

Prog. Theor. Phys. Vol. 60 (1978), Aug.

A Note on a New Effective Field Theory of the Ising Model

Ryosuke Honmura and
Takahito KANEYOSHI

Department of Physics, Nagoya University
Nagoya 464

May 13, 1978

Formerly, Callen¹⁾ applied the Green function method to the Ising model and obtained an exact spin correlation function as follows:

$$\langle \mu_i \rangle = \frac{1}{2} \langle \tanh \frac{1}{2} \beta E_i \rangle \quad (1)$$

with

$$E_i = \sum_j J_{ij} \mu_j, \quad (2)$$

where $\langle \dots \rangle$ indicates an ensemble average:

$$\langle A \rangle = \frac{\text{Tr } A e^{-\beta H}}{\text{Tr } e^{-\beta H}}. \quad (3)$$

Then, if correlation effects are ignored, by replacing $\langle E_i^n \rangle$ by $\langle E_i \rangle^n$, he showed that Eq. (1) became identical to the Weiss molecular field result. However, how to extend it over the Weiss field was not clarified. For this purpose, some authors²⁾ have tried how to extend it over the Weiss field. In this short note we shall discuss a new-type effective field theory of Eq. (1).

Let us transform Eq. (1) as follows:

$$2\langle \mu_i \rangle = \langle \exp(D_i \sum_j t_{ij} \mu_j) \rangle \tanh(\alpha_i) |_{\alpha_i=0} \quad (4)$$

with

$$t_{ij} = \frac{1}{2} \beta J_{ij},$$

where the differential operator $D_i = \partial / \partial \alpha_i$ was introduced. For the evaluation of Eq. (4), we assume that the field at point j is independent of the orientation of μ_j , or the molecular field approximation. Given the approximation, the Hamiltonian H is replaced by $H = \sum_j H_j \mu_j$ with $H_j = -\sum_i$

$\times J_{ji} \langle \mu_i \rangle$. In this case, Eq. (4) reduces to

$$2\langle \mu_i \rangle = \prod_j \left[\frac{\cosh \{ \sum_l t_{jl} (2\langle \mu_l \rangle + \delta_{li} D_l) \}}{\cosh \{ \sum_l 2t_{jl} \langle \mu_l \rangle \}} \right] \times \tanh(\alpha_i) |_{\alpha_i=0}. \quad (5)$$

For simplification, we assume that the exchange interaction is effective only to nearest neighbours. By putting $2\langle \mu_i \rangle = \sigma$, Eq. (5) reduces to

$$\sigma = \left[\frac{\cosh(zt\sigma + tD_i)}{\cosh(zt\sigma)} \right]^z \cdot \tanh(\alpha_i) |_{\alpha_i=0}, \quad (6)$$

where z is the number of nearest neighbours. Here, Eq. (6) has more refined properties in comparison with the usual molecular field equation. For example, in the case $z=4$, Eq. (6) reduces to

$$\sigma = \frac{1}{8 [\cosh(4t\sigma)]^4} \{ \tanh(4t) \sinh(16t\sigma) + 4 \tanh(2t) \sinh(8t\sigma) \}.$$

Then, the Curie temperature $t_c = (1/2) \times (J/k_B T_c)$ is determined by

$$4t_c \tanh(2t_c) = 1 - \frac{1}{2 + \tanh^2(2t_c)}$$

in contrast with the usual Curie temperature given by $4t_c = 1$. In this way, we can easily understand that the transition temperature of Eq. (6) is given by a smaller value than that of the usual molecular field equation. In general, the transition temperature can be for arbitrary z obtained from

$$\frac{1}{zt_c} = \frac{1}{2^z} \sum_{r=0}^z \binom{z}{r} \tanh \{ (2r-z)t_c \} \times (2r-z). \quad (7)$$

For clarification, the transition temperatures T_c (Eq. (7)) are tabulated in comparison with those T_c (Weiss) of Weiss approximation as follows:

z	2	4	6	6
$\frac{T_c(\text{Eq. (7)})}{T_c(\text{Weiss})}$	0.83	0.91	0.93	0.95 (Oguchi method)

As was expected, as the coordination number increases, the transition temperature of Eq. (7) reduces to the usual molecular value. Also, it is a surprising fact that Eq. (7) gives the slightly lower transition temperature than that predicted by the Oguchi method,³⁾ in which short range order effects are taken account of. Thus, in spite of the essential molecular-field assumption given in Eq. (5), the effective

field equations (6) and (7) are different in a desirable direction from the usual molecular field theory.

- 1) H. B. Callen, Phys. Letters **4** (1963), 161.
- 2) T. Kaneyoshi, J. of Phys. C; Solid State Phys. **8** (1975), 3415.
B. Frank and O. Mitran, J. of Phys. C; Solid State Phys. **10** (1977), 2641.
- 3) J. S. Smart, *Effective Field Theories of Magnetism*, 1966, 37.