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A note on a discrete analytic function

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An unsolved problem in discrete analytic function theory has been to find a suitable analogue of the function $\sqrt{z}$. An analogue $z^{(\alpha)}$, of the function z^{α} , is found here for discrete analytic functions of the first kind (or *monodiffric* functions). This function resolves a conjecture of Isaacs in the negative, and at the same time it introduces multi-valued functions into the discrete analytic theory.

1. Introduction

In [*3*, Problem 1] it was stated that a discrete analytic analogue of the function $\sqrt{z}$ had not been found, and a result has been obtained by the author [*7*] in connection with a discrete analytic theory for q-difference functions. In this note the monodiffric analogue $z^{(\alpha)}$, of the classical function z^{α} , is found.

Isaacs [*9*, *10*] conjectured that there were no rational monodiffric functions other than polynomials, and in the following, it is shown that the function $z^{(\alpha)}$ resolves this conjecture in the negative. Also, multi-valued functions are introduced into the theory.

Monodiffric functions were defined by Isaacs [*9*, *10*] and further developed by Kurowski [*11*] and Berzsenyi [*1*, *2*]. They are defined on the set of gaussian integers and satisfy the forward-difference equation,

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$$f(z+1) - f(z) = \frac{f(z+i)-f(z)}{i} . \tag{1.1}$$

The monodiffric function corresponding to z^n (n a non-negative integer) is denoted by $z^{(n)}$ and was found by Isaacs to be

$$z^{(n)} = \sum_{j=0}^{n} \binom{n}{j} x^{(n-j)} i^j y^{(j)} \ ; \quad z^{(0)} = 1 , \tag{1.2}$$

where $z = (x, y)$ and $x^{(j)} = x(x-1) \ldots (x-j+1)$; $x^{(0)} = 1$.

Discrete analytic functions of the second kind were introduced by Ferrand [5] and extensively developed by Duffin [4] and others. They are defined by the difference quotient equality,

$$\frac{f(z+1+i)-f(z)}{1+i} = \frac{f(z+1)-f(z+i)}{1-i} .$$

In this latter theory however, the corresponding function $z^{(n)}$ does not have a simple algebraic form.

2. Preliminaries

In monodiffric theory it is usual to employ forward-differences but it will be more convenient here to utilize backward-differences, the results obtained applying equally well to the standard monodiffric scheme. Accordingly the following definitions of the operators E_1, E_2, Δ_1 and Δ_2 are made:

$$E_1 f(z) = f(z-1) , \quad E_2 f(z) = f(z-i) , \tag{2.1}$$

$$\Delta_1 f(z) = (1-E_1) f(z) = f(z) - f(z-1) ,$$

$$\Delta_2 f(z) = \frac{(1-E_2)}{i} f(z) = \frac{f(z)-f(z-i)}{i} .$$

If $\Delta_1 f(z) = \Delta_2 f(z)$, then f is said to be *monodiffric* at the point z , and a common operator Δ can be used, where

$$\Delta \equiv \Delta_1 = \Delta_2 . \tag{2.2}$$

The domain of definition is to be restricted to the set G of gaussian integers. Hence,

$$G = \{z;\ z = x+iy\ ,\ \text{where}\ \ x, y\ \ \text{are integers}\}\ .$$

Subsequently a complex number z will be used synonymously with its components (x, y) .

Subsets of G in the four quadrants of the complex plane are defined as follows:

$$G_1 = \{z;\ z \in G,\ x > 0,\ y > 0\}\ ,\quad G_2 = \{z;\ z \in G,\ x < 0,\ y > 0\}\ ,$$

$$G_3 = \{z;\ z \in G,\ x < 0,\ y < 0\}\ ,\quad G_4 = \{z;\ z \in G,\ x > 0,\ y < 0\}\ ,$$

and on the axes,

$$X^+ = \{z;\ z \in G,\ x \geq 0,\ y = 0\}\ ,\quad X^- = \{z;\ z \in G,\ x \leq 0,\ y = 0\}\ ,$$

$$Y^+ = \{z;\ z \in G,\ x = 0,\ y \geq 0\}\ ,\quad Y^- = \{z;\ z \in G,\ x = 0,\ y \leq 0\}\ .$$

Before proceeding to the derivation of $z^{(\alpha)}$ it will be shown that if a function is defined for points of G on the axes (on X^+, X^-, Y^+ and Y^-), then it can be extended as a monodiffric function into G .

Kurowski [11] constructed an operator E which provides the extension of a function, defined on the X-axis, to a monodiffric function defined on a half plane.

Defining the operators $(1-i\Delta_1)^m$ and $(1-\Delta_2)^m$ by

$$(1-i\Delta_1)^m = \sum_{k=0}^{m} \binom{m}{k} (-i)^k \Delta_1^k\ ;\quad (1-i\Delta_1)^0 = I\ ,$$

$$(1-\Delta_2)^m = \sum_{k=0}^{m} \binom{m}{k} (-1)^k \Delta_2^k\ ;\quad (1-\Delta_2)^0 = I\ ;$$

where m is a positive integer, Δ_1 and Δ_2 are as defined in (2.1), and I is the identity operator; then the following two theorems are equivalent to Kurowski's result and will only be stated.

THEOREM 2.1. *If* $z = (x, y) \in G$ *, and a function* f *is defined for* $z \in X^+ \cup X^-$ *(on the* X*-axis), then a monodiffric function with these* *prescribed values is determined uniquely for* $z \in G_3 \cup G_4 \cup Y^-$ *(below the X-axis) and is given by*

$$f(z) = \left(1-i\Delta_1\right)^{-y} f(x, 0) = \sum_{j=0}^{-y} \binom{-y}{j} (-i)^j \Delta_1^j [f(x, 0)] .$$

THEOREM 2.2. *The function* f *defined by*

$$f(z) = \left(1-\Delta_2\right)^{-x} f(0, y) = \sum_{j=0}^{-x} \binom{-x}{j} (-1)^j \Delta_2^j [f(0, y)]$$

is the unique monodiffric function for $z \in G_2 \cup G_3 \cup X^-$ *(to the left of the* Y*-axis), with prescribed values* $f(0, y)$ *on* $Y^+ \cup Y^-$.

On the other hand, if f is defined on the positive-half X and Y axes, then a monodiffric function is determined explicitly for all $z \in G_1$ in the first quadrant, as the following theorem shows.

THEOREM 2.3. *If a function* f *is defined on* X^+ *and* Y^+ *, then it has a unique monodiffric extension into* G_1 *, and in fact for* $z = (x, y) \in G_1$ *,*

$$f(z) = (1-i)^{-(x+y)} \left\{ \sum_{j=0}^{x} \binom{x+y}{j} (-i)^j \left(1-i\Delta_1\right)^{x-j} [f(x-j , 0)] + \sum_{j=x+1}^{x+y} \binom{x+y}{j} (-i)^j \left(1-\Delta_2\right)^{j-x} [f(0, j-x)] \right\} .$$

Proof. From (2.2), if f is monodiffric at $z \in G_1$, then $\Delta_1 f(z) = \Delta_2 f(z)$, and so by (2.1),

$$\begin{aligned} f(z) &= (1-i)^{-1}[f(z-i)-if(z-1)] \\ &= (1-i)^{-1}\left[E_2-iE_1\right]f(z) . \end{aligned}$$

Similarly,

$$f(z) = (1-i)^{-2}\left[E_2-iE_1\right]^2 f(z) ,$$

and in general,

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$$f(z) = (1-i)^{-n}[E_2-iE_1]^n f(z) \ ,$$

so that for $n = x + y$,

$$\begin{aligned} f(z) &= (1-i)^{-(x+y)} \sum_{j=0}^{x+y} \binom{x+y}{j} (-i)^j E_2^{x+y-j} E_1^j f(z) \\ &= (1-i)^{-(x+y)} \sum_{j=0}^{x+y} \binom{x+y}{j} (-i)^j f(x-j,\ j-x) \ . \end{aligned}$$

When $0 < j \le x$, the argument of f lies in G_4 and when $x \le j \le x+y$ it lies in G_2 . Hence by Theorems 2.1 and 2.2 the expression for $f(z)$ in the above statement of the theorem is obtained. Uniqueness follows from the constructive method used, completing the proof of the theorem.

As a consequence of the above three theorems, it follows that if a function is defined on the axes, then it has a monodiffric extension to all points of G , the resulting function being unique at least in G_1, G_2 and G_4 .

An alternative form of the above theorem, which is given by the case $n = x + y - 1$, proves useful and is now stated.

THEOREM 2.4. *For $z \in G_1$ and with f defined on the positive-half axes as in the above theorem, then*

$$\begin{aligned} f(z) = (1-i)^{-(x+y-1)} \Bigg\{ &\sum_{j=0}^{x-1} \binom{x+y-1}{j} (-i)^j \left(1-i\Delta_1\right)^{x-j-1} f(x-j\,,\ 0) \\ &+ \sum_{j=x}^{x+y-1} \binom{x+y-1}{j} (-i)^j \left(1-\Delta_2\right)^{j-x} f(0,\ 1+j-x) \Bigg\} \ . \end{aligned}$$

3. The function $z^{(\alpha)}$

A monodiffric function $z^{(\alpha)}$ is said to be an analogue of the classical function z^{α} if

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$$
(3.1)\qquad \begin{cases} \text{(i)} & \Delta z^{(\alpha)} = \alpha z^{(\alpha-1)} , \\ \text{(ii)} & 0^{(\alpha)} = 0 \ ; \quad \alpha > 0 , \\ \text{(iii)} & z^{(0)} = 1 , \end{cases}
$$

where Δ is defined by (2.2).

When $\alpha = n$, a non-negative integer, $z^{(n)}$ is given by Isaacs's function (1.2), where for the backward-difference case, $x^{(j)} = x(x+1) \ldots (x+j-1)$. A general form is now obtained for $z^{(\alpha)}$ (α not a negative integer - but otherwise an arbitrary constant) which is consistent with Isaacs's function when α is a non-negative integer.

For $x \in X^{+} \cup X^{-}$ on the X-axis, the function $x^{(\alpha)}$ is defined by

$$
(3.2)\qquad x^{(\alpha)} = \frac{\Gamma(x+\alpha)}{\Gamma(x)} ,
$$

where α is not a negative integer, and where Γ is the classical gamma function. This function satisfies (3.1) with $\Delta = \Delta_1$.

When $\alpha = n$, a non-negative integer, $x^{(\alpha)}$ reduces to,

$$
x^{(n)} = x(x+1) \ldots (x+n-1) \ ; \quad x^{(0)} = 1 ,
$$

so that $x^{(n)} = 0 \ ; \quad x = 0, -1, -2, \ldots, -n+1$. When α is not an integer, then since $\Gamma(x)$ has a pole when x is a negative integer, it follows that

$$
(3.3)\qquad x^{(\alpha)} = 0 \ ; \quad x \in X^{-} .
$$

For points of G on the Y-axis $\left(iy \in Y^{+} \cup Y^{-}\right)$, the function $(iy)^{(\alpha)}$ is defined by

$$
(3.4)\qquad (iy)^{(\alpha)} = i^{\alpha} y^{(\alpha)} ,
$$

where $y^{(\alpha)}$ is given by (3.2). This function satisfies (3.1) with $\Delta = \Delta_2$.

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The case when α is a non-negative integer has been solved by Isaacs. When α is a negative integer, the function $x^{(\alpha)}$, as defined by (3.2), has a singularity at each of the points $x = 1, 2, \ldots, -\alpha$. Subsequently it will be assumed that α is not an integer.

From Theorem 2.1 it follows that the function

$$z^{(\alpha)} = (1-i\Delta_1)^{-y} x^{(\alpha)} \ ; \quad z = (x, y) \in G_3 \cup G_4 \cup Y^- \ ;$$

is the unique monodiffric function in this region, with prescribed values $x^{(\alpha)}$ on the X-axis. Hence,

$$z^{(\alpha)} = \sum_{j=0}^{-y} \binom{-y}{j} (-i)^j \Delta_1^j x^{(\alpha)} \ ,$$

and since $\Delta_1 x^{(\alpha)} = \alpha x^{(\alpha-1)}$, $\Delta_1^j x^{(\alpha)} = \alpha(\alpha-1), \ldots (\alpha-j+1) x^{(\alpha-j)}$, it follows on simplification that

$$z^{(\alpha)} = \sum_{j=0}^{-y} \binom{\alpha}{j} x^{(\alpha-j)} i^j y^{(j)} \ . \tag{3.5}$$

From (3.3) it follows that

$$z^{(\alpha)} = 0 \ ; \quad z \in G_3 \cup Y^- \cup X^- \ . \tag{3.6}$$

Similarly,

$$z^{(\alpha)} = (1-\Delta_2)^{-x} (iy)^{(\alpha)} \ ; \quad z = (x, y) \in G_2 \cup G_3 \cup X^- \ ,$$

is the unique monodiffric function with values $(iy)^{(\alpha)}$ on the Y-axis, and this reduces to

$$z^{(\alpha)} = \sum_{j=0}^{-x} \binom{\alpha}{j} x^{(j)} i^{\alpha-j} y^{(\alpha-j)} \ . \tag{3.7}$$

Once again $z^{(\alpha)} = 0$; $z \in G_3 \cup Y^- \cup X^-$, and so this definition is consistent with (3.5).

By construction, the function $z^{(\alpha)}$ (given by (3.5), (3.6) and (3.7)), is monodiffric and it can be readily verified that it satisfies

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conditions (i) and (ii) of (3.1).

For the remaining region G_1, it follows from Theorem 2.3 that for $z \in G_1$,

$$z^{(\alpha)} = (1-i)^{-(x+y)}\left\{\sum_{j=0}^{x} \binom{x+y}{j}(-i)^j(1-i\Delta_1)^{x-j}(x-j)^{(\alpha)} + \sum_{j=x+1}^{x+y} \binom{x+y}{j}(-i)^j(1-\Delta_2)^{j-x}[i(j-x)]^{(\alpha)}\right\},$$

and so by (3.5) and (3.7),

$$z^{(\alpha)} = (1-i)^{-(x+y)} \sum_{j=0}^{x+y} \binom{x+y}{j}(-i)^j(x-j,\ j-x)^{(\alpha)}, \tag{3.8}$$

where for $j = 0, 1, \ldots, x$, $(x-j, j-x)^{(\alpha)}$ is given by (3.5), and by (3.7) when $j = x+1, x+2, \ldots, x+y$. The function is monodiffric by Theorem 2.3 and it satisfies $0^{(\alpha)} = 0$. It remains to be verified that $\Delta z^{(\alpha)} = \alpha z^{(\alpha-1)}$.

THEOREM 3.1. *For $z \in G_1$, the function $z^{(\alpha)}$ as defined by* (3.8) *satisfies the condition*

$$\Delta z^{(\alpha)} = \alpha z^{(\alpha-1)}.$$

Proof. Let $z \in G_1$. From (2.1),

$$\Delta_1 z^{(\alpha)} = z^{(\alpha)} - (z-1)^{(\alpha)},$$

and so by (3.8),

$$\Delta_1 z^{(\alpha)} = (1-i)^{-(x+y)}\left\{\sum_{j=0}^{x+y} \binom{x+y}{j}(-i)^j(x-j,\ j-x)^{(\alpha)} - (1-i) \sum_{j=0}^{x+y-1} \binom{x+y-1}{j}(-i)^j(x-1-j,\ j+1-x)^{(\alpha)}\right\}$$

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$$= (1-i)^{-(x+y)}\Big\{(x,\ -x)^{(\alpha)} + \sum_{j=0}^{x+y-1} \binom{x+y}{j+1}(-i)^{j+1}(x-j-1,\ j+1-x)^{(\alpha)}$$

$$- \sum_{j=0}^{x+y-1} \binom{x+y-1}{j}(-i)^{j+1}(x-j-1,\ j+1-x)^{(\alpha)}$$

$$- \sum_{j=0}^{x+y-1} \binom{x+y-1}{j}(-i)^{j}(x-j-1,\ j+1-x)^{(\alpha)}\Big\} ,$$

and on combining the first two sums,

$$\Delta_1 z^{(\alpha)} = (1-i)^{-(x+y)}\Big\{(x,\ -x)^{(\alpha)} + \sum_{j=0}^{x+y-2} \binom{x+y-1}{j+1}(-i)^{j+1}(x-j-1,\ j+1-x)^{(\alpha)}$$

$$- \sum_{j=0}^{x+y-1} \binom{x+y-1}{j}(-i)^{j}(x-1-j,\ j+1-x)^{(\alpha)}\Big\}$$

$$= (1-i)^{-(x+y)} \sum_{j=0}^{x+y-1} \binom{x+y-1}{j}(-i)^{j}\left[(x-j,\ j-x)^{(\alpha)}-(x-1-j,\ j+1-x)^{(\alpha)}\right] .$$

Now if $Z \in G_2$ or G_4 then it has been shown previously that $\Delta_1 Z^{(\alpha)} = \Delta_2 Z^{(\alpha)} = \alpha Z^{(\alpha-1)}$, and from this it follows that

$$(Z-i)^{(\alpha)} - (Z-1)^{(\alpha)} = \alpha(1-i)Z^{(\alpha-1)} .$$

Hence with $Z = (x-j,\ j+1-x)$,

$$\Delta_1 z^{(\alpha)} = (1-i)^{-(x+y)} \sum_{j=0}^{x+y-1} \binom{x+y-1}{j}(-i)^{j}\alpha(1-i)(x-j,\ j+1-x)^{(\alpha-1)}$$

$$= \alpha z^{(\alpha-1)}$$

(from Theorem 2.4). Since $z^{(\alpha)}$ is monodiffric it follows that $\Delta z^{(\alpha)} = \Delta_1 z^{(\alpha)} = \alpha z^{(\alpha-1)}$, completing the proof of the theorem.

Hence $z^{(\alpha)}$ has been specified for all points of G and it can be seen that (3.5) and (3.7) demonstrate an analogy with the binomial expansion of the function $z^{\alpha} = (x+iy)^{\alpha}$. In the region G_1 the expression for $z^{(\alpha)}$ given by (3.8) is a little more complicated, but it can be shown that (3.8) becomes

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$$(3.9)\qquad z^{(\alpha)} = \sum_{j=0}^{\infty} \binom{\alpha}{j} x^{(\alpha-j)} i^j y^{(j)} + \sum_{j=0}^{\infty} \binom{\alpha}{j} x^{(j)} i^{\alpha-j} y^{(\alpha-j)} ,$$

where the two divergent series are summable (E, q) for $q > 0$ in the Euler sense (see Hardy [6]). This represents a remarkable analogy with the binomial expansion of z^{α} and the proof, being lengthy, is given in [8].

To summarize the preceding results:- the monodiffric function corresponding to z^{α} is given by:

$$(3.10)\qquad z^{(\alpha)} = \begin{cases} \sum_{j=0}^{\infty} \binom{\alpha}{j} x^{(\alpha-j)} i^j y^{(j)} + \sum_{j=0}^{\infty} \binom{\alpha}{j} x^{(j)} i^{\alpha-j} y^{(\alpha-j)} ; & z \in G_1 , \\ \sum_{j=0}^{-x} \binom{\alpha}{j} x^{(j)} i^{\alpha-j} y^{(\alpha-j)} ; & z \in G_2 \cup Y^{+} , \\ 0 ; & z \in G_3 \cup X^{-} \cup Y^{-} , \\ \sum_{j=0}^{-y} \binom{\alpha}{j} x^{(\alpha-j)} i^j y^{(j)} ; & z \in G_4 \cup X^{+} . \end{cases}$$

4. Properties and discussion

Isaacs [9, 10] conjectured that there were no monodiffric functions, rational in x and y, other than polynomials. Now from the definition of $x^{(\alpha)}$, it follows that if $z \in G_4$ $(x > 0,\ y < 0)$, then

$$\binom{\alpha}{j} x^{(\alpha-j)} y^{(j)} = \frac{(\alpha-j+1)(\alpha-j+2)\dots(\alpha-j+x-1)y(y+1)\dots(y+j-1)}{(x-1)!} ,$$

and so for $z \in G_4$, the function $z^{(\alpha)}$, given in (3.10) by

$$z^{(\alpha)} = \sum_{j=0}^{-y} \binom{\alpha}{j} x^{(\alpha-j)} i^j y^{(j)} ,$$

is both monodiffric and rational in x and y. This shows that Isaacs's conjecture is false.

In the classical case, the function z^{α} is multi-valued, and in fact if z_1^{α} denotes a particular branch, then it is well known that

$$z^{\alpha} = e^{2i\alpha n\pi} z_1^{\alpha} ;\quad n = 0, \pm 1, \pm 2, \dots .$$

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In the region G_2 , $z^{(\alpha)}$ (given in 3.10) is multi-valued due to the presence of the factor i^{α} , and in fact if $z_1^{(\alpha)}$ denotes a branch, then

$$z^{(\alpha)} = e^{2i\alpha n\pi} z_1^{(\alpha)} \ ; \quad n = 0, \pm 1, \pm 2, \ldots \ ;$$

demonstrating a close analogy with the function z^{α} .

The representation of $z^{(\alpha)}$ in G_1 is promising. It is derived from a combination of values of $z^{(\alpha)}$ on the positive half axes; it is multi-valued; it is a very good approximation to x^{α} on the positive X-axis and to $(iy)^{\alpha}$ on the positive Y-axis.

On the other hand the representation of $z^{(\alpha)}$ in G_3 and G_4 illustrates a certain lack of symmetry in the usage of monodiffric functions. The function is single-valued in G_3 and G_4 and in fact is zero in G_3 - a poor analogy with the classical function z^{α} .

In view of the above observations, a need is suggested for an alternative method of defining discrete analyticity, which retains the algebraic simplicity of monodiffric functions, and which at the same time introduces a symmetry similar to the Schwarz Reflection Principle. Such a theory is discussed in [8].

In the preceding analysis, a backward-difference scheme was used to define a monodiffric function. However if the more usual forward-difference scheme is used, all of the corresponding results, comments and criticisms apply without loss of generality, where $x^{(\alpha)}$ is defined by $(-1)^{\alpha}\Gamma(\alpha-x)/\Gamma(-x)$.

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