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A NORMALIZED MODAL EIGENVALUE APPROACH FOR RESOLVING MODAL INTERACTION

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ABSTRACT

Modal interaction refers to the way that the modes of a structure interact when its geometry and material properties are perturbed. The amount of interaction between the neighboring modes depends on the closeness of the natural frequencies, the mode shapes, and the magnitude and distribution of the perturbation. By formulating the structural eigenvalue problem as a normalized modal eigenvalue problem, it is shown that the amount of interaction in two modes can be simply characterized by six normalized modal parameters and the difference between the normalized frequencies. In this paper, the statistical behaviors of the normalized frequencies and modes are investigated based on a perturbation analysis. The results are independently verified by Monte Carlo simulations.

INTRODUCTION

Variations in manufacturing, measurement, and material properties always cause engineering structures to vary a certain amount from their nominal design. Whether or not this variation significantly affects the structure's dynamic response depends on the magnitude of the variation and the characteristics of the original design. For systems with well separated natural frequencies, first-order perturbation theory (Fox and Kapoor, 1968) shows that the changes in natural frequencies and modes are small if the variation is small. The statistical behavior of the frequencies and modes of systems with well separated natural frequencies have been extensively studied, for example, by Collins and Thomson (1969) and Kiefling (1970). However, it is not unusual for a two- or three-dimensional structure to have closely spaced natural frequencies. In this case, simple perturbation theory indicates that the contributions from the unperturbed neighboring modes to a perturbed mode are significant and can result in large variations in the actual mode shapes.

This result has practical implications to gas turbine blading. Modern low aspect ratio blades often have natural frequencies that are close together. As a result, the modal stress fields of these modes could be highly variable from one blade to the next. This has clear implications to vibration testing and fatigue prediction. If the modal stress fields are highly variable then it becomes more difficult to characterize the stress ratios for a blade, more strain gages may be required, and it may be necessary to test more blades to assess scatter. For engineers encountering systems with frequencies that are close together an important concern is how close do the frequencies have to be before the modes become highly sensitive to structural variations. This concern provides the motivation for this study.

Papers by Sobczyk (1972) and by Schiff and Bogdanoff (1972) addressed the issue of predicting the frequency variation that occurs when the structure is perturbed, but did not discuss the variation in the mode shapes. More recently, the Stochastic Finite Element Method has been widely applied to this class of problems, for example refer to Vanmarcke and Grigoriu (1983), Shinozuka and Yamazaki (1988), or Ramu and Ganesan (1993). The drawbacks of this method are that it is computationally intensive and case specific. Consequently, the results of an analysis on one structure cannot be readily transferred to another.

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In this paper, the problem of modal interaction is formulated in terms of a normalized modal eigenvalue problem. A first-order perturbation solution is presented for the case of two modes. The statistical behavior of the normalized frequencies and modes are then determined from the results of the perturbation analysis. The range of validity of the perturbation solution is then examined by independent Monte Carlo simulations.

NORMALIZED MODAL EIGENVALUE PROBLEM

Consider the structural eigenvalue problem for an unperturbed system,

$$K^o \Phi^o = M^o \Phi^o {\Omega^o}^2 \tag{1}$$

where K^o and M^o are the stiffness and mass matrices. If ϕ_i^o and ω_i^o are the i-th mode shape and natural frequency of the unperturbed system, then Φ^o and Ω^o are the unperturbed modal and frequency matrices defined as:

$$\Phi^o = \left[\phi_1^o \ \phi_2^o \cdots \phi_N^o\right] \tag{2}$$

$$\Omega^o = diag\left(\omega_1^o \ \omega_2^o \cdots \omega_N^o\right) \tag{3}$$

where N is the number of degrees of freedom of the system. When the system is perturbed by variations in its structural properties, the stiffness and mass matrices are assumed to change by ΔK and ΔM, respectively. The perturbed structural eigenvalue problem can then be written as

$$\left(K^o + \Delta K\right)\Phi = \left(M^o + \Delta M\right)\Phi\Omega^2 \tag{4}$$

where Φ and Ω are the modal and the frequency matrices of the perturbed system with the following expressions,

$$\Phi = \left[\phi_1 \ \phi_2 \cdots \phi_N\right] \tag{5}$$

$$\Omega = diag(\omega_1 \ \omega_2 \cdots \omega_3) \tag{6}$$

where ϕ_i and ω_i are the i-th perturbed mode and its associated natural frequency. Since the unperturbed modes form a complete basis, it is possible to express the perturbed mode ϕ_j in terms of the unperturbed modes ϕ_i^o.

$$\phi_j = \sum_i \phi_i^o \alpha_{ij} \tag{7}$$

where α_{ij} is the component of the i-th unperturbed mode in the j-th perturbed mode. By substituting (7) in (4) and premultiply (4) by ${\Phi^o}^T$, (4) implies, because of the orthogonality of modes,

$$\Omega^o(I + \Delta I_k)\Omega^o C_\alpha = (I + \Delta I_m)C_\alpha \Omega^2 \tag{8}$$

where $C_\alpha = \left(\alpha_{ij}\right)$ is the coefficient matrix of αs. Under the assumption of unit modal mass, the perturbations in the normalized modal stiffness and mass matrices are

$$\Delta I_k = {\Omega^o}^{-1} {\Phi^o}^T \Delta K \Phi^o {\Omega^o}^{-1} \tag{9}$$

$$\Delta I_m = {\Phi^o}^T \Delta M \Phi^o \tag{10}$$

Now, if the normalized frequency matrix is defined as

$$\Gamma = \frac{1}{\overline{\omega}}\Omega \tag{11}$$

where $\overline{\omega}$ is a frequency reference, $\Gamma = diag(\gamma_1 \ \gamma_2 \cdots \gamma_N)$, and $\gamma_i = \omega_i/\overline{\omega}$, then, by combining (11) and (8), the normalized modal eigenvalue problem is formulated as

$$\Gamma^o(I + \Delta I_k)\Gamma^o C_\alpha = (I + \Delta I_m)C_\alpha \Gamma^2 \tag{12}$$

It should be noted that, in the case of zero perturbations, that is, when ΔI_k and ΔI_m are zero, the eigen-solution for (12) is

$$\Gamma = \Gamma^o \quad \text{and} \quad C_\alpha = C_\alpha^o = I \tag{13}$$

which means that the natural frequencies remain the same and each "perturbed" mode has only the component of the corresponding unperturbed mode. In general, when ΔI_k and ΔI_m are non-zero, the coefficient matrix C_α will not be a diagonal matrix. This indicates that the perturbed modes have non-zero components from several unperturbed modes. Clearly, the amount of modal interaction depends on how the coefficient matrix C_α changes. Note that C_α is only dependent on the perturbations in the normalized modal matrices (ΔI_k and ΔI_m) and on the distribution of the normalized frequencies Γ^o. As a result, equation (12) has resulted in isolating the issue of how close the unperturbed frequencies must be to have significant modal interaction from the effect of the perturbations in the modal stiffness ΔI_k and the modal mass ΔI_m.

PERTURBATION ANALYSIS

In order to gain better insight into the modal interaction problem consider the case of a system that consists of two closely spaced modes. Assume the two modes have unperturbed natural frequencies ω_1^o and ω_2^o. Then, by letting

$$\overline{\omega} = \frac{1}{2}\left(\omega_1^o + \omega_2^o\right) \tag{14}$$

$$\delta = \frac{1}{2\overline{\omega}}\left|\omega_1^o - \omega_2^o\right| \tag{15}$$

$$\Delta I_k = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \tag{16}$$

$$\Delta I_m = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \tag{17}$$

the normalized modal eigenvalue problem can be written as

$$\begin{bmatrix} 1-\delta & 0 \\ 0 & 1+\delta \end{bmatrix}\begin{bmatrix} 1+k_{11} & k_{12} \\ k_{12} & 1+k_{22} \end{bmatrix}\begin{bmatrix} 1-\delta & 0 \\ 0 & 1+\delta \end{bmatrix}\begin{Bmatrix} \alpha_{1j} \\ \alpha_{2j} \end{Bmatrix}$$

$$= \gamma_j^2 \begin{bmatrix} 1+m_{11} & m_{12} \\ m_{12} & 1+m_{22} \end{bmatrix}\begin{Bmatrix} \alpha_{1j} \\ \alpha_{2j} \end{Bmatrix} \tag{18}$$

where γ_j and $\left[\alpha_{1j}\ \alpha_{2j}\right]^T$ are the j-th normalized frequency and mode shape of the perturbed system. Assuming that k_{ij} and m_{ij} are small quantities and neglecting higher order terms in k_{ij} and m_{ij}, the solution for (18) is

$$\gamma_1 \cong (1-\delta)\left(1 + \frac{k_{11}}{2} - \frac{m_{11}}{2}\right) \tag{19}$$

$$\gamma_2 \cong (1+\delta)\left(1 + \frac{k_{22}}{2} - \frac{m_{22}}{2}\right) \tag{20}$$

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & \dfrac{\left(1-\delta^2\right)k_{12} - (1+\delta)^2 m_{12}}{4\delta} \\ -\dfrac{\left(1-\delta^2\right)k_{12} - (1-\delta)^2 m_{12}}{4\delta} & 1 \end{bmatrix} \tag{21}$$

where γ_1 and γ_2 are the perturbed frequencies, α_{21} is the component of the second unperturbed mode in the first perturbed mode, and α_{12} is the component of the first unperturbed mode in the second perturbed mode. Note that when δ is small, (21) implies that the interaction between neighboring modes can be quite large, even though the structural perturbations are small.

Once the expressions in (19)-(21) were developed, it was possible to define the statistical behaviors of γ_1, γ_2, α_{21}, and α_{12}. Assume that the small quantities k_{ij} and m_{ij} are normally distributed about their mean values $\overline{k}_{ij}$ and $\overline{m}_{ij}$ with standard deviations $\sigma_{k_{ij}}$ and $\sigma_{m_{ij}}$, respectively. The mean values for the frequencies and the amount of modal interaction can be obtained by simply substituting k_{ij} and m_{ij} in (19)-(21) by $\overline{k}_{ij}$ and $\overline{m}_{ij}$. By applying basic multivariate statistical analysis, the standard deviations of γ_1, γ_2, α_{21}, and α_{12} are

$$\sigma_{\gamma_1} = \frac{(1-\delta)}{2}\sqrt{\sigma_{k_{11}}^2 + \sigma_{m_{11}}^2 - 2\operatorname{cov}(k_{11}, m_{11})} \tag{22}$$

$$\sigma_{\gamma_2} = \frac{(1+\delta)}{2}\sqrt{\sigma_{k_{22}}^2 + \sigma_{m_{22}}^2 - 2\operatorname{cov}(k_{22}, m_{22})} \tag{23}$$

$$\sigma_{\alpha_{21}} = \frac{1}{4\delta}\Big[\left(1-\delta^2\right)^2 \sigma_{k_{12}}^2 + (1-\delta)^4 \sigma_{m_{12}}^2$$

$$-2\left(1-\delta^2\right)(1-\delta)^2 \operatorname{cov}(k_{12}, m_{12})\Big]^{1/2} \tag{24}$$

$$\sigma_{\alpha_{12}} = \frac{1}{4\delta}\Big[\left(1-\delta^2\right)^2 \sigma_{k_{12}}^2 + (1+\delta)^4 \sigma_{m_{12}}^2$$

$$-2\left(1-\delta^2\right)(1+\delta)^2 \operatorname{cov}(k_{12}, m_{12})\Big]^{1/2} \tag{25}$$

Equations (22)–(25) give a simple way of calculating the standard deviations of the normalized natural frequencies γ_1 and γ_2 and the amounts of the modal interaction α_{21} and α_{12} for a given frequency difference 2δ when the standard deviations and covariances of the six normalized modal perturbations are small.

MONTE CARLO SIMULATION

Two-Mode Case Study

In order to explore the limitations of the perturbation analysis, a Monte Carlo simulation is conducted for the normalized modal eigenvalue problem defined by (18). The mean values and the covariances of the normalized modal perturbations are assumed to be zero. The mass and stiffness standard deviations are assumed to be the same, that is,

$$\sigma \equiv \sigma_{k_{ij}} = \sigma_{m_{ij}} \qquad \forall i, j \tag{26}$$

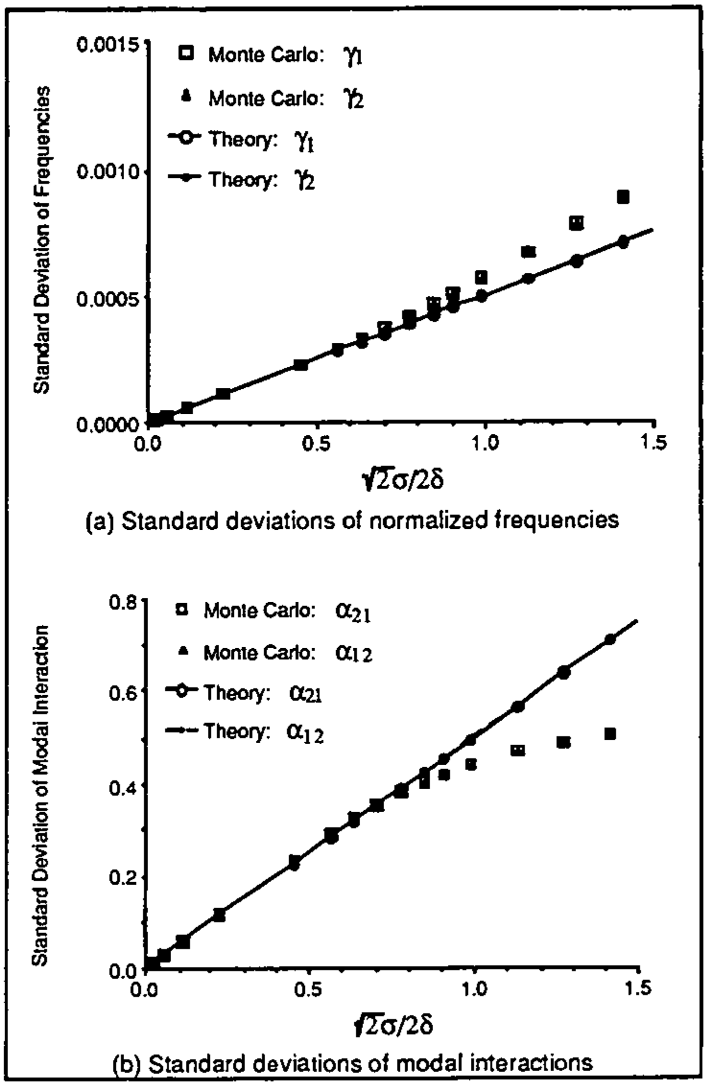
Since the case of closely spaced modes is of primary interest, the frequency difference 2δ will be assigned values significantly less than one. The results of the Monte Carlo simulation will be compared with that predicted by equation (22)–(25). Under the above assumptions, equations (22)–(25) may be simplified to:

$$\sigma_{\gamma_1} = \sigma_{\gamma_2} = \frac{1}{2}\left(\sqrt{2}\sigma\right) \tag{27}$$

$$\sigma_{\alpha_{21}} = \sigma_{\alpha_{12}} = \frac{1}{2}\left(\frac{\sqrt{2}\sigma}{2\delta}\right) \tag{28}$$

Equations (27) and (28) imply that the standard deviation in the natural frequencies and in the modal interactions should increase linearly with σ when δ is fixed. Monte Carlo results[1] depicted in Figure 1(a) and 1(b) show the linear relationship holds reasonably well for $\sqrt{2}\sigma/2\delta$ less than 0.8. Note from Figure 1(b) that the linear relationship holds for values of $\sigma_{\alpha_{12}}$ up to 0.4 and, consequently, can be used to predict relatively large amounts of modal interaction. The Monte Carlo simulations confirm that the perturbation results also hold when δ is varied and σ is held fixed, Figures 2(a) and 2(b). From Figure 2(a) the standard deviation in the frequency is relatively independent of δ when δ is small — a result which is consistent with equation (27).

[1] Based on 10,000 simulations.



(a) Standard deviations of normalized frequencies

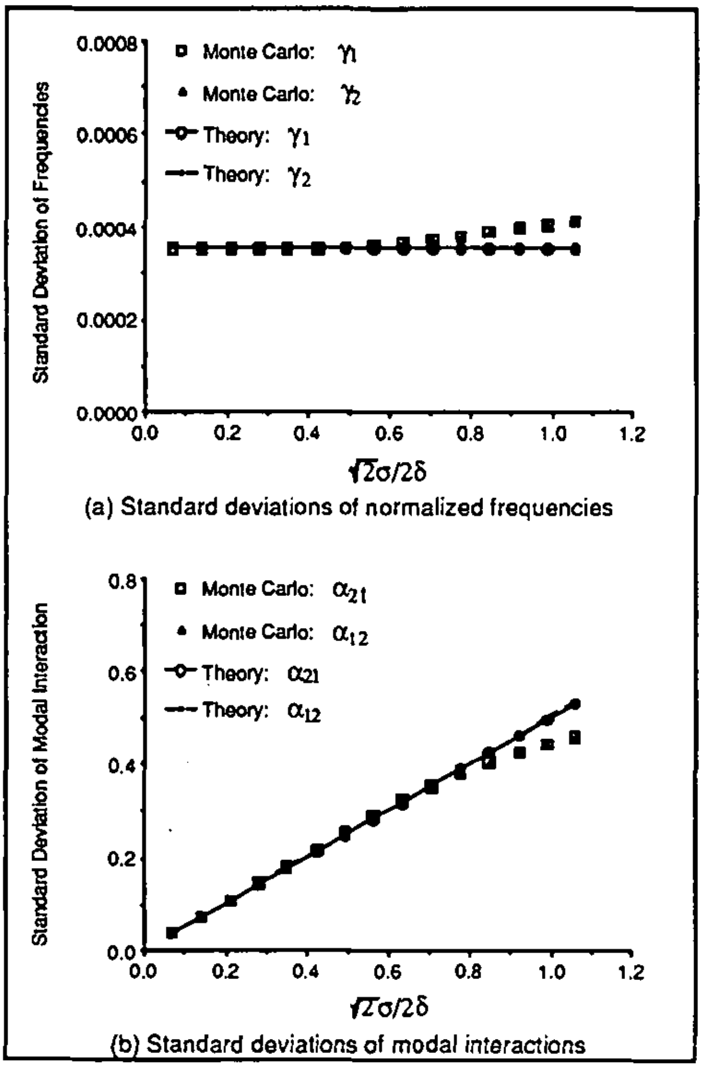
(b) Standard deviations of modal interactions

Figure 1: Two mode case study: $\delta = 0.0005$, σ varies

Similarly, from Figure 2(b) it can be seen that $\sigma_{\alpha_{12}}$ is linearly proportional to $\sqrt{2}\sigma / 2\delta$ which is consistent with equation (28).

Four-Mode Case Study

A practical concern is to what extent the perturbation results developed for the two mode case can be applied to a system that has more than two modes that are close together. This concern is investigated by performing a Monte Carlo simulation of a system with four closely spaced modes. Figure 3 indicates the spacing of the unperturbed frequencies. This study concentrates on the representative case of the interaction between the second and the third modes and how it is affected by the closeness of the first and fourth modes. The selection of the normalized modal parameters is essentially the same as in the two-mode case study. In addition, it is assumed that the standard deviations σ_{k_y} and σ_{m_y} associated with the first and fourth modes are the same as those of the two center modes.



(a) Standard deviations of normalized frequencies

(b) Standard deviations of modal interactions

Figure 2: Two mode case study: δ *varies*, $\sigma = 0.0005$

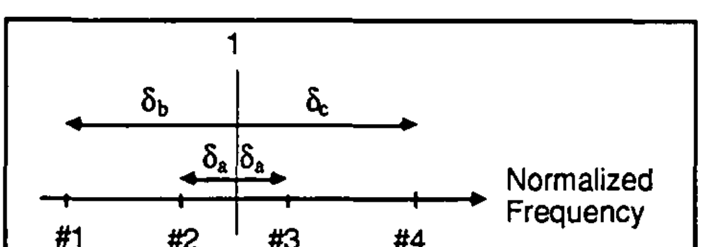
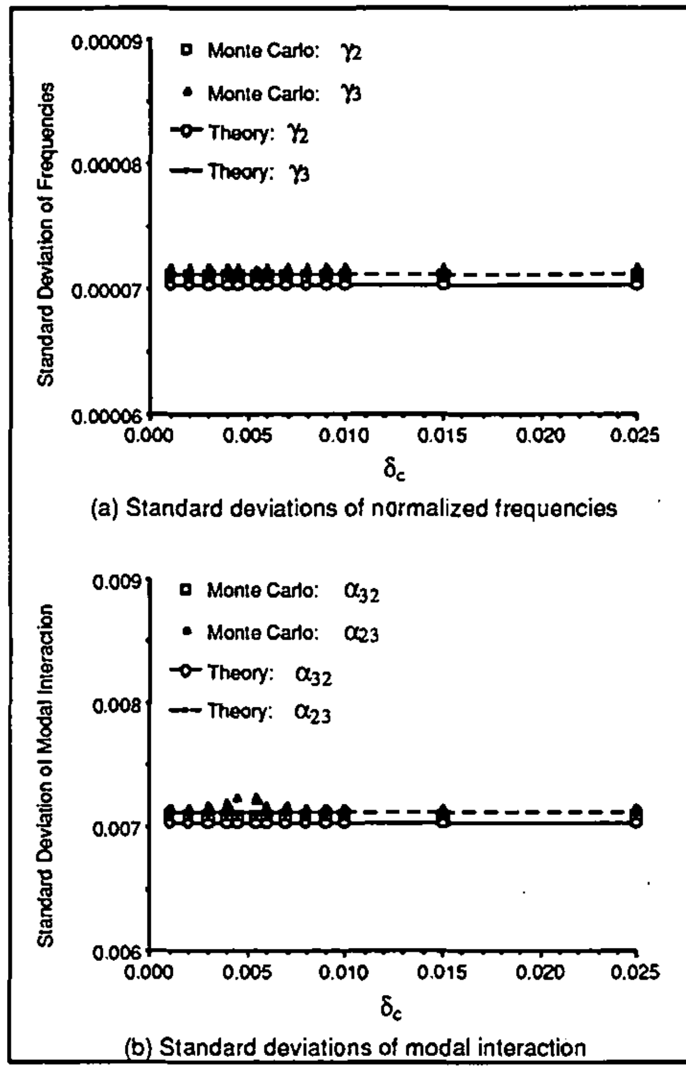


Figure 3: Frequencies of the four mode case



(a) Standard deviations of normalized frequencies

(b) Standard deviations of modal interaction

Figure 4: Four mode case study: $\delta_a = 0.005$, $\delta_b = 0.010$, δ_c *varies*, $\sigma = 0.0001$

Representative results from the Monte Carlo simulation are depicted in Figures 4(a) and 4(b). The results in Figure 4(a) indicate that the standard deviations in the second and third frequencies are essentially independent of the distance, δ_c. This is not surprising since this result is consistent with the two mode case (Figure 2(a)) and the behavior predicted by equations (22) and (23). Figure 4(b) indicates that the Monte Carlo generated σ_αs agree reasonably well with the theory (equations (24) and (25)) when the interaction between the third and fourth modes is small. Figure 4(b) also indicates that there is some deviation from the two-mode theory when the third and fourth modes are sufficiently close together.

Consequently, the results of the four mode Monte Carlo simulation appear to infer that the perturbation results developed for the two mode case is applicable to structures where several modes have frequencies that are close together provided that the neighboring modes do not significantly interact with the center modes. This result significantly simplifies the determination of the likelihood of modal interaction in a complex structure since it means that modes can be dealt with a pair at a time.

CONCLUSIONS

In this paper the perturbed structural eigenvalue problem is formulated as a normalized modal eigenvalue problem. The advantage of this formulation is that it more clearly separates frequency and structural effects in the modal interaction problem. As a result, a perturbation analysis of the normalized problem yields the results that the standard deviation in the interaction between two modes is approximately given by:

$$\sigma_{\alpha_{ij}} = \frac{1}{4\delta}\sqrt{2}\,\sigma \tag{29}$$

where δ characterizes the closeness of the frequencies $(\Delta f/\bar{f})$ for the nominal geometry and σ characterizes the variation in the structural properties that corresponds to the modes in question. Thus, for example, if an engineer wants to insure that a perturbed mode will contain no more than 10% of an unperturbed neighboring mode, then $\sigma_{\alpha_{ij}}$ could be limited to a third of that value, i.e., 0.0333. The frequency parameter δ could be determined from the natural frequencies of the nominal geometry using a standard finite element analysis. Then equation (29) would yield a maximum allowable value of σ.

The structural parameter σ depends on the mode shapes of the specific modes of interest, as well as, the amount of variability that occurs in the geometry and in the material properties. It may be possible to develop estimates of σ for certain classes of problems (cast compressor blades, for example) in which the types of structures, modes, and manufacturing processes are limited. This will be the next area of research in this research program. If it is possible to establish estimates for σ, then equation (29) and the natural frequencies of the nominal geometry could be used to quickly determine which modes would be likely to have high variability.

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