-	-	-
	4	1168	0.180	0.0459	0.0193	0.298	1.519	631.5
riveted lap joint	5	116	0.059	0.0092	0.0097	0.219	3.806	73.5
	6	1019	0.098	0.0135	0.0127	0.258	2.521	602.6

The probability density functions of the Weibull distributions are shown in Figure 4. The effect of the shape parameter b is evident. A typical shape of the Weibull probability density function can be observed for test series 4, for which the normal distribution function is a poor approximation. The b-value of series 5 (3.81) is the value most close to the value $b \approx 3.23$, for which the Weibull distribution function becomes very much similar to the normal distribution function. The similarity is illustrated by Fig.4, while it can also be observed from a comparison of Fig.1e with Fig.2e.

From a physical point of view, it should be expected that the Weibull distribution function is more appropriate than the normal distribution function. It has a lower limit, which appears to be physically necessary. On the other hand, a lower limit is associated with very low probabilities of failure. The lowest probability of failure in Figures 1 and 2 is 0.02 % (1 in 5000), and the graphs do not suggest dramatic differences for the two distribution functions. The only exception is test series 4, where the normal distribution predicts much lower fatigue lives than the Weibull distribution for a very low level of failure probability. Test series 3 is the other one of the two test series on edge-notched specimens. It shows a completely different behaviour, compare Figs. 1c and 1d. There is no obvious explanation why such an unsystematic difference could occur. As a matter of fact, we must be very cautious in deriving statistical information of laboratory test series. There may be several sources for scatter, which are irrelevant for practical problems. Part of the scatter may be due to the fatigue test itself (fatigue machine, operator) or specimen production. We will not deal with this issue any further here to avoid speculations. The prime reason for the present analysis was to obtain Weibull distribution functions for some relatively large test series with a least square approach and to see how the results would compare to the usual normal distribution approach.

SUMMARY AND CONCLUSIONS

1. Six relatively large test series (18 to 30 similar tests) on 2024-T3 specimens were statistically analyzed. For this purpose a least square deviation procedure was developed for the 3-parameter Weibull distribution function.
2. On the average, the Weibull distribution function agreed equally well of better with the test data in comparison to the normal distribution function. There was one noteworthy exception.
3. The data confirm relatively more scatter for higher endurances.
4. The practical significance of statistical analysis of large test series should be considered with extreme caution in view of possible sources of scatter in laboratory tests, which may be irrelevant in practice.

REFERENCES

[1] Schijve,J. and Jacobs,F.A., Fatigue tests on notched and unnotched clad 24 S-T sheet specimens to verify the cumulative damage hypothesis. Nat. Aerospace Lab. NLR, Amsterdam, Report M.1982, 1955.

[2] Schijve,J. and Jacobs,F.A., Research on cumulative damage in fatigue of riveted aluminum alloy joints. Nat. Aerospace Lab. NLR, Amsterdam, Report M.1999, 1956.

[3] Hald,A., Statistical tables and formulas. John Wiley & Sons, New York, 1958.

[4] Dwight,H.B., Tables of integrals and other mathematical data. Macmillan, New York, 1961.

Appendix A: Fatigue lives obtained in six test series.

Unnotched specimens, table 3.1 of [1], S_{max} = 225 MPa (R = 0)

i	N_i (kc)
1	110
2	128
3	151
4	160
5	160
6	170
7	171
8	172
9	199
10	203
11	204
12	205
13	211
14	212
15	233
16	248
17	259
18	268
19	285
20	296

Unnotched specimens, table 3.1 of [1], S_{max} = 157 MPa (R = 0)

i	N_i (kc)
1	636
2	707
3	801
4	1038
5	1090
6	1102
7	1160
8	1167
9	1262
10	1265
11	1404
12	1418
13	1461
14	1511
15	2119
16	2132
17	2158
18	2368

Edge-notched specimens, table 3.2 of [1], S_{max} = 103 MPa (R = 0)

i	N_i (kc)
1	122
2	145
3	153
4	161
5	163
6	163
7	184
8	187
9	194
10	194
11	195
12	199
13	202
14	204
15	205
16	212
17	212
18	216
19	216
20	222

Edge-notched specimens, table 3.2 of [1], S_{max} = 64 MPa (R = 0)

i	N_i (kc)
1	655
2	678
3	707
4	734
5	740
6	841
7	867
8	884
9	915
10	930
11	988
12	1000
13	1011
14	1018
15	1060
16	1087
17	1095
18	1184
19	1238
20	1241
21	1264
22	1395
23	1430
24	1469
25	1685
26	1726
27	1920
28	2391
29	2947
30	3348

Riveted lap joints, table 4.1 of [2], S_m = 88 MPa, S_a = 71 MPa

i	N_i (kc)
1	87
2	96
3	100
4	104
5	105
6	105
7	106
8	112
9	115
10	115
11	121
12	121
13	122
14	123
15	124
16	126
17	128
18	130
19	139
20	157

Riveted lap joint, table 4.1 of [2], S_m = 88 MPa, S_a = 31 MPa

i	N_i (kc)
1	655
2	774
3	790
4	830
5	884
6	906
7	908
8	930
9	953
10	1004
11	1012
12	1041
13	1070
14	1128
15	1154
16	1155
17	1252
18	1259
19	1516
20	1664

Appendix B: The normal distribution function

The equation of the normal distribution function is:

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{v-\mu}{\sigma}\right)^2} dv \tag{17}$$

Substitution of:

$$u = \frac{v-\mu}{\sigma} \tag{18}$$

leads to the normalized equation:

$$P(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\mu+u\sigma} e^{-\frac{1}{2}u^2} du \tag{19}$$

which is tabulated in many handbooks, see for instance [3].
An approximation is given in the formula book by Dwight [4] (formula 585):

$$\frac{1}{\sqrt{2\pi}} \int_{-x}^{x} e^{-t^2/2} dt = x\left(\frac{2}{\pi}\right)^{1/2}\left[1 - \frac{x^2}{2\cdot 1!3} + \frac{x^4}{2^2\cdot 2!5} - \frac{x^6}{2^3\cdot 3!7} + \dots\right] \tag{20}$$

Combining Eqs. (19) and (20) gives:

$$P(x) = 0.5 + \frac{x}{\sqrt{2\pi}}\left[1 - \frac{x^2}{2\cdot 1!3} + \frac{x^4}{2^2\cdot 2!5} - \frac{x^6}{2^3\cdot 3!7} + \dots\right] \tag{21}$$

The number of terms in [ ] should be larger for larger $|x|$ values. For x up to ± 3 a number of 25 terms is sufficient, for x approaching ± 5 a number of 50 terms should be used.

A normal probability paper plot requires that plotting positions are calculated for specific P-levels (in Figs 1a to 1f at P = 0.02, 0.05, 0.1, 0.2, 0.5, 1, 2, 3, 4, 5, 10, 15, 20, 30, 40, 50, and symmetric values). P-levels must also be calculated to plot the test data ($P_i = (i-½)/n$, Eq.(9)). It implies that x must be calculated for specified P(x) values. This has been done with Eq.(21). It requires an iteration procedure, which was started at x = -5, and using increasing x-values until Eq.(21) is satisfied for a given P-value.

The normal distribution in Figs. 1a to 1f is a linear function, passing through the point (μ,0.5) with a slope $1/\sigma$.

Appendix C: The Weibull distribution function

Solving x_o and a from Eqs.(13) and (14) and substitution in Eq.(15) leads to:

$(BV =)\ x_o \cdot SLPP + a \cdot SLP2P - SXLPP = 0$

with: $a = (SXLP \cdot n - SXI \cdot SLP) / (SLP2 \cdot n - SLP^2)$

$x_o = (SXI - a \cdot SLP) / n$

$SXI = \Sigma x_i$

$TE_i = \{- \log(1 - P_i)\}^{1/b}$

$TEL_i = b \log(TE_i)$

$SLP = \Sigma TE_i$

$SLP2 = \Sigma TE_i^2$

$SXLP = \Sigma x_i TE_i$

$SLPP = \Sigma TE_i \cdot TEL_i$

$SLP2P = \Sigma TE_i^2 TEL_i$

$SXLPP = \Sigma x_i TE_i TEL_i$

The solution of b can now be obtained by an iteration process. Starting with b = 1 the value of b is increased until BV = 0 in the above equation is satisfied.

The plotting positions for specified P-values can easily be derived from the distribution function:

$$P(x) = 1 - e^{-\left(\frac{x - x_o}{a}\right)^b} \qquad (22)$$

by rewritting the equation as:

$$x - x_o = a[-\log(1 - P)]^{1/b} \qquad (23)$$

By selecting a probability scale with the y-coördinate as a linear function of the right hand term of Eq.(23), the Weibull distribution function will become a straight line. It illustrates that the scale is depending on a and b.

Fig. 1a

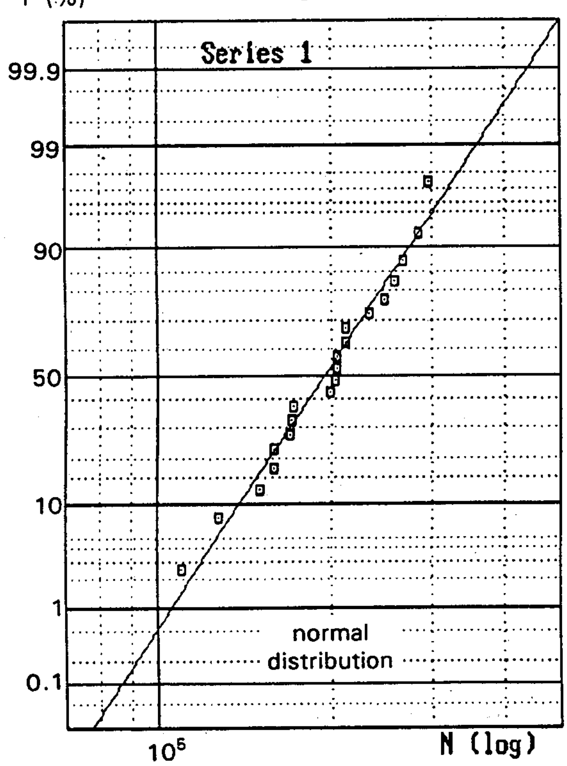


Fig. 1b

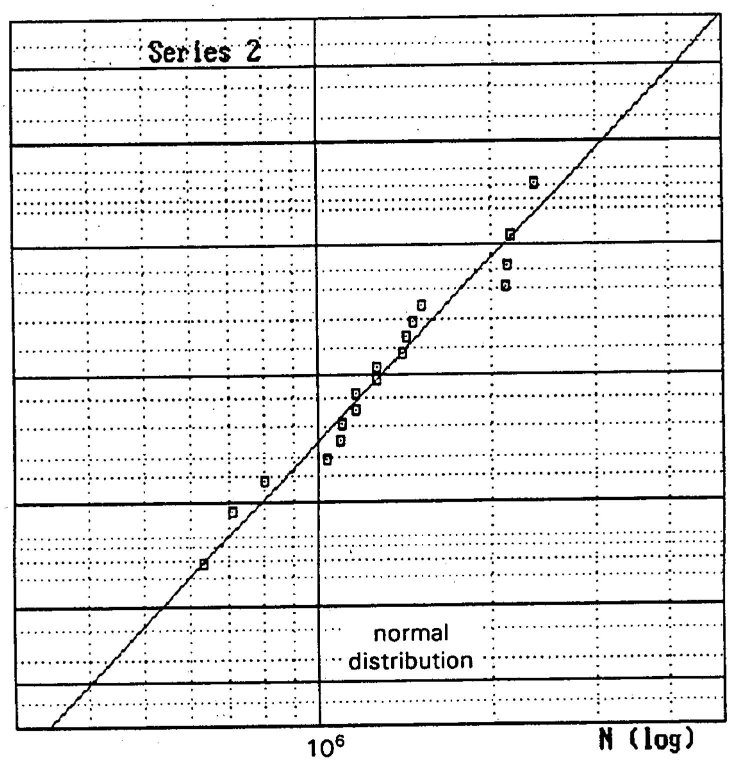


Fig. 2a

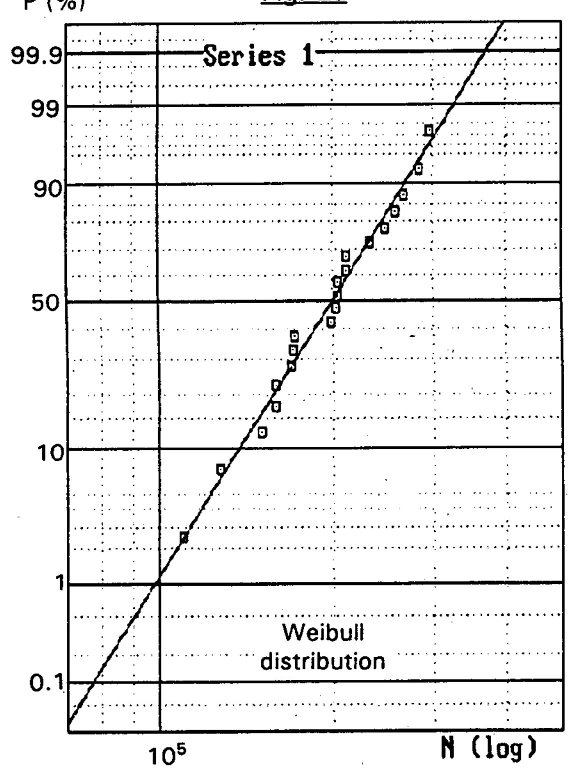


Fig. 2b

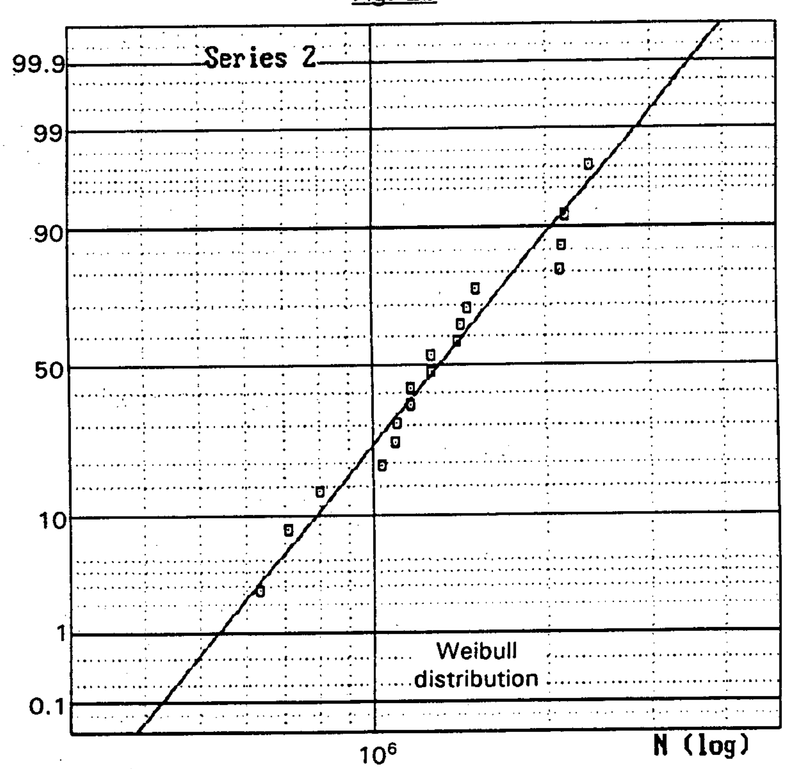


Fig. 1c

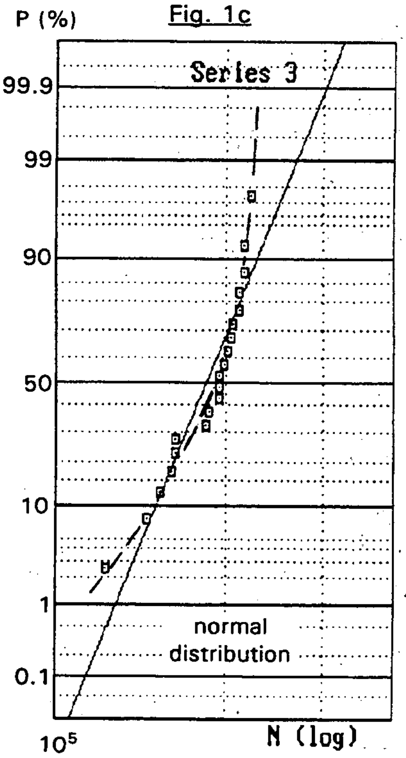


Fig. 1d

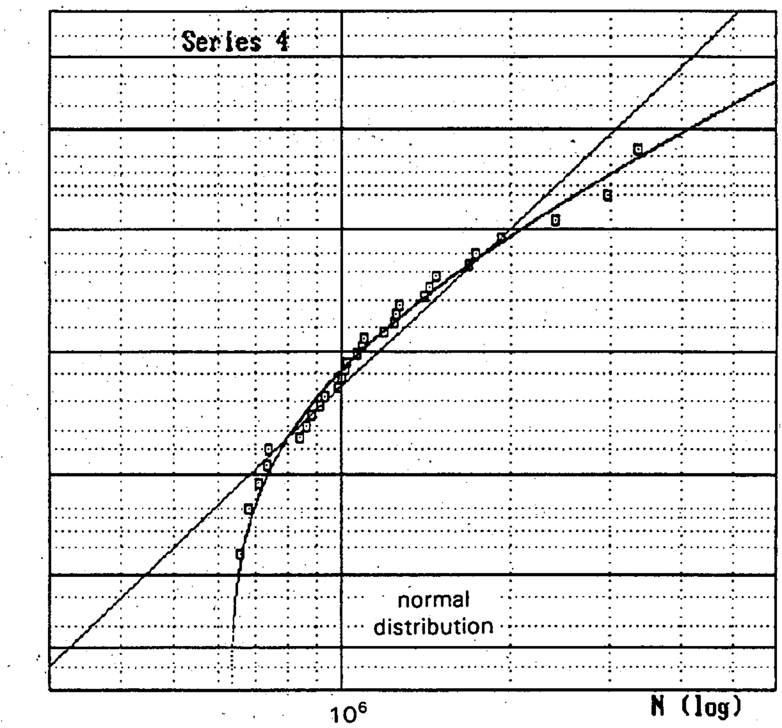


Fig. 2d

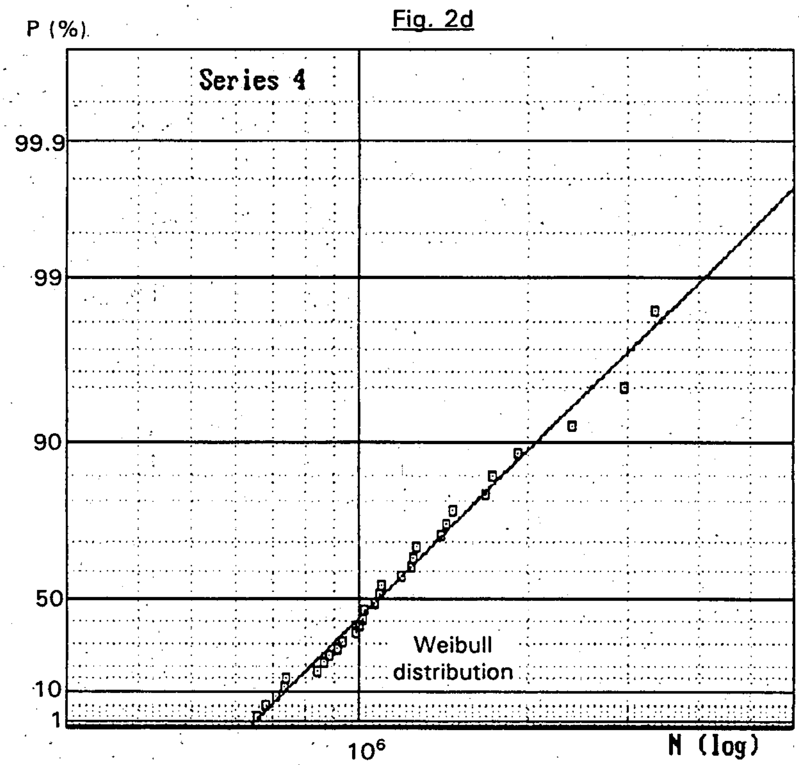


Fig. 1e

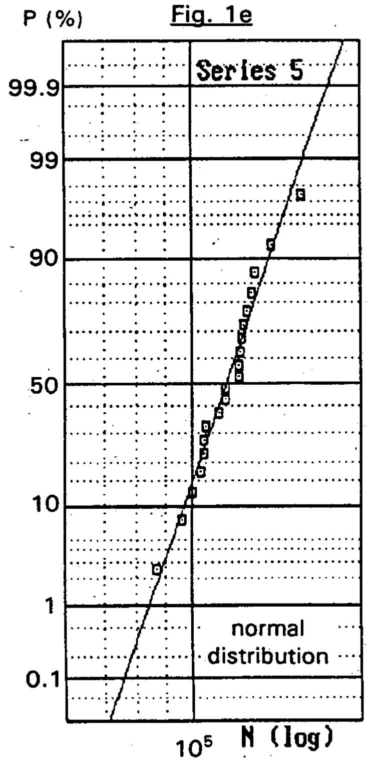


Fig. 1f

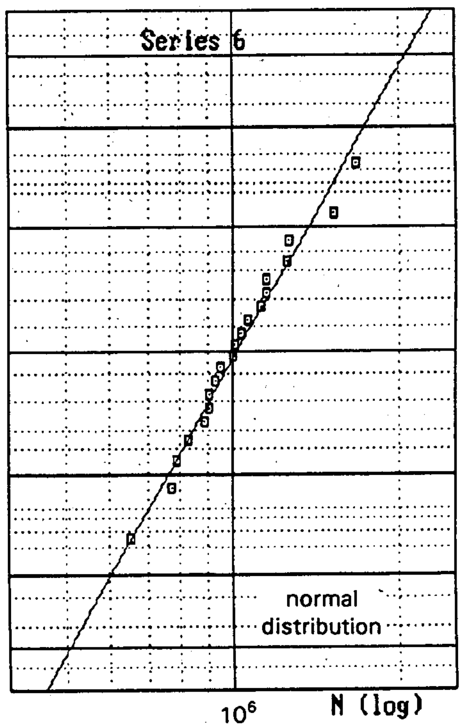


Fig. 2e

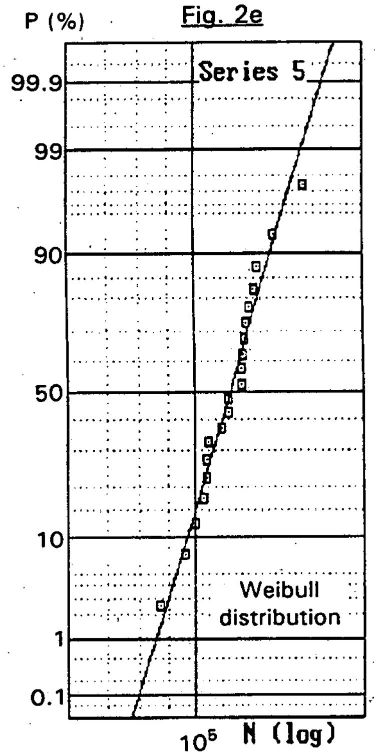
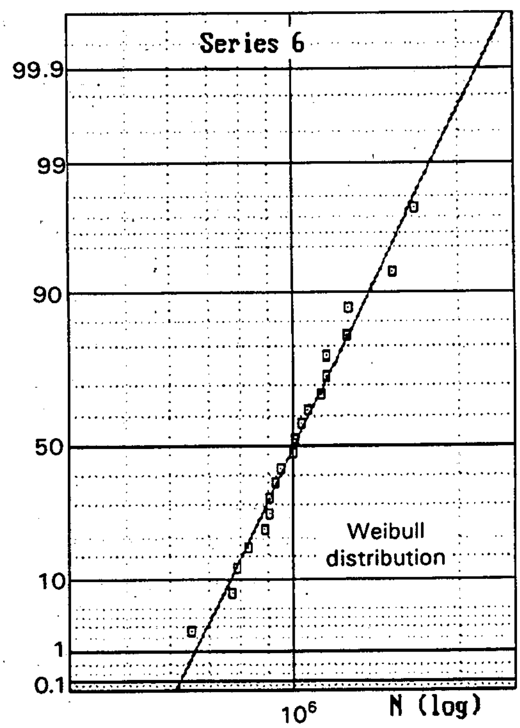


Fig. 2f



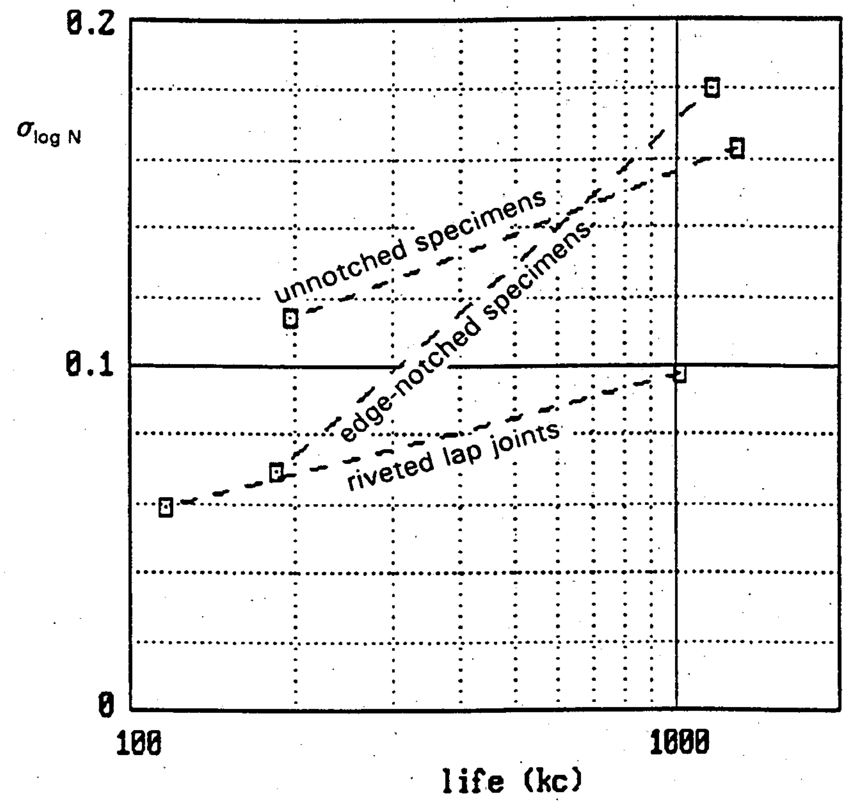


Figure 3. The standard deviations for the three types of specimens.

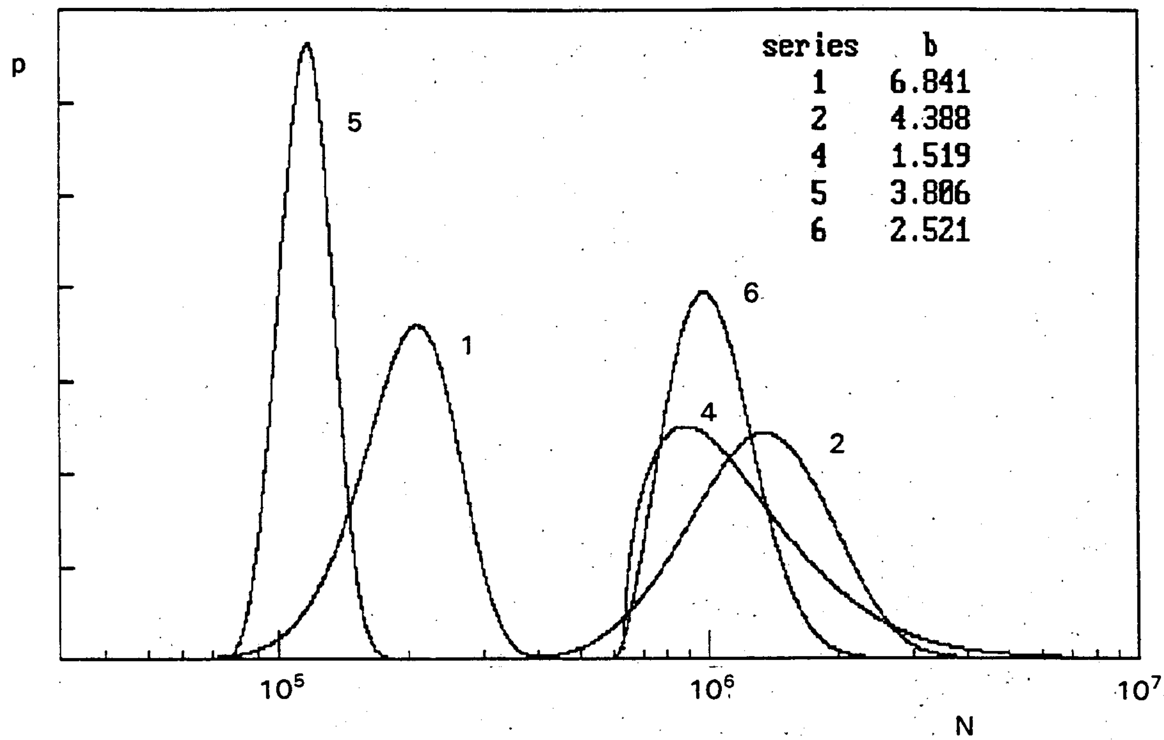


Figure 4. Weibull probability density functions for 5 test series.

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