CLOSED FORMULA FOR POLY-BERNOULLI NUMBERS

Roberto Sánchez-Peregrino
Dipartimento di Matematica Pura ed Applicata, Via Belzoni 7, I-35131 Padova, Italy
e-mail: sanchez@math.unipd.it
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1. INTRODUCTION AND BACKGROUND

In the present note we shall give two proofs of a property of the poly-Bernoulli numbers, the closed formula for negative index poly-Bernoulli numbers given by Arakawa and Kaneko [1]. The first proof uses weighted Stirling numbers of the second kind (see [2], [3]). The second, much simpler, proof is due to Zeilberger.

In Kaneko's paper, "On Poly-Bernoulli Numbers" [5], the poly-Bernoulli numbers, which generalize the classical Bernoulli numbers, are defined and studied. For every integer k, called the index, we define a sequence of rational numbers B_n^k $(n = 0, 1, 2, \ldots)$, which we refer to as poly-Bernoulli numbers, by

$$\frac{1}{z}\mathrm{Li}_k(z)\big|_{z=1-e^{-x}} = \sum_{n=0}^{\infty}\mathrm{B}_n^k\frac{x^n}{n!}. \tag{1}$$

Here, for any integer k, $\mathrm{Li}_k(z)$ denotes the formal power series $\sum_{m=1}^{\infty} z^m/m^k$, which is the k^{th} polylogarithm if $k \geq 1$ and a rational function if $k \leq 0$. When $k = 1$, B_n^1 is the usual Bernoulli number (with $\mathrm{B}_1^1 = 1/2$). In [4] Kaneko obtained an explicit formula for B_n^k:

$$\mathrm{B}_n^k = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m!}{(m+1)^k} \left\{ {n \atop m} \right\}, \tag{2}$$

where $\left\{ {n \atop m} \right\}$ is an integer referred to as a Stirling number of the second kind [6].

2. CLOSED FORMULA

Theorem 2.1 (Closed Formula): For any $n, k \geq 0$, we have

$$\mathrm{B}_n^{-k} = \sum_{j=0} (j!)^2 \left\{ {n+1 \atop j+1} \right\} \left\{ {k+1 \atop j+1} \right\}. \tag{3}$$

We need two lemmas. We use the notation and numeration of the equations in Carlitz's paper [3].

Lemma 2.1:

$$\sum_{m=0}^{n} (-1)^m m! \binom{m}{\ell} \left\{ {n \atop m} \right\} = (-1)^n \ell! \left\{ {n+1 \atop \ell+1} \right\} = (-1)^n \ell!\,\mathrm{R}(n, \ell, 1), \tag{4}$$

where

$$\mathrm{R}(n, k, \lambda) = \sum_{m=0}^{n-k} \binom{n}{m} \left\{ {n-m \atop k} \right\} \lambda^m.$$

Proof: In order to prove this lemma, we calculate the generating function:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} (-1)^m m! \binom{m}{\ell} \left\{ {n \atop m} \right\} \frac{z^n}{n!} = \sum_{m=0}^{\infty} (-1)^m \binom{m}{\ell} m! \sum_{n=0}^{\infty} \left\{ {n \atop m} \right\} \frac{z^n}{n!}$$

$$= \sum_{m=0}^{\infty} (-1)^m \binom{m}{\ell} m! \frac{(e^z-1)^m}{m!} = \frac{(1-e^z)^\ell}{(1-(1-e^z))^{\ell+1}}, \text{ by the generalized binomial theorem,}$$

$$= e^{-z}(e^{-z}-1)^\ell = \sum_{n=0}^{\infty} \ell!\,\mathrm{R}(n,\ell,1)(-1)^n \frac{z^n}{n!}, \text{ by [3], (3.9).}$$

Lemma 2.2:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathrm{B}_n^{-k} x^n y^k = \sum_{j=0}^{\infty} p_j(x)p_j(y) \tag{5}$$

where $p_j(x) = j!\sum_{n=0}^{\infty} \mathrm{R}(n,j,1)x^n$.

Proof: By (2), we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathrm{B}_n^{-k} x^n y^k = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left((-1)^n(-1)^m m!\left\{{n \atop m}\right\}(m+1)^k\right)x^n y^k, \text{ by [3], (3.4),}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty} \ell!\binom{m}{\ell}\mathrm{R}(k,\ell,1)\left((-1)^n(-1)^m m!\left\{{n \atop m}\right\}\right)x^n y^k$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty} \ell!\binom{m}{\ell}\left((-1)^n(-1)^m m!\left\{{n \atop m}\right\}\right)\frac{p_\ell(y)}{\ell!}x^n$$

$$= \sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty} p_\ell(y)(-1)^n \sum_{m=0}^{\infty}\binom{m}{\ell}(-1)^m m!\left\{{n \atop m}\right\}x^n, \text{ by Lemma 2.1,}$$

$$= \sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty} p_\ell(y)(-1)^n(-1)^n \ell!\,\mathrm{R}(n,\ell,1)\,x^n = \sum_{\ell=0}^{\infty} p_\ell(y)\ell!\sum_{n=0}^{\infty}\mathrm{R}(n,\ell,1)\,x^n = \sum_{\ell=0}^{\infty} p_\ell(x)p_\ell(y).$$

Proof of (3): To prove (3), we compare the coefficients on both sides of (5). In the course of Arakawa and Kaneko's proof they prove the following proposition.

Proposition 2.1: For $n > 0$,

$$\sum_{\ell=0}^{n} (-1)^\ell \mathrm{B}_{n-\ell}^{-\ell} = 0.$$

Proof: We offer a more direct proof:

$$\sum_{\ell=0}^{n} (-1)^\ell \mathrm{B}_{n-\ell}^{-\ell} = \sum_{\ell=0}^{n}(-1)^\ell(-1)^{n-\ell}\sum_{m=0}^{n-\ell}(-1)^m m!(m+1)^\ell\left\{{n-\ell \atop m}\right\}$$

$$= (-1)^n \sum_{m=0}^{n}\sum_{\ell=0}^{n}(-1)^m m!(m+1)^\ell\left\{{n-\ell \atop m}\right\}, \text{ by [4], (6.20),}$$

$$= (-1)^n \sum_{m=0}^{n}(-1)^m m!\left\{{n+1 \atop m+1}\right\} = (-1)^n \delta_{1\,n+1} = 0.$$

3. ANOTHER PROOF

In Kaneko's paper [4], he obtained the *symmetric* formula:

$$\sum_{k\ge 0}\sum_{n\ge 0} \mathrm{B}_n^{-k}\frac{x^n}{n!}\frac{y^k}{k!} = \frac{e^{x+y}}{e^x+e^y-e^{x+y}}. \tag{6}$$

By using (6), D. Zeilberger gives a much simpler proof of (3) as follows:

$$\sum_{k\geq 0}\sum_{n\geq 0} B_n^{-k}\frac{x^n}{n!}\frac{y^k}{k!} = \frac{e^{x+y}}{e^x+e^y-e^{x+y}} = e^{x+y}\sum_{j\geq 0}(1-e^x)^j(1-e^y)^j$$

$$= \sum_{j\geq 0}\frac{1}{(1+j)^2}(j+1)(1-e^x)^j(-e^x)(j+1)(1-e^y)^j(-e^y)$$

$$= \sum_{j\geq 0}\frac{1}{(j+1)^2}D_x[(1-e^x)^{j+1}]D_y[(1-e^y)^{j+1}].$$

Now using the usual generating function for the Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$, i.e.,

$$\sum_{n\geq k}\left\{ {n \atop k} \right\}\frac{u^n}{n!} = \frac{(e^u-1)^k}{k!},$$

he obtains:

$$\sum_{n\geq 0}\sum_{k\geq 0} B_n^{-k}\frac{x^n}{n!}\frac{y^k}{k!} = \sum_{j\geq 0}\frac{1}{(j+1)^2}D_x\left[(-1)^{j+1}(j+1)!\sum_{n\geq j+1}\left\{ {n \atop j+1} \right\}\frac{x^n}{n!}\right]$$

$$\times D_y\left[(-1)^{j+1}(j+1)!\sum_{k\geq j+1}\left\{ {k \atop j+1} \right\}\frac{y^k}{k!}\right]$$

$$= \sum_{j\geq 0} j!^2\sum_{n\geq j}\left\{ {n+1 \atop j+1} \right\}\frac{x^n}{n!}\sum_{k\geq j}\left\{ {k+1 \atop j+1} \right\}\frac{y^k}{k!}$$

$$= \sum_{n\geq 0}\sum_{k\geq 0}\frac{x^n}{n!}\frac{y^k}{k!}\sum_{j\geq 0} j!^2\left\{ {n+1 \atop j+1} \right\}\left\{ {k+1 \atop j+1} \right\}.$$

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