A NOTE CONCERNING THE L_1 CONVERGENCE OF A CLASS OF GAMES WHICH BECOME FAIRER WITH TIME

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(Received 4 March, 1970; revised 1 August, 1970)

Throughout this note, let $(\Omega, \mathfrak{A}, P)$ be a probability space with $\{\mathfrak{A}_n\}_{n \geqq 1}$ an increasing sequence of sub σ-fields of $\mathfrak{A}$ whose union generates $\mathfrak{A}$. Let $\{X_n\}_{n \geqq 1}$ be a sequence of random variables adapted to $\{\mathfrak{A}_n\}_{n \geqq 1}$ (see [3], p. 65) and henceforth be referred to as a game. As in [1], the game $\{X_n\}_{n \geqq 1}$ will be said to become fairer with time if, for every $\varepsilon > 0$,

$$P[|E(X_n \mid \mathfrak{A}_m) - X_m| > \varepsilon] \to 0$$

as $n, m \to \infty$ with $n \geqq m$.

The purpose of this note is to establish the

THEOREM. *If $\{X_n\}_{n \geqq 1}$ is a game which becomes fairer with time and if there exists $Z \in L_1(\Omega, \mathfrak{A}, P)$ such that $|X_n| \leqq Z$ for all $n \geqq 1$, then $\{X_n\}_{n \geqq 1}$ converges in the L_1 norm.*

Proof. The result will be established by exhibiting a martingale game, $\{Y_n\}_{n \geqq 1}$, adapted to $\{\mathfrak{A}_n\}_{n \geqq 1}$, which converges in the L_1 norm to a random variable Y and showing that the process $\{X_n - Y_n\}_{n \geqq 1}$ converges in the L_1 norm to zero.

First, the martingale $\{Y_n\}_{n \geqq 1}$ is produced. Let $\{X_n\}_{n \geqq 1}$ be used to define a sequence of signed measures on each $\mathfrak{A}_n$ as follows:

$$\mu_n(A) \equiv \int_A X_n \, dP \quad \text{for all} \quad A \in \mathfrak{A}_n.$$

Observe that μ_n is a signed measure on $\mathfrak{A}_k$ for all $k \leqq n$ and also that μ_n is a finite signed measure inasmuch as $|X_n| \leqq Z$ all n.

Observe also that, for n arbitrary but fixed and $A \in \mathfrak{A}_n$, the sequence $\{\mu_p(A)\}_{p \geqq n}$ is Cauchy. This follows immediately from the inequality

$$|\mu_p(A) - \mu_q(A)| \leqq \int |(E(X_p \mid \mathfrak{A}_q) - X_p)| \, dP \tag{1}$$

and the facts that $|X_n| \leqq Z$ for all n and that $\{X_n\}_{n \geqq 1}$ becomes fairer with time.

Thus, let ν_n be a set function on $\mathfrak{A}_n$ defined by

$$\nu_n(A) \equiv \lim_{\substack{p \to \infty \\ p \geqq n}} \mu_p(A) \quad \text{for each} \quad A \in \mathfrak{A}_n.$$

Where $|X_k| \leqq Z$ for all $k \geqq 1$, it is immediate that $|\nu_n(A)| < \infty$ for all $A \in \mathfrak{A}_n$ and hence that ν_n is a signed measure on $\mathfrak{A}_n$. (See [2], p. 159.)

It follows from the construction that, for any n and m with $n \geqq m$, $\nu_n = \nu_m$ on $\mathfrak{A}_m$ and also that, for every n, $\nu_n \ll P$. Hence there exists a martingale game $\{Y_n\}_{n \geqq 1}$ adapted to $\{\mathfrak{A}_n\}_{n \geqq 1}$ with $\nu_n(A) = \int_A Y_n \, dP$ for all $A \in \mathfrak{A}_n$.

https://doi.org/10.1017/S0017089500001348 Published online by Cambridge University Press

It will next be established that, for every n, $Y_n^+ \leqq E(Z|\mathfrak{A}_n)$ a.e. and $Y_n^- \leqq E(Z|\mathfrak{A}_n)$ a.e. This will show that $\{Y_n\}_{n\geqq 1}$ is a uniformly integrable martingale and hence that it converges in the L_1 norm.

Let

$$K_n \equiv \{w\colon Y_n^+ > E(Z|\mathfrak{A}_n)\} \quad \text{and}$$

$$D_n \equiv \text{the positive set for } \nu_n.$$

Without loss of generality, assume $P(D_n) > 0$. It follows that

$$\int_{K_nD_n} Y_n^+ \, dP = \nu_n(K_nD_n) = \lim_{\substack{p\to\infty \\ p\geqq n}} |\mu_p(K_nD_n)| = \lim_{\substack{p\to\infty \\ p\geqq n}} \left|\int_{K_nD_n} X_p \, dP\right| \leqq \int_{K_nD_n} Z \, dP.$$

Hence $P(K_nD_n) = 0$; otherwise,

$$\int_{K_nD_n} Y_n^+ \, dP > \int_{K_nD_n} E(Z|\mathfrak{A}_n) \, dP = \int_{K_nD_n} Z \, dP,$$

which gives a contradiction. Also, where $Z \geqq 0$ it follows that $E(Z|\mathfrak{A}_n) \geqq 0$ a.e. and thus $P(K_nD_n^c) = 0$. Thus $P(K_n) = 0$ and $Y_n^+ \leqq E(Z|\mathfrak{A}_n)$ a.e.

A similar argument shows that $Y_n^- \leqq E(Z|\mathfrak{A}_n)$ a.e.

The proof will be finished by proving that $\{X_n - Y_n\}_{n\geqq 1}$ converges to zero in the L_1 norm. To do so, one need only show that

$$\lim_{n\to\infty} |\mu_n(A_n) - \nu_n(A_n)| = 0 \tag{2}$$

for any sequence $\{A_n\}_{n\geqq 1}$ of sets from $\{\mathfrak{A}_n\}_{n\geqq 1}$; that is, that $A_n \in \mathfrak{A}_n$ all n. For, in particular, let B_n be the positive set for $(X_n - Y_n)$ and C_n be the negative set for $(X_n - Y_n)$, for all n. Then

$$\int |X_n - Y_n| \, dP \leqq |\mu_n(C_n) - \nu_n(C_n)| + |\mu_n(B_n) - \nu_n(B_n)|$$

and by (2) it is immediate that $\lim_{n\to\infty} \|X_n - Y_n\|_1 = 0$.

To establish (2), let $\varepsilon > 0$ be given. For n and ε given, there exists an integer $k_{n,\varepsilon}$, which may, without loss of generality, be taken greater than n, such that

$$|\nu_n(A_n) - \mu_{k_{n,\varepsilon}}(A_n)| < \varepsilon/2. \tag{3}$$

Also, for all n greater than some sufficiently large integer

$$|\mu_n(A_n) - \mu_{k_{n,\varepsilon}}(A_n)| < \varepsilon/2; \tag{4}$$

this follows by again using the inequality set forth in (1) and the facts that the game $\{X_n\}_{n\geqq 1}$ is uniformly (a.e.) dominated in absolute value by Z and becomes fairer with time. Inequalities (3) and (4) establish (2) and the proof is complete.

https://doi.org/10.1017/S0017089500001348 Published online by Cambridge University Press

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