A NOTE ON ABSOLUTE CESÀRO SUMMABILITY OF FOURIER SERIES

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1. Let $f(x)$ be an integrable function in Lebesgue sense, and periodic of period 2π, and let

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

$$\sigma_n^{\alpha}(x) = \frac{1}{2}a_0 + \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha}(a_\nu \cos \nu x + b_\nu \sin \nu x)/A_n^{\alpha},$$

where $\alpha > -1$, and $A_n^{\alpha} = \binom{\alpha + n}{n}$.

DEFINITION 1. If $\alpha > -1$, and

$$\sum_{n=1}^{\infty} |\sigma_n^{\alpha}(x) - \sigma_{n-1}^{\alpha}(x)| < \infty,$$

then the Fourier series of $f(t)$ is said to be absolutely summable (C, α), or briefly summable $|C,\alpha|$ at the point x.

Various theorems concerning the absolute Cesàro summability of Fourier series have been obtained by many authors.

Supposing that $p \geqq 1$ and $f \in L^p$, we write

$$w_p(t) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right)^{1/p} \qquad (t > 0). \tag{1. 1}$$

Recently, Chow [3] has proved that

(I) If $1 \leqq p \leqq 2$, $f \in L^p$, and

$$\int_0^{\pi} \frac{w_p(t)}{t} dt < \infty, \tag{1. 2}$$

then the Fourier series of f is summable $|C, \alpha|$ almost everywhere for $\alpha > 1/p$.

(II) If $1 \leqq p \leqq 2$, $f \in L^p$, and

$$w_p(t) = O\left(\log \frac{1}{t}\right)^{-(1+1/p+\epsilon)} \qquad (t \to 0),$$

for some $\varepsilon > 0$, then the conclusion in (I) is true for $\alpha = 1/p$.

We can show that the condition (1.2) itself implies the conclusion in (II) when $1 < p \leqq 2$, under some additional condition, and to do it is the purpose of this note.

DEFINITION 2. We define $\lambda(x)$ such as

1° $\lambda(x) > 0$ for all $x \geqq x_0 > 0$,

2° $\lambda(x) \uparrow \infty$ as $x \uparrow \infty$,

3° $H < \lambda(x^\delta)/\lambda(x) \leqq 1$ for $0 < \delta < 1$ and $x \geqq x_0$, where H is a positive constant depending only on δ.

We may take as $\lambda(x)$, e. g.,

$$(\log x)^\alpha,\ (\log x)^\alpha/\log\log x,\ (\log\log x)^\alpha, \ldots\ldots \qquad (\alpha > 0).$$

After this definition, we see easily that $\lambda(x) = o(x^\varepsilon)$ as $x \uparrow \infty$, for every $\varepsilon > 0$.

Now, the theorem to be proved is as follows:

THEOREM 1. *If $1 < p \leqq 2$, $f \in L^p$, and for a function $w_p^*(t) \geqq w_p(t)$,*

$$\int_0^\pi \frac{w_p^*(t)}{t} dt < \infty, \tag{1. 3}$$

then the Fourier series of f is summable $|C, 1/p|$ almost everywhere, provided that

$$[w_p^*(1/x) \log x]^{-1}$$

is a function λ defined by Definition 2.

We have the "allied Fourier series"-analogue, cf. loc. cit. [3].

COROLLARY 1. *The conclusion in Theorem 1 is true, if $1 < p \leqq 2$, and for some $\varepsilon > 0$,*

$$w_p(t) = O\left(\log \frac{1}{t}\right)^{-(1+\varepsilon)} \qquad (t \to 0).$$

2. Proof of Theorem 1. We write for the sake of convenience,

$$\alpha = 1/p.$$

Employing the identity

$$\sigma_n^\alpha(x) - \sigma_{n-1}^\alpha(x) = \frac{\alpha}{n}[\sigma_n^{\alpha-1}(x) - f(x)] - \frac{\alpha}{n}[\sigma_n^\alpha(x) - f(x)],$$

in order to prove Theorem 1, it is sufficient to show that

$$\sum_{n=1}^\infty \frac{1}{n} |\sigma_n^{\alpha-1}(x) - f(x)| < \infty \tag{2. 1}$$

for almost every x, since (2. 1) implies, as it may be easily verified, the convergence of $\sum n^{-1}|\sigma_n^\alpha(x)-f(x)|$.

We have

$$\sigma_n^{\alpha-1}(x)-f(x)=\frac{2}{\pi}\int_0^\pi \varphi_x(t)K_n^{\alpha-1}(t)dt, \tag{2. 2}$$

where

$$\varphi_x(t)=\frac{1}{2}[f(x+t)+f(x-t)-2f(x)], \tag{2. 3}$$

and $K_n^{\alpha-1}(t)$ is the n-th Fejér kernel of order $\alpha-1$. And, as it is well known,

$$K_n^{\alpha-1}(t)=\Lambda_n^{\alpha-1}(t)+R_n^{\alpha-1}(t),$$

where

$$\Lambda_n^{\alpha-1}(t)=\frac{\cos(n\,t+\alpha(t-\pi)/2)}{A_n^{\alpha-1}(2\sin(t/2))^\alpha}, \tag{2. 4}$$

$$K_n^{\alpha-1}(t)=O(n) \qquad (0\leqq t\leqq\pi), \tag{2. 5}$$

$$R_n^{\alpha-1}(t)=O(1/nt^2) \qquad \left(\frac{\pi}{n}\leqq t\leqq\pi\right), \tag{2. 6}$$

O being uniform in n and t.

(2. 2) is written as

$$\begin{aligned}\frac{\pi}{2}&[\sigma_n^{\alpha-1}(x)-f(x)]\\ &=\int_0^\pi \varphi_x(t)\Lambda_n^{\alpha-1}(t)dt+\int_0^{\pi/n}\varphi_x(t)K_n^{\alpha-1}(t)dt\\ &\quad-\int_0^{\pi/n}\varphi_x(t)\Lambda_n^{\alpha-1}(t)dt+\int_{\pi/n}^\pi\varphi_x(t)R_n^{\alpha-1}(t)dt\\ &=I_n(x)+I_n^{(1)}(x)+I_n^{(2)}(x)+I_n^{(3)}(x).\end{aligned} \tag{2. 7}$$

Here, for the proof, supposing that $[w_p(1/x)\log x]^{-1}$ is a function λ defined by Definition 2, we may use the function $w_p(t)$ itself in place of $w_p^*(t)$, since the conclusion remains unchanged by the assumption $w_p^*(t)\geqq w_p(t)$. Besides, then,

$$[w_p(1/x)]^{-1}=[w_p(1/x)\log x]^{-1}\log x$$

is also a function λ, and the condition (1. 3) replaced w_p^* by w_p, i. e.,

$$\int_0^\pi \frac{w_p(t)}{t}dt<\infty \tag{2. 8}$$

is equivalent to

$$\sum_{n=1}^{\infty}\frac{1}{n}w_p\Big(\frac{1}{n}\Big)<\infty. \tag{2. 8$'$}$$

In these circumstances, by (2. 3) and (2. 5) we have

$$\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{2\pi}|I_n^{(1)}(x)|dx = O\Big(\sum_{n=1}^{\infty}\int_0^{\pi/n}dt\int_0^{2\pi}|\varphi_x(t)|dx\Big)$$
$$= O\Big(\sum_{n=1}^{\infty}\int_0^{\pi/n}w_p(t)dt\Big)=O\Big(\int_0^{\pi}\frac{du}{u^2}\int_0^{u}w_p(t)dt\Big)$$
$$= O\Big(\int_0^{\pi}w_p(t)dt\int_t^{\pi}\frac{du}{u^2}\Big)=O\Big(\int_0^{\pi}\frac{w_p(t)}{t}dt\Big),$$

which is finite by (2. 8). Similarly, by (2. 4),

$$\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{2\pi}|I_n^{(2)}(x)|dx = O\Big(\int_0^{\pi}\frac{w_p(t)}{t}dt\Big)<\infty.$$

Next, by (2. 6),

$$\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{2\pi}|I_n^{(3)}(x)|dx = O\Big(\sum_{n=1}^{\infty}\frac{1}{n^2}\int_{\pi/n}^{\pi}\frac{dt}{t^2}\int_0^{2\pi}|\varphi_x(t)|dx\Big)$$
$$= O\Big(\sum_{n=1}^{\infty}\frac{1}{n^2}\int_{\pi/n}^{\pi}\frac{w_p(t)}{t^2}dt\Big)=O\Big(\int_0^{\pi}du\int_u^{\pi}\frac{w_p(t)}{t^2}dt\Big)$$
$$= O\Big(\int_0^{\pi}\frac{w_p(t)}{t^2}dt\int_0^{t}du\Big)=O\Big(\int_0^{\pi}\frac{w_p(t)}{t}dt\Big)<\infty.$$

Further, by (2. 4),

$$\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{2\pi}|I_n(x)|dx \leqq \sum_{n=1}^{\infty}\frac{1}{\alpha n^{\alpha}}\int_0^{2\pi}\Big|\int_0^{\pi}\frac{\varphi_x(t)e^{i(nt+\alpha(t-\pi)/2)}}{(2\sin(t/2))^{\alpha}}dt\Big|dx.$$

Hence, letting

$$\rho_n(x)=\Big|\frac{2}{\pi}\int_0^{\pi}G_x(t)e^{int}dt\Big|,$$

where

$$G_x(t)=\frac{\varphi_x(t)}{|2\sin(t/2)|^{\alpha}}\qquad(0<|t|\leqq\pi),$$

the proof is, by 2.1) and (2. 7), completed if it be shown that

$$\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}\int_0^{2\pi}\rho_n(x)dx<\infty, \tag{2.9}$$

since the conclusion is unchanged using $\varphi_x(t)$ in place of $\varphi_x(t)e^{i\alpha(t-\pi)/2}$.

Supposing that $G_x(t)$ considered as a function of t is periodic of period 2π, we have, since $\alpha=1/p$,

$$\int_0^{2\pi}dx\int_0^{2\pi}|G_x(t)|^p dt=O\Big(\int_0^{\pi}\frac{[w_p(t)]^p}{t}dt\Big)<\infty,$$

which implies $G_x(t)\in L^p$ in $0\leqq t\leqq 2\pi$, for almost every x. So, in view of $1<p\leqq 2$, by a Paley's theorem, cf. Zygmund [5, p. 203], we see that, for almost every x,

$$\sum_{n=1}^{\infty}\left|\rho_n(x)\sin\frac{1}{2}nh\right|^p n^{p-2}\leqq A_p\int_0^{2\pi}|G_x(t)-G_x(t+h)|^p dt, \tag{2.10}$$

where A_p is a constant depending only on p. And, it is seen with no difficulty that, for $0<t\leqq\pi$,

$$G_x(t)-G_x(t+h)=\frac{\varphi_x(t)-\varphi_x(t+h)}{(2\sin 2^{-1}(t+h))^{\alpha}}+\varphi_x(t)\cdot O\Big(\frac{h}{t^{\alpha}(t+h)}\Big),$$

where O is independent of x, t and h. Hence, neglecting the constant factors, and since $\alpha=1/p$,

$$\begin{aligned}&\int_0^{2\pi}dx\int_0^{\pi}|G_x(t)-G_x(t+h)|^p dt\\ &<\int_0^{\pi}\frac{dt}{t+h}\int_0^{2\pi}|\varphi_x(t)-\varphi_x(t+h)|^p dx+h^p\int_0^{\pi}\frac{dt}{t(t+h)^p}\int_0^{2\pi}|\varphi_x(t)|^p dx\\ &<\int_0^{\pi}\frac{[w_p(h)]^p dt}{t+h}+h^p\int_0^{\pi}\frac{[w_p(t)]^p dt}{t(t+h)^p}.\end{aligned}$$

It is analogous to $\int_0^{2\pi}dx\int_{\pi}^{2\pi}|G_x(t)-G_x(t+h)|^p dt$.

Integrating both sides of (2.10) with respect to x over $(0, 2\pi)$, and again neglecting the constant factor and the term $O(h^p)$, we have

$$\sum_{n=1}^{\infty}\left|\sin\frac{1}{2}nh\right|^p\int_0^{2\pi}\frac{[\rho_n(x)]^p dx}{n^{2-p}}<[w_p(h)]^p\log\frac{\pi}{h}+h^p\int_0^1\frac{[w_p(t)]^p dt}{t(t+h)^p}. \tag{2.11}$$

By the assumption,

$$\lambda\Big(\frac{1}{h}\Big)=\Big[w_p(h)\log\frac{\pi}{h}\Big]^{-(p-1)} \tag{2.12}$$

is a function λ defined by Definition 2. Multiplying both sides of (2. 11) by

$$\frac{\lambda(1/h)}{h[\log(\pi/h)]^{2-p}}=\frac{1}{h[w_p(h)]^{p-1}\log(\pi/h)},$$

and then integrating them with respect to h over $(0,1)$, we obtain

$$\begin{aligned}(2.13)\qquad &\sum_{n=1}^{\infty}\int_0^1\frac{\lambda(1/h)|\sin(nh/2)|^p dh}{h[\log(\pi/h)]^{2-p}}\int_0^{2\pi}\frac{[\rho_n(x)]^p dx}{n^{2-p}}\\ &<\int_0^1\frac{w_p(h)}{h}dh+\int_0^1\frac{h^{p-1}\lambda(1/h)dh}{[\log(\pi/h)]^{2-p}}\int_0^1\frac{[w_p(t)]^p dt}{t(t+h)^p}=J_1+J_2.\end{aligned}$$

J_1 is clearly finite by (2. 8). And

$$J_2=\int_0^1\frac{[w_p(t)]^p}{t}dt\left(\int_0^{t^2}+\int_{t^2}^1\right)\frac{h^{p-1}\lambda(1/h)dh}{(t+h)^p[\log(\pi/2)]^{2-p}}=J_2^{(1)}+J_2^{(2)}.$$

As it is noticed before, $\lambda(x)=o(x^{\varepsilon})$ as $x\to\infty$ for every $\varepsilon>0$. So, taking $\varepsilon=1/2$, and observing that $1<p\leqq 2$.

$$\begin{aligned}J_2^{(1)}&<\int_0^1\frac{[w_p(t)]^p}{t}dt\int_0^{t^2}\frac{h^{p-1-1/2}}{(t+h)^p}dh\\ &<\int_0^1\frac{[w_p(t)]^p}{t^{p+1}}(t^2)^{p-1/2}dt\\ &=\int_0^1\frac{[w_p(t)]^p}{t^{2-p}}dt=O\left(\int_0^1\frac{w_p(t)}{t}dt\right)<\infty.\end{aligned}$$

Further, taking into account the property of the function λ, and $p>1$,

$$\begin{aligned}J_2^{(2)}&<\int_0^1\frac{[w_p(t)]^p}{t}dt\int_{t^2}^1\frac{\lambda(1/h)dh}{h[\log(\pi/h)]^{2-p}}\\ &<\int_0^1\frac{[w_p(t)]^p}{t}\lambda\left(\frac{1}{t^2}\right)dt\int_{t^2}^1\frac{dh}{h[\log(\pi/h)]^{2-p}}\\ &=O\left(\int_0^1\frac{[w_p(t)]^p}{t}\lambda\left(\frac{1}{t}\right)\left(\log\frac{\pi}{t}\right)^{p-1}dt\right)\\ &=O\left(\int_0^1\frac{w_p(t)}{t}dt\right), \qquad\text{by (2. 12)},\end{aligned}$$

which is finite by (2. 8). On the other hand, the coefficient of $\int_0^{2\pi}[\rho_n(x)]^p n^{p-2}dx$ in the first member of (2. 13) is, since $p\leqq 2$ and $|\sin(nh/2)|^p\geqq|\sin(nh/2)|^2$,

$$\int_0^1 \frac{\lambda(1/h)|\sin(nh/2)|^p dh}{h[\log(\pi/h)]^{2-p}} > \int_{1/n}^{1/\sqrt{n}}$$

$$> \frac{\lambda(\sqrt{n})}{[\log(n\pi)]^{2-p}} \int_{1/n}^{1/\sqrt{n}} \frac{1-\cos nh}{2h} dh$$

$$> \frac{\lambda(\sqrt{n})}{[\log(n\pi)]^{2-p}} \left(\frac{1}{4}\log n - 1\right)$$

$$> K\lambda(n)[\log(n\pi)]^{p-1} = \frac{K}{[w_p(1/n)]^{p-1}}, \qquad \text{by (2. 12)},$$

for $n \geqq n_0$, where K is a positive constant independent of n. Thus, observing that J_1 and J_2 are finite we see, from (2. 13),

$$\sum_{n=1}^{\infty} \int_0^{2\pi} \frac{[\rho_n(x)]^p n^{p-2}}{[w_p(1/n)]^{p-1}} dx < \infty. \tag{2. 14}$$

Letting $q = p/(p-1)$, we now obtain by Hölder's inequality,

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/p}} \int^{2\pi} \rho_n(x) dx = \sum_{n=1}^{\infty} \left(\frac{w_p(1/n)}{n}\right)^{1/q} \int_0^{2\pi} \left(\frac{[\rho_n(x)]^p n^{p-2}}{[w_p(1/n)]^{p-1}}\right)^{1/p} dx$$

$$\leqq \left(2\pi \sum_{n=1}^{\infty} \frac{w_p(1/n)}{n}\right)^{1/q} \left(\sum_{n=1}^{\infty} \int_0^{2\pi} \frac{[\rho_n(x)]^p n^{p-2}}{[w_p(1/n)]^{p-1}} dx\right)^{1/p},$$

which is finite by (2. 8)$'$ and (2. 14), and we get (2. 9). This completes the proof.

3. REMARK 1. Using the notations in § 1, and applying the argument employed in the preceding proof to the Parseval's equation

$$[w_2(t)]^2 = \frac{1}{2\pi} \int_0^{2\pi} [f(x+t) - f(x)]^2 dx = 2\sum_{n=1}^{\infty} (a_n^2 + b_n^2) \left(\sin \frac{1}{2} nt\right)^2,$$

where $f \in L^2$, we see that one of the two expressions

$$\int_0^1 \frac{1}{t} \lambda\left(\frac{1}{t}\right) [w_2(t)]^2 dt$$

and

$$\sum_{n=1}^{\infty} (a_n^2 + b_n^2) \lambda(n) \log n$$

converges, then the other does.

Hence, if $f(x)$ satisfies the condition

(3. 1) $$w_2(t) = O\left(\log\frac{1}{t}\right)^{-(a+\epsilon)} \left(a \geqq \frac{1}{2},\ \varepsilon > 0\right),$$

then, taking $\lambda(1/t) = (\log(1/t))^{2a-1+\epsilon}$, we have

(3. 2) $$\sum_{n=1}^{\infty}(a_n^2 + b_n^2)(\log n)^{2a+\epsilon} < \infty.$$

In particular, we see that by a theorem of Wang [1], also cf. Tsuchikura [2], the condition (3. 2) and so (3. 1) implies the summability $|C, \alpha|$, a. e., of the Fourier series of f for $\alpha > 1/2$, or $\alpha = 1/2$, according as $a = 1/2$ or $a = 1$. Thus, Corollary 1 stated in § 1 is a result from the Wang's theorem with $a = 1$, when $p = 2$.

REMARK 2. Using the Parseval's equation in place of the Paley's inequality we can prove the following theorem quite analogously as Theorem 1.

THEOREM 2. *Let by $w(t)$ denote the modulus of continuity of the function f in $(0, 2\pi)$. If for a function $w^*(t) \geqq w(t)$,*

$$\int_0^{\pi} \frac{w^*(t)}{t} dt < \infty,$$

then the Fourier series of f is summable $|C, 1/2|$ everywhere, provided that

$$[w^*(1/x)\log x]^{-1}$$

is a function λ defined by Definition 2.

COROLLARY 2. *The conclusion in Theorem 2 is true, if for some $\varepsilon > 0$,*

$$w(t) = O\left(\log\frac{1}{t}\right)^{-(1+\epsilon)} \qquad (t \to 0).$$

This corollary improves a result of Chow [4, Theorem 3].

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