PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, Number 3, August 1971

A NOTE CONCERNING $A^* = L_1(\mu)$

H. ELTON LACEY[1]

ABSTRACT. It is shown that there are exactly two abstract L-spaces which are duals of infinite dimensional separable Banach spaces.

A problem of long standing is to characterize those Banach spaces of the type $L_1(\mu)$ which are isomorphic to a dual Banach space. One of the early negative results was Gelfand's theorem that the space $L_1[0, 1]$ is not isomorphic to a dual space. Recently Pelczynski has given an extension of this result to the case $L_1(\mu)$ for μ a σ-finite non-purely atomic measure [6]. In a recent paper [8], Rosenthal has given some deep results on the structure of $C(X)^*$ for certain compact Hausdorff spaces X and a generalization of Pelczynski's result.

The main purpose of this note is to show that there are exactly two possible spaces of the type $L_1(\mu)$ which can be the dual of an infinite dimensional separable Banach space. This is closely related to some results in [8, p. 242].

NOTATION. Let m be a cardinal number and S a set of cardinal m. The L-space $l_1(m)$ is the Banach space of all absolutely summable real-valued functions on S with $\|f\| = \sum_{s\in S} |f(s)|$ for all $f \in l_1(m)$. The L-space $L_1[0, 1]$ is the classical space of all Lebesgue integrable functions on $[0, 1]$ with the norm $\|f\| = \int_0^1 |f(t)|\, dt$ for all $f \in L_1[0, 1]$. If $\{X_i\}_{i\in I}$ is a family of Banach spaces, $[\oplus \sum_I X_i]_1$ denotes the Banach space whose elements are the indexed sets $\{x_i\}_{i\in I}$ with $x_i \in X_i$ for all $i \in I$ and $\sum_{i\in I} \|x_i\| < \infty$. This last number is the norm of $\{x_i\}_{i\in I}$. All Banach spaces in this note are over the real number field and infinite dimensional. By an L-space it is meant a space of the type $L_1(\mu)$.

The following lemma gives a linear norm preserving extension of A^* into B^* when $A \subset B$ and A^* is an L-space.

LEMMA 1. *Let A and B be Banach spaces with $A \subset B$ and A^* an L-space. Then there is a linear isometry $T: A^* \to B^*$ such that $T(x^*)(x) = x^*(x)$ for all $x^* \in A^*$ and $x \in A$.*

Presented to the Society, August 25, 1970; received by the editors July 20, 1970 and, in revised form, December 13, 1970.

AMS 1970 *subject classifications*. Primary 46E15, 46E30; Secondary 46E05.

Key words and phrases. L-space, dual L-space, spaces whose duals are L-spaces.

[1] The author was partially supported by AFSOR Grant 69-1829.

PROOF. Let $T:A^*\to A^{***}$, $K:B\to B^{**}$, $k:A\to B$ be the natural maps. Since A^* is an L-space, A^{**} is a P_1 space. Hence there is a contractive projection P on B^{**} with range $k^{**}(A^{**})$. Let $T = [(k^{**})^{-1}PK]^* J$. Then T has the required properties.

Before stating the theorem, the essential property of the compact Hausdorff spaces under consideration is discussed. Let X be a compact Hausdorff space such that for each regular Borel measure μ on X, $L_1(\mu)$ is separable. Examples of such spaces are the compact metric spaces and the weakly compact subsets of Banach spaces (see [**3**]). This last class includes spaces of arbitrarily high cardinal. Also, the dispersed spaces have this property since every regular Borel measure on a dispersed space is purely atomic.

THEOREM. *Let X be a compact Hausdorff space with the above property and A a closed linear subspace of X. If $A^*=L_1(\mu)$ where μ is not purely atomic, then A^* is linearly isometric to $(l_1(m)\oplus\sum_n L_1[0,1])_1$ for some cardinals m and n with $m, n\geqq c$, the cardinality of the continuum.*

PROOF. By a standard argument involving Zorn's lemma and the decomposition of L-spaces into direct sums of subspaces with weak order units, it can be assumed that

$$A^* = \left[l_1(m) + \left(\oplus \sum_{n\in N} L_1(\mu_n)\right)_1\right]_1$$

where m is a cardinal and N is a set of cardinals and μ_n is a finite nonatomic measure for each $n\in N$. If $L_1(\mu_n)$ is not separable, then it contains a copy of some nonseparable Hilbert space H [**9**], and by the lemma $C(X)^*$ contains a copy of H. This is impossible by Lemma 1.3 of [**8**]. Thus $L_1(\mu_n)$ is separable and since μ_n is nonatomic, $L_1(\mu_n) = L_1[0,1]$ for each $n\in N$.

Since A^* is not of the type $l_1(a)$, it follows from [**1**] that there is a separable subspace B of A such that $B^*=L_1(\nu)$ and ν is not purely atomic. Hence, by [**2**], B contains a subspace C such that $C^* = C[0,1]^*$. Since every extreme point of the unit sphere of C^* is the image of an extreme point in the unit sphere of A^* and $m=$ cardinality of the extreme points of the unit sphere of A^*, $m\geqq c$.

Now, it is well known that

$$C[0,1]^* = \left[l_1(c) + \left(\oplus \sum_c L_1[0,1]\right)_1\right]$$

(in fact, see [**8**] for a more general result). Since each $L_1[0,1]$ contains a copy of a separable Hilbert space H, by the lemma there is a linear isometry T of $(\oplus \sum_{s\in S} H_s)_1$ into A^* where S is a set of cardinality c and H_s is the same isomorphic copy of H for all $s\in S$. Let P be the natural projection in A^* with range $l_1(m)$ and null space $(\oplus \sum_{n\in N} L_1(\mu_n))_1$. Then $PT|H_s$ is compact for each $s\in S$ from the appendix in [9]. Thus there is an infinite-dimensional subspace H_s' of H_s such that $\|PT|H_s'\|<\frac{1}{2}$. Hence $\|(I-P)Th\|\geqq\frac{1}{2}\|h\|$ for all h in $(\oplus \sum_{s\in S} H_s')_1$. That is, $(\oplus \sum_{n\in N} L_1(\mu_n))_1$ contains an isomorphic copy of $(\oplus \sum_{s\in S} H_s')_1$. Since the smallest cardinal of a dense set in $(\oplus \sum_{s\in S} H_s')$ is c, it follows that card $N\geqq c$.

SUMMARY. If A is a separable Banach such that $A^*=L_1(\mu)$, then

(a) $A^*=l_1=l_1(\aleph_0)$ if and only if the set of extreme points in the unit sphere of A is countable;

(b) $A^*=[l_1(c)+(\oplus \sum_c L_1[0, 1])_1]$ if and only if the set of extreme points of the unit sphere of A^* is uncountable.

The necessity of (a) is clear. The sufficiency follows from the fact that the unit sphere of A^* is weak* metrizable and, hence, every element of the unit sphere is "represented" by a measure supported on the extreme points (see [7]). It is noted that an incorrect proof of this result appears in [5].

The necessity of (b) is clear and the sufficiency follows from the theorem since $A\subset C[0, 1]$.

Using a different argument, it can actually be shown that if

$$A^* = \left[l_1(m) + \left(\oplus \sum_{n\in N} L_1(\mu_n)\right)_1\right]_1$$

with $N\neq\emptyset$, then $m\geqq c$ and card $N\geqq c$.

The author thanks the referee for pointing out how the proof could be given without resorting to the order structure of L-spaces.

REFERENCES

1. H. Elton Lacey and Peter D. Morris, *On spaces of type $A(K)$ and their duals*, Proc. Amer. Math. Soc. **23** (1969), 151–157.

2. A. J. Lazar and J. Lindenstrass, *Banach spaces whose duals are L spaces and their representing matrices*, Acta. Math. (to appear).

3. J. Lindenstrass, *Weakly compact sets-their topological properties and the Banach spaces they generate*, Proc. Sympos. Infinite Dimensional Topology, Louisiana State University (to appear).

4. D. Maharam, *On homogeneous measure algebras*, Proc. Nat. Acad. Sci. U.S.A. **28** (1942), 108–111. MR **4**, 12.

5. R. Nirenberg and P. Panzone, *On the spaces L^1 which are isomorphic to a B^**, Rev. Un. Math. Argentina **21** (1963), 119–130. MR **29** #5090.

6. A. Pełczynski, *On Banach spaces containing* $L_1(\mu)$, Studia Math. **30** (1968), 231–246. MR **38** #521.

7. R. R. Phelps, *Lectures on Choquet's theorem,* Van Nostrand, Princeton, N.J., 1966. MR **33** #1690.

8. H. P. Rosenthal, *On injective Banach spaces and the spaces* $L^\infty(\mu)$ *for finite measures,* Acta Math. **124** (1970), 205–248.

9. ———, *On quasi-complemented subspaces of Banach spaces, with an appendix on compactness of operators from* $L^p(\mu)$ *to* $L^r(\nu)$, J. Functional Analysis **4** (1969), 176–214. MR **40** #3277.

University of Texas, Austin, Texas 78712