Title
A Note on Algebraic Semantics for S5 with Propositional Quantifiers

Author
Holliday, Wesley Halcrow

Publication Date
2017-04-01

Peer reviewed

A Note on Algebraic Semantics for S5 with Propositional Quantifiers

Wesley H. Holliday

University of California, Berkeley

Preprint of April 2017. Forthcoming in *Notre Dame Journal of Formal Logic*.

Abstract

In two of the earliest papers on extending modal logic with propositional quantifiers, R. A. Bull and K. Fine studied a modal logic $\mathsf{S5}\Pi$ extending $\mathsf{S5}$ with axioms and rules for propositional quantification. Surprisingly, there seems to have been no proof in the literature of the completeness of $\mathsf{S5}\Pi$ with respect to its most natural algebraic semantics, with propositional quantifiers interpreted by meets and joins over all elements in a complete Boolean algebra. In this note, we give such a proof. This result raises the question: for which normal modal logics L can one axiomatize the quantified propositional modal logic determined by the complete modal algebras for L?

Keywords: modal logic, propositional quantifiers, algebraic semantics, monadic algebras, MacNeille completion

MSC: 03B45, 03C80, 03G05

1 Introduction

The idea of extending the language of propositional modal logic with propositional quantifiers $\forall p$ and $\exists p$ was first investigated in Kripke 1959, Bull 1969, Fine 1970, and Kaplan 1970.[1] The language $\mathcal{L}\Pi$ of quantified propositional modal logic is given by the grammar

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \forall p\varphi,$$

where p comes from a countably infinite set Prop of propositional variables. The other connectives $\vee$, $\to$, and $\leftrightarrow$ are defined as usual, and we let $\Diamond\varphi := \neg\Box\neg\varphi$ and $\exists p\varphi := \neg\forall p\neg\varphi$.

A focus of the papers cited above was on extending the modal logic $\mathsf{S5}$ with propositional quantification. As usual, one can think about natural extensions syntactically or semantically. Syntactically, arguably the most natural extension among those considered is the system $\mathsf{S5}\Pi$ studied by Bull and Fine, which extends the axioms and rules of $\mathsf{S5}$ (see, e.g., Chellas 1980, §1.2) with the following axioms and rule for the propositional quantifiers:

- Universal distribution axiom: $\forall p(\varphi \to \psi) \to (\forall p\varphi \to \forall p\psi)$.

[1] Earlier a first-order language with strict implication was extended with propositional quantifiers in Barcan 1947.

- Universal instantiation axiom: $\forall p\varphi \to \varphi^p_\psi$ where ψ is substitutable for p in φ, and φ^p_ψ is the result of replacing all free occurrences of p in φ by ψ.[2]

- Vacuous quantification axiom: $\varphi \to \forall p\varphi$ where p is not free in φ.

- Rule of universal generalization: if φ is a theorem, then $\forall p\varphi$ is a theorem.

Fine and Kaplan also considered a system $\mathsf{S5\Pi}^+$ extending $\mathsf{S5\Pi}$ with the axiom

$$\exists q(q \wedge \forall p(p \to \Box(q \to p))), \tag{W}$$

expressing that there is a true proposition q that necessarily entails every other true proposition p. This axiom is needed for certain completeness results (see Theorem 2.1 below). But if one is interested in a propositionally quantified extension of $\mathsf{S5}$ that can be taken seriously philosophically, as capturing valid principles for reasoning about necessity and quantification over propositions, then the axiom (W) asserting the existence of the true "world proposition" q that entails all other truths is much more questionable than the rest of $\mathsf{S5\Pi}$.

If we think semantically about natural extensions of modal logics with propositional quantifiers, there are various semantics to consider. One natural idea is that since we are quantifying over propositions, our models should consist of an algebra of propositions, plus a function taking propositions to propositions to interpret the modal operator. Our focus here will be on such an algebraic semantics. We will assume familiarity with the basic notions of Boolean algebra (see, e.g., Halmos 1963, Givant and Halmos 2009).

Definition 1. A *modal algebra* is a pair $(B, \Box)$ where B is a Boolean algebra and $\Box$ is a unary function on B such that $\Box 1 = 1$ and $\Box(a \wedge b) = \Box a \wedge \Box b$ for all $a, b \in B$. The modal algebra is *complete* if B is a complete Boolean algebra.

For convenience, we conflate the notation for an algebra and its carrier set. We also trust that no confusion will arise by using '$\Box$' for the function in our algebras and for the modal operator in our formal language. Similarly, we will use '$\neg$', '$\wedge$', and '$\vee$' for the *complement*, *meet*, and *join* in our Boolean algebras; and we define $a \to b := \neg(a \wedge \neg b)$. If B is a complete Boolean algebra and $X \subseteq B$, then $\bigwedge X$ is the meet of X and $\bigvee X$ is the join of X. Sometimes for clarity we will write '1_A' and '0_A' for the top and bottom elements of an algebra A. We will also use the defined operator $\Diamond$ given by $\Diamond a := \neg\Box\neg a$.

We can interpret the language $\mathcal{L}\Pi$ in complete modal algebras as follows.

Definition 2. Let A be a complete modal algebra. A *valuation* on A is a function $\theta\colon \mathsf{Prop} \to A$, which extends to an $\mathcal{L}\Pi$-*valuation* $\tilde{\theta}\colon \mathcal{L}\Pi \to A$ by:

1. $\tilde{\theta}(p) = \theta(p)$;

2. $\tilde{\theta}(\neg\varphi) = \neg\tilde{\theta}(\varphi)$;

3. $\tilde{\theta}(\varphi \wedge \psi) = \tilde{\theta}(\varphi) \wedge \tilde{\theta}(\psi)$;

[2]We can speak of *free* and *bound* propositional variables p in a formula with propositional quantifiers, just as we speak of free and bound individual variables in an ordinary first-order formula. A formula ψ is *substitutable for p in φ* if no free propositional variable in ψ becomes bound by a quantifier in φ^p_ψ.

4. $\tilde{\theta}(\Box\varphi) = \Box\tilde{\theta}(\varphi)$;

5. $\tilde{\theta}(\forall p\varphi) = \bigwedge\{\tilde{\gamma}(\varphi) \mid \gamma$ a valuation on A differing from θ at most at $p\}$.

A formula φ is *valid* in A if $\tilde{\theta}(\varphi) = 1$ for every valuation θ. Otherwise it is *refutable* in A.

It is easy to see that for any class of complete modal algebras, the formulas valid in every algebra in the class will be what we call a *normal* Π-*logic*: a set of $\mathcal{L}\Pi$ formulas containing all instances of classical tautologies, all instances of $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, and all instances of the axioms for propositional quantifiers given for $\mathsf{S5}\Pi$ above, while being closed under modus ponens, necessitation (if φ is in the set, so is $\Box\varphi$), and universal generalization.[3] Below when we speak of *extensions* of a logic L, we mean normal Π-logics extending L.

To validate the axioms of $\mathsf{S5}$, which we may take to be $\Box\varphi \to \varphi$ (T) and $\Diamond\varphi \to \Box\Diamond\varphi$ (5) (see, e.g., Chellas 1980, §1.2), we need the following subclass of modal algebras.

Definition 3. An *S5 algebra* is a modal algebra A such that $\Box a \leq a$ and $\Diamond a \leq \Box\Diamond a$ for all $a \in A$.

Thus, complete S5 algebras validate the logic $\mathsf{S5}\Pi$ according to Definition 2.

S5 algebras are also known as *monadic algebras* [Halmos, 1955, 1962], which are often defined with the operator $\Diamond$ as the primitive instead of its dual $\Box$. In the literature on monadic algebras, the symbols '$\exists$' and '$\forall$' are typically used instead of '$\Diamond$' and '$\Box$', but here we reserve the former symbols for the propositional quantifiers.

Our interest here will be in the following special S5 algebras.

Definition 4 (Halmos 1955)**.** A *simple S5 algebra* is a pair $(B, \Box)$ where B is a Boolean algebra and $\Box$ is the unary function on B defined for $a \in B$ by:[4]

$$\Box a = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{otherwise.} \end{cases}$$

A simple S5 algebra $(B, \Box)$ is *complete* if B is a complete Boolean algebra.

Obviously (complete) simple S5 algebras are a special case of (complete) S5 algebras. Note that for the operator $\Diamond$, we have:

$$\Diamond a = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Also note that a simple S5 algebra is uniquely determined by its underlying Boolean algebra.

It is well known that $\mathsf{S5}$ is complete with respect to finite simple S5 algebras [Scroggs, 1951]. Surprisingly, however, there seems to be no proof in the literature of the following fact.

Theorem 1. $\mathsf{S5}\Pi$ is sound and complete with respect to complete simple S5 algebras: for all $\varphi \in \mathcal{L}\Pi$, φ is a theorem of $\mathsf{S5}\Pi$ iff φ is valid in all complete simple S5 algebras.

[3] Note that universal generalization, universal instantiation, and modus ponens imply the usual closure under uniform substitution for pure modal formulas (without propositional quantifiers).

[4] A simple algebra is defined as an algebra in which $\{0\}$ is its only proper ideal, which is equivalent to the condition in Definition 4 in the case of S5 algebras [Halmos, 1955, p. 226]. Simple S5 algebras are also known as *Henle algebras* [Dunn and Hardegree, 2001, p. 367].

By contrast, Bull and Fine proved that $\mathsf{S5}\Pi$ is sound and complete with respect to possible world models with the propositional quantifiers ranging over a distinguished algebra of sets of worlds, not necessarily a complete Boolean algebra. The purpose of the present note is to prove Theorem 1 with the help of a result in Fine 1970.

Before proving Theorem 1, we should note an apparent obstacle. Based on an analogy with algebraic semantics for predicate logic (for an overview, see Scott 2008), a natural strategy to prove such a theorem for S5 algebras[5] would be to embed the Lindenbaum algebra of $\mathsf{S5}\Pi$, which is a Boolean algebra, into its MacNeille completion (defined in Section 2) and then argue that any non-theorem of $\mathsf{S5}\Pi$ is refuted in the completion by the valuation θ where $\theta(p)$ is the image under the embedding of the equivalence class of p. As expected, in the Lindenbaum algebra of $\mathsf{S5}\Pi$, the equivalence class of $\forall p\varphi$ is the meet of the equivalence classes of φ^p_ψ for each $\psi \in \mathcal{L}\Pi$ that is substitutable for p in φ; and the MacNeille completion preserves all existing meets. The problem, however, is that in the completion, the semantic value of $\forall p\varphi$ is defined *not* as the meet of the elements $\tilde{\theta}(\varphi^p_\psi)$ for each such ψ, but rather as the meet of the elements $\tilde{\gamma}(\varphi)$ for each valuation γ differing from θ at most at p. So for all we know, the set of which we are taking the meet may become bigger, so that $\tilde{\theta}(\forall p\varphi)$ is not the image under the embedding of the equivalence class of $\forall p\varphi$. This does not happen in the predicate case, because the meet for $\forall x$ is taken over a set of objects that remains fixed as one goes from the Lindenbaum algebra to its completion. The problem for propositional quantifiers is that our domain of quantification enlarges when we go to the completion.[6]

The point of this note is that despite the apparent obstacle just described, we can prove Theorem 1 using MacNeille completion after all thanks to a result in Fine 1970. Adapting the proof of Theorem 1, we also easily obtain the following in Sections 2.1-2.2.

Theorem 2.

1. [Fine, 1970, Kaplan, 1970] The extension $\mathsf{S5}\Pi^+$ of $\mathsf{S5}\Pi$ with the axiom (W) is sound and complete with respect to complete and *atomic* simple S5 algebras.

2. The extension $\mathsf{S5}\Pi^\star$ of $\mathsf{S5}\Pi$ with the negation of axiom (W) is sound and complete with respect to complete and *atomless* simple S5 algebras.

3. The extension $\mathsf{S5}\Pi_\infty$ (resp. $\mathsf{S5}\Pi^+_\infty$) of $\mathsf{S5}\Pi$ (resp. $\mathsf{S5}\Pi^+$) with an infinite schema of axioms given in Section 2.2 is sound and complete with respect to *infinite* complete (resp. complete and atomic) simple S5 algebras.

Much of the recent research on quantified propositional modal logic has focused on possible world semantics (see, e.g., Kaminski and Tiomkin 1996, Antonelli and Thomason 2002, ten Cate 2006, Belardinelli and van der Hoek 2015, 2016, Kuusisto 2015, Fritz 2017).[7]

[5] *Simple* S5 algebras require more work, as we shall see in Section 2.

[6] The sketched proof strategy involving MacNeille completion was used in a claimed proof of the completeness of a quantified propositional intuitionistic logic with respect to complete Heyting algebras in Geuvers 1994. As explained by Zdanowski [2009, p. 160], the claimed proof is flawed, due to precisely the same problem that the domain of propositional quantification enlarges when we go to the MacNeille completion. I thank an anonymous reviewer for bringing these references to my attention.

[7] As usual, a choice point for possible world semantics concerns the interpretation of $\Box$, e.g., whether by relational semantics, neighborhood semantics, topological semantics, etc. For neighborhood semantics for $\Box$ plus a substitutional semantics for $\forall p$, see Gabbay 1971. For topological semantics for $\Box$ plus full quantification over the powerset for $\forall p$, see Kremer 1997b.

Similarly, much of the research on intuitionistic logic with propositional quantifiers has focused on Kripke semantics (see, e.g., Gabbay 1974, Sobolev 1977, Kremer 1997a, Skvortsov 1997, Zach 2004).[8] We hope that the present note may provide some impetus for the further study of algebraic semantics based on complete Boolean or Heyting algebras.

In particular, Theorem 1 raises the following question, where $\mathsf{L}\Pi$ is the least normal Π-logic extending the normal modal logic L.

Question 1. For which normal modal logics L is $\mathsf{L}\Pi$ complete with respect to the complete modal algebras for L? For which normal modal logics L is the quantified propositional modal logic determined by the complete modal algebras for L recursively axiomatizable? Similarly, for which superintuitionistic logics L is the quantified propositional superintuitionistic logic determined by the complete Heyting algebras for L recursively axiomatizable?

In the case of normal modal logics properly extending $\mathsf{S5}$, an affirmative answer to Question 1 is easily obtained using Scroggs' [1951] theorem that each such logic is complete with respect to a finite simple S5 algebra. Indeed, the proof is even easier in this case than in the case of $\mathsf{S5}$, since we can avoid MacNeille completion altogether thanks to the finiteness of the relevant algebras. In Section 2.2, we will prove the following.

Theorem 3. For every normal modal logic L extending $\mathsf{S5}$, the logic $\mathsf{L}\Pi$ is complete with respect to the complete simple S5 algebras for L.

We will briefly return to other cases of Question 1 in our concluding Section 3.

2 Proofs of Theorems 1-3

The soundness of $\mathsf{S5}\Pi$ (resp. $\mathsf{S5}\Pi^+$, $\mathsf{S5}\Pi^\star$) with respect to complete (resp. complete and atomic, complete and atomless) simple S5 algebras is easy to check.

For completeness, we begin with a proof sketch. We will make essential use of a result from Fine [1970] that every formula φ of $\mathcal{L}\Pi$ can be translated into a *quantifier-free* formula $qf(\varphi)$ in a language $\mathcal{L}$Mg with infinitely many new modal operators and a new propositional constant, such that the equivalence $\varphi \leftrightarrow qf(\varphi)$ is provable in a conservative extension $\mathsf{S5}\Pi$Mg of $\mathsf{S5}\Pi$, which has a sound interpretation in complete simple S5 algebras. If φ is not a theorem of $\mathsf{S5}\Pi$, then $qf(\varphi)$ is not a theorem of $\mathsf{S5}\Pi$Mg, so in the Lindenbaum algebra of $\mathsf{S5}\Pi$Mg, the equivalence class of $qf(\varphi)$ is not the top element. We will show that by taking an appropriate quotient of the Lindenbaum algebra, we obtain a simple S5 algebra in which $qf(\varphi)$ is refuted according to the intended semantics for the new modal operators and propositional constant. Then we will take the MacNeille completion of the quotient and show that the new modal operators and propositional constant are appropriately preserved, so that we obtain a complete simple S5 algebra in which $qf(\varphi)$ is refuted according to the intended semantics. It follows that φ is also refuted in this complete simple S5 algebra, because $\varphi \leftrightarrow qf(\varphi)$ is valid in such algebras. This completes the proof. The argument is diagrammed in Figure 1 with some added details to be introduced below.

[8] Topological semantics for intuitionistic logic with propositional quantification over open sets has been considered in, e.g., Połacik 1998. Other non-classical logics have also been considered, such as relevance logic with propositional quantifiers in Kremer 1993.

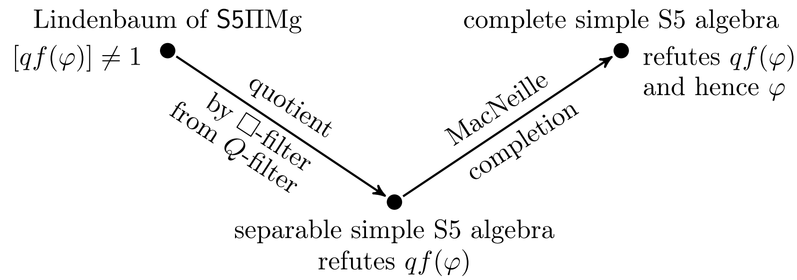

Figure 1: Diagram of the completeness proof.

We define the language $\mathcal{L}$Mg by extending the basic modal language, *without propositional quantifiers*, with new unary operators $\mathrm{M}_1, \mathrm{M}_2, \dots$, one for each $n \in \mathbb{N} \setminus \{0\}$, and a new propositional constant g, as in Fine 1970, §1.3. We can evaluate formulas of $\mathcal{L}$Mg in what Fine [1970, §1.4] calls separable Boolean algebras, as in Definition 5. Recall that an *atom* in a Boolean algebra is a non-zero element with no distinct non-zero element below it.

Definition 5. A Boolean algebra is *separable* if its set of atoms has a join. A modal algebra is separable if it is based on a separable Boolean algebra.

Definition 6. Let A be a separable modal algebra. An $\mathcal{L}$Mg-*valuation* for A is defined as in Definition 2 for the Boolean and $\Box$ cases, plus:

1. $\tilde{\theta}(\mathrm{M}_n\varphi) = \begin{cases} 1 & \text{if there are at least } n \text{ distinct atoms below } \tilde{\theta}(\varphi) \\ 0 & \text{otherwise;} \end{cases}$

2. $\tilde{\theta}(\mathrm{g}) = \bigvee\{a \in A \mid a \text{ an atom in } A\}$.

Separability ensures that we can interpret g in the desired way.

Let $\mathcal{L}\Pi$Mg be the full modal language with propositional quantifiers, the M_n operators, and g. If a modal algebra is complete—and hence separable—then we can evaluate $\mathcal{L}\Pi$Mg formulas in it by combining all of the semantic clauses above.

Definition 7. Let A be a complete modal algebra. An $\mathcal{L}\Pi$Mg-*valuation* for A is defined by combining the clauses from Definitions 2 and 6.

For any formula α, define

$$atom(\alpha) := \Diamond\alpha \wedge \forall q(\Box(\alpha \to q) \vee \Box(\alpha \to \neg q)), \tag{1}$$

where q is the first variable that does not appear in α, assuming some fixed enumeration of Prop (our $atom(\alpha)$ is Fine's [1970, p. 339] QA). As shown in the proof of Lemma 1 below, $atom(\alpha)$ will evaluate to 1 if α evaluates to an atom, and 0 otherwise.

As in §1.3 of Fine 1970, let us define a logic S5ΠMg just as we defined S5Π in Section 1 but with the following new axioms, where $p_1, \dots, p_n$ do not appear in φ:

$$\mathrm{M}_n\varphi \leftrightarrow \exists p_1 \dots \exists p_n (\bigwedge_{1 \le i < j \le n} \Box(p_i \to \neg p_j) \wedge \bigwedge_{1 \le i \le n} (atom(p_i) \wedge \Box(p_i \to \varphi))) \tag{2}$$

$$\mathrm{g} \leftrightarrow \exists p(p \wedge atom(p)). \tag{3}$$

One can think of (2)–(3) as capturing syntactically the intended interpretations of M_n and g as in Definitions 6–7, as confirmed by Lemma 1.

Lemma 1. S5ΠMg is sound with respect to complete simple S5 algebras according to the semantics of Definition 7.

Proof. Let A be such an algebra. All we need to check is the two new axioms (2) and (3). First, we show that for any valuation θ, we have:

$$\text{if } \theta(p) \text{ is an atom in } A, \text{ then } \tilde{\theta}(atom(p)) = 1; \text{ otherwise } \tilde{\theta}(atom(p)) = 0. \tag{4}$$

Where Γ is the set of all valuations that differ from θ at most at q, we have:

$$\tilde{\theta}(atom(p)) = \Diamond\theta(p) \wedge \bigwedge\{\Box(\theta(p) \to \gamma(q)) \vee \Box(\theta(p) \to \neg\gamma(q)) \mid \gamma \in \Gamma\}. \tag{5}$$

Recall that in a Boolean algebra,

$$\text{a non-zero element } a \text{ is an atom iff for every } b, \text{ either } a \leq b \text{ or } a \leq \neg b. \tag{6}$$

Now suppose $\theta(p)$ is an atom. Then $\theta(p) \neq 0$, so $\Diamond\theta(p) = 1$. Moreover, for any element $\gamma(q)$, either $\theta(p) \leq \gamma(q)$ or $\theta(p) \leq \neg\gamma(q)$, in which case either $\theta(p) \to \gamma(q) = 1$ or $\theta(p) \to \neg\gamma(q) = 1$, in which case either $\Box(\theta(p) \to \gamma(q)) = 1$ or $\Box(\theta(p) \to \neg\gamma(q)) = 1$, which implies $\Box(\theta(p) \to \gamma(q)) \vee \Box(\theta(p) \to \neg\gamma(q)) = 1$, which implies that the meet in (5) is 1. Hence $\tilde{\theta}(atom(p)) = 1$ when $\theta(p)$ is an atom. Now suppose $\theta(p)$ is not an atom. If $\theta(p) = 0$, then $\Diamond\theta(p) = 0$, whence $\tilde{\theta}(atom(p)) = 0$. So suppose $\theta(p) \neq 0$. Then since $\theta(p)$ is not an atom, there is a b such that $\theta(p) \not\leq b$ and $\theta(p) \not\leq \neg b$. It follows that $\Box(\theta(p) \to b) \vee \Box(\theta(p) \to \neg b) = 0$, which implies that the meet in (5) is 0. Hence $\tilde{\theta}(atom(p)) = 0$ when $\theta(p)$ is not an atom.

It follows from (4) that for any valuation γ, if $\gamma(p)$ is an atom, then $\tilde{\gamma}(p \wedge atom(p)) = \gamma(p)$, while if $\gamma(p)$ is not an atom, then $\tilde{\gamma}(p \wedge atom(p)) = 0$. Thus, where Γ is the set of all valuations that differ from θ at most at p, we have

$$\begin{aligned} \tilde{\theta}(\exists p(p \wedge atom(p))) &= \bigvee\{\tilde{\gamma}(p \wedge atom(p)) \mid \gamma \in \Gamma\} \\ &= \bigvee\{a \in A \mid a \text{ an atom in } A\} \\ &= \tilde{\theta}(\mathrm{g}), \end{aligned}$$

so axiom (3) is valid. Similar reasoning shows that the right hand side of (2) evaluates to 1 iff there are at least n distinct atoms below $\tilde{\theta}(\varphi)$, so (2) is valid. □

Next we make an obvious syntactic observation.

Lemma 2. S5ΠMg is a conservative extension of S5Π: for all $\varphi \in \mathcal{L}\Pi$, if φ is a theorem of S5ΠMg, then φ is a theorem of S5Π.

Proof. Given a proof of φ using S5ΠMg, we can clearly replace M_n and g according to (2)–(3) to obtain a proof of φ using only S5Π. □

The key fact about S5ΠMg for our purposes is Fine's [1970, §1.3] result, proven in Fine 1969, pp. 50-54, that it allows quantifier elimination in the following sense.

Lemma 3 (Fine 1970)**.** For each $\varphi \in \mathcal{L}\Pi$, there is a formula $qf(\varphi) \in \mathcal{L}$Mg (for *q*uantifier *f*ree) such that $\varphi \leftrightarrow qf(\varphi)$ is a theorem of $\mathsf{S5}\Pi$Mg.

Proof. For the reader's convenience, we outline the proof of Lemma 3 in the Appendix. $\square$

From now on, we assume that $qf(\varphi)$ has been fixed for each formula $\varphi \in \mathcal{L}\Pi$. Putting together Lemmas 1 and 3, we have the following.

Lemma 4. For each $\varphi \in \mathcal{L}\Pi$, $\varphi \leftrightarrow qf(\varphi)$ is valid in complete simple S5 algebras according to Definition 7.

The next important lemma we need is the following.

Lemma 5. For every $\varphi \in \mathcal{L}\Pi$, if φ is not a theorem of $\mathsf{S5}\Pi$, then $qf(\varphi)$ is refutable in a separable simple S5 algebra according to Definition 6.

Lemma 5 follows from results stated in Fine 1970 (Propositions 1 and 3) and proved in Fine 1969 (Chs. 4–5), but here we prefer to give an independent algebraic proof of Lemma 5.

First part of proof. Suppose $\xi \in \mathcal{L}\Pi$ is not a theorem of $\mathsf{S5}\Pi$, so by Lemma 2, it is not a theorem of $\mathsf{S5}\Pi$Mg, which with Lemma 3 implies that $qf(\xi)$ is not a theorem of $\mathsf{S5}\Pi$Mg. Then in the Lindenbaum algebra of $\mathsf{S5}\Pi$Mg, defined below, the equivalence class of $qf(\xi)$ is not the top element. We will show that by taking an appropriate quotient of the Lindenbaum algebra, we obtain a separable simple S5 algebra in which $qf(\xi)$ is refutable.

As usual, the elements of the Lindenbaum algebra of $\mathsf{S5}\Pi$Mg are the equivalence classes

$$[\varphi] = \{\psi \in \mathcal{L}\Pi\text{Mg} \mid \vdash_{\mathsf{S5}\Pi\text{Mg}} \varphi \leftrightarrow \psi\}$$

for each $\varphi \in \mathcal{L}\Pi$Mg, ordered by $[\varphi] \leq [\psi]$ iff $\vdash_{\mathsf{S5}\Pi\text{Mg}} \varphi \to \psi$, with the meet, complement, and modal box given by $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$, $\neg[\varphi] = [\neg\varphi]$, and $\Box[\varphi] = [\Box\varphi]$. The following lemma is straightforward to prove using the axioms and rules of the logic.

Lemma 6. The Lindenbaum algebra of $\mathsf{S5}\Pi$Mg is an S5 algebra such that for all $\varphi \in \mathcal{L}\Pi$Mg and $p \in \mathsf{Prop}$:

$$[\forall p\varphi] = \bigwedge\{[\varphi^p_\psi] \mid \psi \in \mathcal{L}\Pi\text{Mg free for } p \text{ in } \varphi\} \quad (7)$$

$$[\exists p\varphi] = \bigvee\{[\varphi^p_\psi] \mid \psi \in \mathcal{L}\Pi\text{Mg free for } p \text{ in } \varphi\}. \quad (8)$$

In Lemma 6 and the following, *free for p in φ* is short for *substitutable for p in φ*. When the relevant formula φ is clear from context, we simply write *free for p*.

The Lindenbaum algebra of $\mathsf{S5}\Pi$Mg is atomless (because for any φ such that $[\varphi] \neq 0$ and p not in φ, we have $0 \neq [\varphi \wedge p] < [\varphi]$) and not simple (because, e.g., $[\Box p] \neq 0$ while $[p] \neq 1$). Given our formula $qf(\xi)$ such that $[qf(\xi)] \neq 1$, we will go from the Lindenbaum algebra to a separable and simple quotient algebra in which the meets and joins in (7)–(8) are preserved. This will be a quotient obtained from a filter of a special kind.

Definition 8. Let A be a Boolean algebra and $Q = \langle \{X_n\}_{n\in\mathbb{N}}, \{Y_n\}_{n\in\mathbb{N}} \rangle$ a pair of countable subsets of $\wp(A)$ such that $\bigwedge X_n \in A$ and $\bigvee Y_n \in A$ for all $n \in \mathbb{N}$. An ultrafilter F in A is a *Q-filter* iff for all $n \in \mathbb{N}$:

1. if $X_n \subseteq F$, then $\bigwedge X_n \in F$;

2. if $\bigvee Y_n \in F$, then $Y_n \cap F \neq \varnothing$.

The famous Rasiowa-Sikorski lemma [Rasiowa and Sikorski, 1950, 1963] ensures that elements can be separated by such filters. A proof of exactly the following statement of the lemma may be found in Tanaka 1999, p. 328.

Lemma 7 (Rasiowa-Sikorski)**.** Let A be a Boolean algebra and $Q = \langle \{X_n\}_{n\in\mathbb{N}}, \{Y_n\}_{n\in\mathbb{N}} \rangle$ a pair of countable subsets of $\wp(A)$ such that $\bigwedge X_n \in A$ and $\bigvee Y_n \in A$ for all $n \in \mathbb{N}$. Then for any $a, b \in A$ such that $a \not\leq b$, there is a Q-filter F such that $a \in F$ but $b \notin F$.

Let $\langle \varphi_1, p_1 \rangle$, $\langle \varphi_2, p_2 \rangle$, ... be an enumeration of all pairs of a $\varphi_i \in \mathcal{L}\Pi\text{Mg}$ and $p_i \in \mathsf{Prop}$. From the Lindenbaum algebra of $\mathsf{S5}\Pi\text{Mg}$, define $Q = \langle \{X_n\}_{n\in\mathbb{N}}, \{Y_n\}_{n\in\mathbb{N}} \rangle$ where

$$X_n = Y_n = \{[(\varphi_n)^{p_n}_{\psi}] \mid \psi \in \mathcal{L}\Pi\text{Mg free for } p_n \text{ in } \varphi_n\},$$

so by Lemma 6, Q satisfies the condition in Definition 8. Henceforth we keep this Q fixed.

For the next lemma, recall that a filter F in a Boolean algebra A induces a congruence relation on A by: $a \equiv_F b$ iff there is an $x \in F$ such that $a \wedge x = b \wedge x$. Let A/F be the quotient of A with respect to $\equiv_F$, and note that the quotient map sends an element $a \in A$ to $1_{A/F}$ iff $a \in F$. Finally, a $\Box$-*filter* is a filter F such that $a \in F$ implies $\Box a \in F$.

Lemma 8. For any Q-filter F in the Lindenbaum algebra A of $\mathsf{S5}\Pi\text{Mg}$:

1. $\Box^{-1}F = \{a \in A \mid \Box a \in F\}$ is a maximal $\Box$-filter;

2. the quotient $A/\Box^{-1}F$ is a simple S5 algebra;

3. the function π mapping each $a \in A$ to its equivalence class $\pi(a)$ in $A/\Box^{-1}F$ is a Boolean homomorphism from A onto $A/\Box^{-1}F$ such that $\pi(\Box a) = \Box\pi(a)$ and

$$\pi(\bigwedge X_n) = \bigwedge\{\pi(x) \mid x \in X_n\} \tag{9}$$
$$\pi(\bigvee X_n) = \bigvee\{\pi(x) \mid x \in X_n\}. \tag{10}$$

4. in $A/\Box^{-1}F$, the following properties hold:[9]

   (a) $\{\pi[\alpha \wedge atom(\alpha)] \mid \alpha \in \mathcal{L}\Pi\text{Mg}\} \setminus \{0\}$ is the set of atoms;

   (b) $\pi[\text{g}]$ is the join of the set of atoms, so $A/\Box^{-1}F$ is separable;

   (c) $\pi[\text{M}_n\varphi] = \begin{cases} 1 & \text{if there are at least } n \text{ distinct atoms below } \pi[\varphi] \\ 0 & \text{otherwise.} \end{cases}$

Proof. For part 1, it is easy to verify that if F is a maximal Boolean filter in an S5 algebra, then $\Box^{-1}F$ is a maximal $\Box$-filter (see Halmos and Givant 1998, p. 120).[10] In addition, the

[9] To avoid clutter, we omit parentheses when applying π to equivalence classes in square brackets.

[10] As in Halmos 1955, in Halmos and Givant 1998, S5 algebras are called monadic algebras; what we call $\Box$-filters are their *monadic filters*; and Boolean homomorphisms that preserve $\Box$ are their *monadic homomorphisms*.

quotient of any S5 algebra by a maximal $\Box$-filter is a simple S5 algebra (see Halmos and Givant 1998, p. 119). Thus, part 1 implies part 2.

For part 3, the map sending each element of an S5 algebra to its equivalence class in the quotient by a $\Box$-filter is always an onto Boolean homomorphism that preserves $\Box$ (see Halmos and Givant 1998, p. 118). It only remains to establish (9)–(10). The argument for (10) is given in Tanaka 1999, Lemma 2.6. For the reader's convenience, we will prove (9) in a similar fashion. First, note that any S5 $\Box$ is completely multiplicative: if $\bigwedge X$ exists, then $\bigwedge\{\Box x \mid x \in X\}$ exists and $\Box \bigwedge X = \bigwedge\{\Box x \mid x \in X\}$.[11] Now for (9), clearly $\pi(\bigwedge X_n)$ is a lower bound of $\{\pi(x) \mid x \in X_n\}$. To see that it is the greatest, suppose $\pi(y)$ is also a lower bound. Then we have:

$$\begin{aligned}
\forall x \in X_n\colon \pi(y) \leq \pi(x) \quad &\Leftrightarrow \quad \forall x \in X_n\colon \pi(y) \to \pi(x) = 1_{A/\Box^{-1}F}\\
&\Leftrightarrow \quad \forall x \in X_n\colon \pi(y \to x) = 1_{A/\Box^{-1}F}\\
&\Leftrightarrow \quad \forall x \in X_n\colon y \to x \in \Box^{-1}F\\
&\Leftrightarrow \quad \forall x \in X_n\colon \Box(y \to x) \in F\\
&\Leftrightarrow \quad \bigwedge\{\Box(y \to x) \mid x \in X_n\} \in F \text{ since } F \text{ is a } Q\text{-filter}\\
&\Leftrightarrow \quad \Box \bigwedge\{y \to x \mid x \in X_n\} \in F \text{ by complete multiplicativity}\\
&\Leftrightarrow \quad \Box(y \to \bigwedge X_n) \in F\\
&\Leftrightarrow \quad y \to \bigwedge X_n \in \Box^{-1}F\\
&\Leftrightarrow \quad \pi(y \to \bigwedge X_n) = 1_{A/\Box^{-1}F}\\
&\Leftrightarrow \quad \pi(y) \to \pi(\bigwedge X_n) = 1_{A/\Box^{-1}F}\\
&\Leftrightarrow \quad \pi(y) \leq \pi(\bigwedge X_n).
\end{aligned}$$

For part 4a, we first show that $\pi[\alpha \wedge atom(\alpha)]$ is either an atom or 0 in $A/\Box^{-1}F$. In the Lindenbaum algebra A, using Lemma 6 we have

$$\begin{aligned}
[\alpha \wedge atom(\alpha)] &= [\alpha \wedge \Diamond\alpha \wedge \forall q(\Box(\alpha \to q) \vee \Box(\alpha \to \neg q))]\\
&= [\alpha] \wedge \Diamond[\alpha] \wedge \bigwedge\{\Box([\alpha] \to [\psi]) \vee \Box([\alpha] \to \neg[\psi]) \mid \psi \in \mathcal{L}\Pi\text{Mg}\},
\end{aligned}$$

so in the quotient $A/\Box^{-1}F$, using part 3 of the present lemma, we have

$$\begin{aligned}
\pi[\alpha \wedge atom(\alpha)] &= \pi\big([\alpha] \wedge \Diamond[\alpha] \wedge \bigwedge\{\Box([\alpha] \to [\psi]) \vee \Box([\alpha] \to \neg[\psi]) \mid \psi \in \mathcal{L}\Pi\text{Mg}\}\big)\\
&= \pi[\alpha] \wedge \Diamond\pi[\alpha] \wedge\\
&\quad \bigwedge\{\Box(\pi[\alpha] \to \pi[\psi]) \vee \Box(\pi[\alpha] \to \neg\pi[\psi]) \mid \psi \in \mathcal{L}\Pi\text{Mg}\}\\
&= \pi[\alpha] \wedge \Diamond\pi[\alpha] \wedge\\
&\quad \bigwedge\{\Box(\pi[\alpha] \to a) \vee \Box(\pi[\alpha] \to \neg a) \mid a \in A/\Box^{-1}F\}. \qquad (11)
\end{aligned}$$

[11] To see this, observe that the derivable B axiom $a \leq \Box\Diamond a$ gives us that $\Diamond$ is residual to $\Box$, i.e., $a \leq \Box b$ iff $\Diamond a \leq b$, and as is well known, if $\Box$ has a residual, then it is completely multiplicative: by residuation, $\bigwedge\{\Box x \mid x \in X\} \leq \Box \bigwedge X$ iff $\Diamond \bigwedge\{\Box x \mid x \in X\} \leq \bigwedge X$; then since $\Diamond \bigwedge\{\Box x \mid x \in X\} \leq \bigwedge\{\Diamond\Box x \mid x \in X\}$ and, using residuation again, $\Diamond\Box x \leq x$, we have $\Diamond \bigwedge\{\Box x \mid x \in X\} \leq \bigwedge X$.

By (11), if $\pi[\alpha \wedge atom(\alpha)] \neq 0$, then $\pi[\alpha] \neq 0$, so we have

(i) $\Diamond\pi[\alpha] = 1$

since $A/\Box^{-1}F$ is a simple S5 algebra. In addition, if $\pi[\alpha \wedge atom(\alpha)] \neq 0$, then (11) implies that for every $a \in A/\Box^{-1}F$, we have $\Box(\pi[\alpha] \to a) \vee \Box(\pi[\alpha] \to \neg a) \neq 0$, so either $\Box(\pi[\alpha] \to a) \neq 0$ or $\Box(\pi[\alpha] \to \neg a) \neq 0$, which with simplicity implies that $\Box(\pi[\alpha] \to a) = 1$ or $\Box(\pi[\alpha] \to \neg a) = 1$, which in turns implies that

(ii) $\Box(\pi[\alpha] \to a) \vee \Box(\pi[\alpha] \to \neg a) = 1$ and

(iii) either $\pi[\alpha] \leq a$ or $\pi[\alpha] \leq \neg a$.

Then by (i)–(ii) and (11), $\pi[\alpha \wedge atom(\alpha)] = \pi[\alpha]$, so by (iii) and the characterization of atoms in (6), $\pi[\alpha \wedge atom(\alpha)]$ is an atom.

Conversely, for any atom b in $A/\Box^{-1}F$, since π is onto, $b = \pi[\alpha]$ for some $\alpha \in \mathcal{L}\Pi\text{Mg}$. Then since b is an atom, by reasoning as in the previous paragraph, we have:

$$\pi[\alpha] = \pi[\alpha] \wedge \Diamond\pi[\alpha] \wedge \bigwedge\{\Box(\pi[\alpha] \to a) \vee \Box(\pi[\alpha] \to \neg a) \mid a \in A/\Box^{-1}F\}.$$

Then working backward from (11), we have $\pi[\alpha] = \pi[\alpha \wedge atom(\alpha)]$. This completes the proof that the set of atoms in $A/\Box^{-1}F$ is $\{\pi[\alpha \wedge atom(\alpha)] \mid \alpha \in \mathcal{L}\Pi\text{Mg}\} \setminus \{0\}$.

For part 4b, recall the definition of $atom(\alpha)$ from (1):

$$atom(\alpha) := \Diamond\alpha \wedge \forall q(\Box(\alpha \to q) \vee \Box(\alpha \to \neg q)),$$

where q is the first variable not appearing in α. For any propositional variable r not in α, let $atom(\alpha, r)$ be the result of replacing the three occurrences of q by r:

$$atom(\alpha, r) := \Diamond\alpha \wedge \forall r(\Box(\alpha \to r) \vee \Box(\alpha \to \neg r)).$$

It is easy to see that for any propositional variables q and r not in α, $atom(\alpha, q) \leftrightarrow atom(\alpha, r)$ is a theorem of $\mathsf{S5}\Pi\text{Mg}$. Using this fact and axiom (2) of $\mathsf{S5}\Pi\text{Mg}$, in the Lindenbaum algebra of $\mathsf{S5}\Pi\text{Mg}$ we have

$$\begin{aligned} [\text{g}] &= [\exists p(p \wedge atom(p))] \\ &= \bigvee\{[\exists p(p \wedge atom(p, r))] \mid r \in \mathsf{Prop} \setminus \{p\}\}, \end{aligned} \tag{12}$$

and using Lemma 6 we have

$$[\exists p(p \wedge atom(p, r))] = \bigvee\{[\psi \wedge atom(\psi, r)] \mid \psi \in \mathcal{L}\Pi\text{Mg free for } p \text{ in } atom(p, r)\}. \tag{13}$$

We can always pick an $r \in \mathsf{Prop} \setminus \{p\}$ such that ψ is free for p in $atom(p, r)$, so (12)–(13) are easily seen to imply that

$$[\text{g}] = \bigvee\{[\alpha \wedge atom(\alpha)] \mid \alpha \in \mathcal{L}\Pi\text{Mg}\},$$

so by part 3 of the current lemma, we have

$$\begin{aligned}\pi[\mathrm{g}] &= \pi\big(\bigvee\{[\alpha \wedge atom(\alpha)] \mid \alpha \in \mathcal{L}\Pi\mathrm{Mg}\}\big)\\ &= \bigvee\{\pi[\alpha \wedge atom(\alpha)] \mid \alpha \in \mathcal{L}\Pi\mathrm{Mg}\},\end{aligned}$$

which with part 4a implies that $\pi[\mathrm{g}]$ is the join of the atoms in $A/\Box^{-1}F$, as desired.

Similar reasoning as above may be used to prove part 4c. First, defining

$$M_n(\varphi;\psi_1,\dots,\psi_n) := \bigwedge_{1\leq i<j\leq n} \Box(\psi_i \to \neg\psi_j) \wedge \bigwedge_{1\leq i\leq n} (atom(\psi_i) \wedge \Box(\psi_i \to \varphi)),$$

we can write axiom (2) of $\mathsf{S5}\Pi\mathrm{Mg}$ as

$$\mathrm{M}_n\varphi \leftrightarrow \exists p_1 \dots \exists p_n M_n(\varphi; p_1,\dots,p_n),$$

where $p_1,\dots,p_n$ do not occur in φ. Using Lemma 6 and the same kind of reasoning involving change of variables as for 4b above, in the Lindenbaum algebra of $\mathsf{S5}\Pi\mathrm{Mg}$ we have

$$\begin{aligned}[\mathrm{M}_n\varphi] &= [\exists p_1 \dots \exists p_n M_n(\varphi; p_1,\dots,p_n)]\\ &= \bigvee\{[M_n(\varphi;\psi_1,\dots,\psi_n)] \mid \psi_1,\dots,\psi_n \in \mathcal{L}\Pi\mathrm{Mg}\}\end{aligned}$$

so by part 3 of the present lemma, we have

$$\begin{aligned}\pi[\mathrm{M}_n\varphi] &= \pi\big(\bigvee\{[M_n(\varphi;\psi_1,\dots,\psi_n)] \mid \psi_1,\dots,\psi_n \in \mathcal{L}\Pi\mathrm{Mg}\}\big)\\ &= \bigvee\{\pi[M_n(\varphi;\psi_1,\dots,\psi_n)] \mid \psi_1,\dots,\psi_n \in \mathcal{L}\Pi\mathrm{Mg}\} \qquad (14)\end{aligned}$$

and

$$\begin{aligned}\pi[M_n(\varphi;\psi_1,\dots,\psi_n)] &= \bigwedge_{1\leq i<j\leq n} \Box(\pi[\psi_i] \to \neg\pi[\psi_j])\\ &\quad \wedge \bigwedge_{1\leq i\leq n} (\pi[atom(\psi_i)] \wedge \Box(\pi[\psi_i] \to \pi[\varphi])). \qquad (15)\end{aligned}$$

Since $A/\Box^{-1}F$ is simple, it follows from (15) by reasoning that should by now be familiar that $\pi[M_n(\varphi;\psi_1,\dots,\psi_n)] = 1$ if $\pi[\psi_1],\dots,\pi[\psi_n]$ are distinct atoms of $A/\Box^{-1}F$ under $\pi[\varphi]$, and otherwise $\pi[M_n(\varphi;\psi_1,\dots,\psi_n)] = 0$. Then since π is surjective, (14) implies that $\pi[\mathrm{M}_n\varphi] = 1$ if there are at least n distinct atoms of $A/\Box^{-1}F$ under $\pi[\varphi]$, and otherwise $\pi[\mathrm{M}_n\varphi] = 0$. This completes the proof. □

We now have everything we need to prove Lemma 5.

Last part of proof. As before, suppose ξ is not a theorem of $\mathsf{S5}\Pi$, so $qf(\xi)$ is not a theorem of $\mathsf{S5}\Pi\mathrm{Mg}$. It follows that $\Box qf(\xi)$ is not a theorem of $\mathsf{S5}\Pi\mathrm{Mg}$, so the equivalence class $[\Box qf(\xi)]$ of $\Box qf(\xi)$ in the Lindenbaum algebra A of $\mathsf{S5}\Pi\mathrm{Mg}$ is not the top element. Thus, by Lemma 7, there is a Q-filter F, for the Q fixed after Lemma 7, that does not contain $[\Box qf(\xi)] = \Box[qf(\xi)]$. Hence $\Box^{-1}F$ does not contain $[qf(\xi)]$, which implies that in the quotient $A/\Box^{-1}F$, we have $\pi[qf(\xi)] \neq 1$, where $\pi\colon A \to A/\Box^{-1}F$ is the quotient map. By Lemma 8, $A/\Box^{-1}F$ is a separable simple S5 algebra.

Define a valuation θ on $A/\Box^{-1}F$ by $\theta(p) = \pi[p]$, where $[p]$ is the equivalence class of p in A. Then we claim that for all formulas $\chi \in \mathcal{L}\mathrm{Mg}$:

$$\tilde{\theta}(\chi) = \pi[\chi], \tag{16}$$

where $\tilde{\theta}$ is the $\mathcal{L}\mathrm{Mg}$-valuation extending θ as in Definition 6. The proof is by induction, with the base case given by definition of θ. The Boolean and $\Box$ cases follow from the fact in Lemma 8.3 that π is a Boolean homomorphism preserving $\Box$. The g and M_n cases follow from Lemma 8.4. Then since $\pi[qf(\xi)] \neq 1$, by (16) we have $\tilde{\theta}(qf(\xi)) \neq 1$, so we are done. $\Box$

Now we need to transfer the refutation of $qf(\xi)$ in a separable simple S5 algebra to a refutation in a complete simple S5 algebra. It is well known that for any Boolean algebra B, there is a complete Boolean algebra B^* and a Boolean embedding f of B into B^* such that every element of B^* is a join of images of elements of B (see, e.g., Givant and Halmos 2009, Ch. 25). It follows that f preserves all existing joins and that the atoms in B^* are exactly the images of atoms from B. B^* is unique up to isomorphism and is called the *MacNeille completion* of B. Given a simple S5 algebra $A = (B, \Box)$, let $\overline{A}$ be the complete simple S5 algebra uniquely determined by B^*. Since f is an embedding, $f(x) = 1_{B^*}$ iff $x = 1_B$, and likewise for 0, so $f(\Box x) = \Box f(x)$. Thus, we arrive at the following lemma.

Lemma 9. For any simple S5 algebra A, there is a complete simple S5 algebra $\overline{A}$, the MacNeille completion of A, such that:

1. there is a Boolean embedding f of A into $\overline{A}$ that preserves $\Box$ and preserves all existing joins, i.e., if $\bigvee X$ exists in A, then $f(\bigvee X) = \bigvee\{f(x) \mid x \in X\}$;
2. the set of atoms in $\overline{A}$ is $\{f(a) \mid a \text{ an atom in } A\}$.

Lemma 10. For any $\psi \in \mathcal{L}\mathrm{Mg}$, if ψ is refutable in a separable simple S5 algebra A according to Definition 6, then ψ is also refutable in $\overline{A}$.

Proof. Suppose θ is the valuation on A such that $\tilde{\theta}(\psi) \neq 1_A$. Define a new valuation μ on $\overline{A}$ by $\mu(p) = f(\theta(p))$. Then we claim that for all $\chi \in \mathcal{L}\mathrm{Mg}$:

$$\tilde{\mu}(\chi) = f(\tilde{\theta}(\chi)). \tag{17}$$

The proof is by induction on χ, with the base case given by the definition of μ. The Boolean and $\Box$ cases simply use the fact that f is a Boolean embedding that preserves $\Box$. For the g case:

$$\begin{aligned}
f(\tilde{\theta}(\mathrm{g})) &= f(\bigvee\{a \mid a \text{ an atom in } A\}) \text{ by Definition 6} \\
&= \bigvee\{f(a) \mid a \text{ an atom in } A\} \text{ by Lemma 9.1} \\
&= \bigvee\{a \mid a \text{ an atom in } \overline{A}\} \text{ by Lemma 9.2} \\
&= \tilde{\mu}(\mathrm{g}) \text{ by Definition 6.}
\end{aligned}$$

Finally, given the inductive hypothesis that $\tilde{\mu}(\chi) = f(\tilde{\theta}(\chi))$, it follows by Lemma 9.2 and the fact that f is a Boolean embedding that there are at least n distinct atoms of A below $\tilde{\theta}(\chi)$ iff there are at least n distinct atoms of $\overline{A}$ below $f(\tilde{\theta}(\chi)) = \tilde{\mu}(\chi)$. If both sides hold, then $f(\tilde{\theta}(\mathrm{M}_n\chi)) = f(1_A) = 1_{\overline{A}}$ and $\tilde{\mu}(\mathrm{M}_n\chi) = 1_{\overline{A}}$. If neither side holds, then $f(\tilde{\theta}(\mathrm{M}_n\chi)) = f(0_A) = 0_{\overline{A}}$ and $\tilde{\mu}(\mathrm{M}_n\chi) = 0_{\overline{A}}$. Hence $\tilde{\mu}(\mathrm{M}_n\chi) = f(\tilde{\theta}(\mathrm{M}_n\chi))$.

Finally, since $\tilde{\theta}(\psi) \neq 1_A$, we have $f(\tilde{\theta}(\psi)) \neq f(1_A) = 1_{\overline{A}}$, so $\tilde{\mu}(\psi) \neq 1_{\overline{A}}$ by (17). □

We now have everything we need to prove the completeness part of Theorem 1.

Proof. If φ is not a theorem of $\mathsf{S5\Pi}$, then by Lemma 5, $qf(\varphi)$ is refutable in a separable simple S5 algebra, whence by Lemmas 9-10, $qf(\varphi)$ is refutable in a complete simple S5 algebra. Then by Lemma 4, φ is refutable in a complete simple S5 algebra. □

2.1 Atomic and Atomless Algebras

To prove Theorem 2.1-2 for atomic algebras and atomless algebras, we need one more lemma. Let $\mathsf{S5\Pi}\mathrm{Mg}^+$ be the extension of $\mathsf{S5\Pi}\mathrm{Mg}$ with the axiom (W), and let $\mathsf{S5\Pi}\mathrm{Mg}^\star$ be the extension of $\mathsf{S5\Pi}\mathrm{Mg}$ with the negation of (W). Just as in Lemma 2, it is easy to see that $\mathsf{S5\Pi}\mathrm{Mg}^+$ is a conservative extension of $\mathsf{S5\Pi}^+$, and similarly for $\mathsf{S5\Pi}\mathrm{Mg}^\star$ and $\mathsf{S5\Pi}^\star$. Let Q^+ and $Q^\star$ be defined for $\mathsf{S5\Pi}\mathrm{Mg}^+$ and $\mathsf{S5\Pi}\mathrm{Mg}^\star$, respectively, just as we defined Q for $\mathsf{S5\Pi}\mathrm{Mg}$ after Lemma 7. Then we have the following analogue of Lemma 8.

Lemma 11.

1. For any Q^+-filter F in the Lindenbaum algebra A of $\mathsf{S5\Pi}\mathrm{Mg}^+$, all of the parts of Lemma 8 hold, and in addition $A/\Box^{-1}F$ is *atomic*;

2. For any $Q^\star$-filter F in the Lindenbaum algebra A of $\mathsf{S5\Pi}\mathrm{Mg}^\star$, all of the parts of Lemma 8 hold, and in addition $A/\Box^{-1}F$ is *atomless*;

Proof. For part 1, to see that $A/\Box^{-1}F$ is atomic, consider any non-zero $b \in A/\Box^{-1}F$. By the analogue of Lemma 8.3 for $\mathsf{S5\Pi}\mathrm{Mg}^+$, $b = \pi[\beta]$ for some $\beta \in \mathcal{L}\Pi\mathrm{Mg}$, so we can reason in terms of $\pi[\beta]$. First, it is easy to check that

$$\exists q(q \wedge atom(q) \wedge \forall p(p \to \Box(q \to p)))$$

is a theorem of $\mathsf{S5\Pi}\mathrm{Mg}^+$, which in the Lindenbaum algebra A means that

$$\begin{aligned} 1_A &= [\exists q(q \wedge atom(q) \wedge \forall p(p \to \Box(q \to p)))] \\ &= \bigvee \Big\{ [\alpha \wedge atom(\alpha)] \wedge \\ &\qquad \bigwedge \{[\rho \to \Box(\alpha \to \rho)] \mid \rho \in \mathcal{L}\Pi\mathrm{Mg}\} \,\Big|\, \alpha \in \mathcal{L}\Pi\mathrm{Mg} \text{ free for } q \Big\}, \end{aligned}$$

using Lemma 6. Then by the analogue of Lemma 8.3 for $\mathsf{S5\Pi}\mathrm{Mg}^+$ again, we have

$$\begin{aligned} 1_{A/\Box^{-1}F} &= \bigvee \Big\{ \pi[\alpha \wedge atom(\alpha)] \wedge \\ &\qquad \bigwedge \{\pi[\rho] \to \Box(\pi[\alpha] \to \pi[\rho]) \mid \rho \in \mathcal{L}\Pi\mathrm{Mg}\} \,\Big|\, \alpha \in \mathcal{L}\Pi\mathrm{Mg} \text{ free for } q \Big\}. \end{aligned}$$

Thus, for some α, we have

$$0_{A/\Box^{-1}F} \neq \pi[\alpha \wedge atom(\alpha)] \wedge \bigwedge\{\pi[\rho] \to \Box(\pi[\alpha] \to \pi[\rho]) \mid \rho \in \mathcal{L}\Pi\mathrm{Mg}\}$$

and hence

$$0_{A/\Box^{-1}F} \neq \pi[\alpha \wedge atom(\alpha)] \wedge (\pi[\beta] \to \Box(\pi[\alpha] \to \pi[\beta])).$$

As in the proof of Lemma 8.4a, $\pi[\alpha \wedge atom(\alpha)] \neq 0$ implies that $\pi[\alpha]$ is an atom. In addition, together $\pi[\beta] \to \Box(\pi[\alpha] \to \pi[\beta]) \neq 0$ and $\pi[\beta] \neq 0$ imply $\Box(\pi[\alpha] \to \pi[\beta]) \neq 0$, which with the simplicity of $A/\Box^{-1}F$ implies $\pi[\alpha] \leq \pi[\beta]$. Thus, we have shown that below any non-zero element of $A/\Box^{-1}F$ there is an atom.

For part 2, by reasoning similar to that in the proof of part 1, if $A/\Box^{-1}F$ contains an atom, then

$$\begin{aligned} & \pi[\exists q(q \wedge \forall p(p \to \Box(q \to p)))] \neq 0_{A/\Box^{-1}F} \\ \Rightarrow \quad & \pi[\neg\exists q(q \wedge \forall p(p \to \Box(q \to p)))] \neq 1_{A/\Box^{-1}F}. \end{aligned}$$

Yet where A is the Lindenbaum algebra of $\mathsf{S5}\Pi\mathrm{Mg}^\star$, we have

$$\begin{aligned} & [\neg\exists q(q \wedge \forall p(p \to \Box(q \to p)))] = 1_A \\ \Rightarrow \quad & \pi[\neg\exists q(q \wedge \forall p(p \to \Box(q \to p)))] = 1_{A/\Box^{-1}F}. \end{aligned}$$

Hence $A/\Box^{-1}F$ is atomless. □

Using Lemma 11 in place of Lemma 8, we can prove the following lemma in just the way we proved Lemma 5.

Lemma 12. For every $\varphi \in \mathcal{L}\Pi$:

1. if φ is not a theorem of $\mathsf{S5}\Pi^+$, then $qf(\varphi)$ is refutable in an *atomic* separable simple S5 algebra according to Definition 6;

2. if φ is not a theorem of $\mathsf{S5}\Pi^\star$, then $qf(\varphi)$ is refutable in an *atomless* separable simple S5 algebra according to Definition 6.

Using Lemma 12 in place of Lemma 5, we can prove Theorem 2.1-2 in just the way we proved Theorem 1, but now adding the fact from Lemma 9.2 that if A is atomic (resp. atomless), then so is its MacNeille completion $\overline{A}$.

2.2 Finite and Infinite Algebras

Finally, we will prove Theorem 2.3, which concerns infinite algebras, and Theorem 3, which concerns finite algebras. Let us begin with the finite case.

As shown by Scroggs [1951], the (quasi-)normal modal logics properly extending $\mathsf{S5}$ are exactly the normal modal logics extending $\mathsf{S5}$ with the axiom

$$P_n := \bigvee_{1 \leq i < j \leq n+1} \Box(p_i \leftrightarrow p_j)$$

for some $n = 2^m$, $m \in \mathbb{N}$. Let $\mathsf{S5}_n$ be the normal extension of $\mathsf{S5}$ with P_n. It is easy to see that a simple S5 algebra validates P_n iff the algebra has at most n elements. Scroggs showed that $\mathsf{S5}_n$ is complete with respect to the simple S5 algebra with n elements.

Let $\mathsf{S5}_n\Pi$ be the least normal Π-logic extending $\mathsf{S5}_n$ (recall Section 1). Using the rule of universal generalization, $\mathsf{S5}_n\Pi$ proves

$$\forall p_1 \dots \forall p_{n+1} P_n.$$

Using this fact, we can show that $\mathsf{S5}_n\Pi$ proves the axiom (W) for atomic algebras, by showing that the negation of (W) implies $\neg\forall p_1 \dots \forall p_{n+1} P_n$, roughly as follows:

$$\begin{aligned}
& \forall q(q \to \exists p(p \wedge \Diamond(q \wedge \neg p))) \quad \text{negation of (W)} && (18)\\
\Rightarrow\ & \exists p_1\, p_1 && (19)\\
\Rightarrow\ & p_1 \to \exists p_2(p_2 \wedge \Diamond(p_1 \wedge \neg p_2)) \quad \text{from (18)} && (20)\\
\Rightarrow\ & \exists p_1 \exists p_2(p_1 \wedge p_2 \wedge \Diamond(p_1 \wedge \neg p_2)) \quad \text{from (19) and (20)} && (21)\\
\Rightarrow\ & (p_1 \wedge p_2) \to \exists p_3(p_3 \wedge \Diamond(p_1 \wedge p_2 \wedge \neg p_3)) \quad \text{from (18)} && (22)\\
\Rightarrow\ & \exists p_1 \exists p_2 \exists p_3(p_1 \wedge p_2 \wedge p_3 \wedge \Diamond(p_1 \wedge \neg p_2) \wedge \Diamond(p_1 \wedge p_2 \wedge \neg p_3)) \quad \text{from (21) and (22)} &&\\
& \vdots &&\\
\Rightarrow\ & \exists p_1 \dots \exists p_{n+1}(\Diamond(p_1 \wedge \neg p_2) \wedge \Diamond(p_1 \wedge p_2 \wedge \neg p_3) \wedge \dots \wedge \Diamond(p_1 \wedge \dots \wedge p_n \wedge \neg p_{n+1})), &&
\end{aligned}$$

and the last formula implies $\neg\forall p_1 \dots \forall p_{n+1} P_n$. Thus, (W) is indeed a theorem of $\mathsf{S5}_n\Pi$, so there is no distinction to be made between $\mathsf{S5}_n\Pi$ and $\mathsf{S5}_n\Pi^+$.

With this background, let us now prove Theorem 3 using an argument very similar to that in the proof of Lemma 5, which appeared after the proof of Lemma 8.

Proof. First, exact analogues of Lemma 6 and Lemma 8.1-3 hold for $\mathsf{S5}_n\Pi$ in place of $\mathsf{S5}\Pi\mathrm{Mg}$, with Q now defined in the Lindenbaum algebra A of $\mathsf{S5}_n\Pi$. Suppose that ξ is not a theorem of $\mathsf{S5}_n\Pi$, so that $\Box\xi$ is also not a theorem, so that $[\Box\xi]$ is not the top element of A. Hence by Lemma 7 there is a Q-filter F in A that does not contain $[\Box\xi] = \Box[\xi]$, which means $\Box^{-1}F$ does not contain ξ, so that in the quotient $A/\Box^{-1}F$ from Lemma 8, we have $\pi[\xi] \neq 1$. Since $A/\Box^{-1}F$ is a homomorphic image of A, it validates P_n, and then since $A/\Box^{-1}F$ is a simple S5 algebra, it has at most n elements. Thus, it is finite and hence complete, so we can evaluate propositional quantifiers in $A/\Box^{-1}F$. Defining a valuation θ on $A/\Box^{-1}F$ by $\theta(p) = \pi[p]$, it follows by (the analogues of) Lemma 6 and Lemma 8.3 that for all formulas $\chi \in \mathcal{L}\Pi$, we have $\tilde{\theta}(\chi) = \pi[\chi]$, so that $\pi[\xi] \neq 1$ implies $\tilde{\theta}(\xi) \neq 1$. □

Remark 1. The form of argument in the proof just given applies beyond extensions of $\mathsf{S5}$. For example, let L be any normal extension of $\mathsf{S4}$ that is tabular, i.e., the logic of a single finite algebra, and let $\mathsf{L}\Pi\mathsf{B}$ be the least normal Π-logic that extends L and contains all instances of the Barcan axiom $\forall p\Box\varphi \to \Box\forall p\varphi$, the converse of which is already provable in $\mathsf{L}\Pi$. Lemma 8.1 still holds for $\mathsf{L}\Pi\mathsf{B}$ because L extends $\mathsf{S4}$. In Lemma 8.2, we replace *simple S5* with *subdirectly irreducible* (see, e.g., Bezhanishvili and Bezhanishvili 2011, Lemma 4.1). For Lemma 8.3, complete multiplicativity of $\Box$ is not required. The weaker Barcan equivalences

$\forall p\Box\varphi \leftrightarrow \Box\forall p\varphi$ suffice for the proof. Thus, $A/\Box^{-1}F$ is a subdirectly irreducible algebra for L, which with the tabularity of L implies that $A/\Box^{-1}F$ is finite (see Burris and Sankappanavar 1981, p. 149, Corollary 6.10). We can therefore evaluate propositional quantifiers in $A/\Box^{-1}F$, and the Barcan equivalences are valid in $A/\Box^{-1}F$ due to its finiteness.

The proof of Theorem 3 above shows that $\mathsf{S5}_n\Pi$ is complete with respect to the class of simple S5 algebras with at most n elements. Scroggs showed that $\mathsf{S5}_n$ is complete with respect to the simple S5 algebra with exactly n elements, using the fact that any simple S5 algebra with fewer than n elements is a subalgebra of the one with n elements. But while the validity of modal formulas without propositional quantifiers is preserved by subalgebras, this is not guaranteed for formulas with propositional quantifiers, since by taking a subalgebra we are shrinking the domain of propositional quantification (cf. Section 1). For example, the formula

$$E_n := \forall p_1 \dots \forall p_{n+1} P_n \wedge \neg\forall p_1 \dots \forall p_n P_{n-1}$$

is valid in a simple S5 algebra iff the algebra has exactly n elements.

For any finite set $X \subseteq \mathbb{N}$ of powers of 2, the extension $\mathsf{S5}\Pi_X$ of $\mathsf{S5}\Pi$ with the axiom $\bigvee_{n\in X} E_n$ can be proved complete with respect to the class of simple S5 algebras A with $|A| \in X$, using the same form of argument as in the proof of Theorem 3 above. In particular, $\mathsf{S5}\Pi_n := \mathsf{S5}\Pi_{\{n\}}$ is complete with respect to the simple S5 algebra with n elements.[12]

Let us now return to Theorem 2.3 for infinite algebras. We may define the logic $\mathsf{S5}\Pi_\infty$ (resp. $\mathsf{S5}\Pi^+_\infty$) as the extension of $\mathsf{S5}\Pi$ (resp. $\mathsf{S5}\Pi^+$) with the axiom $\neg E_{2^m}$ for every $m \in \mathbb{N}$. All of these axioms are valid in a simple S5 algebra iff the algebra is infinite (see Kripke 1959, p. 12 for a different axiom schema that forces the algebra to be infinite). Since atomless algebras are infinite, $\mathsf{S5}\Pi^\star$ extends $\mathsf{S5}\Pi_\infty$. We saw above how to use an axiom of $\mathsf{S5}\Pi^\star$, namely the negation of (W), to derive $\neg\forall p_1 \dots \forall p_{n+1} P_n$ and hence $\neg E_n$.

To prove Theorem 2.3, we need only observe that the analogue of Lemma 8 (resp. Lemma 11.1) stating that the quotient $A/\Box^{-1}F$ is infinite holds for $\mathsf{S5}\Pi\mathrm{Mg}_\infty$ (resp. $\mathsf{S5}\Pi\mathrm{Mg}^+_\infty$). Thus, the MacNeille completion of $A/\Box^{-1}F$ is also infinite, so the proof of Theorem 2.3 can follow the same strategy as the proofs of Theorems 1 and 2.1.

Remark 2. While Scroggs showed that the (quasi-)normal modal logics extending $\mathsf{S5}$ are linearly ordered, the foregoing discussion shows that the structure of the normal Π-logics extending $\mathsf{S5}\Pi$ is richer. These extensions may be classified with the help of Fine's quantifier elimination in the Appendix. As a special case, consider those extensions that are *negation-complete* in the sense that for each closed formula φ, i.e., in which all propositional variables are bound by quantifiers, either φ or $\neg\varphi$ is a theorem of the logic. This implies that the logic is not properly extended by any consistent Π-logic. Fine [1970, p. 342] observed that the logics $\mathsf{S5}\Pi^\star$ and $\mathsf{S5}\Pi^+_\infty$ are negation-complete.[13] Fine's quantifier elimination in the Appendix can also be used to show that for any closed formula φ and finite simple S5 algebra A, either φ is valid in A or $\neg\varphi$ is valid in A; thus, by the completeness result for

[12] Note the distinction between $\mathsf{S5}_n\Pi$ and $\mathsf{S5}\Pi_n$: the former is valid in any simple S5 algebras with at most n elements, while the latter is valid only in the simple S5 algebra with exactly n elements.

[13] Cf. the classic results that the first-order theories of atomless BAs and of infinite atomic BAs, respectively, are negation-complete (see, e.g., Poizat 2000, pp. 83-84).

$\mathsf{S5\Pi}_n$ above, each $\mathsf{S5\Pi}_n$ is also negation-complete. It follows that every consistent negation-complete Π-logic L extending $\mathsf{S5\Pi}$ is either $\mathsf{S5\Pi}^\star$, $\mathsf{S5\Pi}^+_\infty$, or $\mathsf{S5\Pi}_{2^m}$ for some $m \in \mathbb{N}$.[14] For either L contains (W) or its negation. In the second case, $\mathsf{L} \supseteq \mathsf{S5\Pi}^\star$, which implies $\mathsf{L} = \mathsf{S5\Pi}^\star$ since $\mathsf{S5\Pi}^\star$ is negation-complete. In the first case, $\mathsf{L} \supseteq \mathsf{S5\Pi}^+$. Then for each $m \in \mathbb{N}$, either L contains E_{2^m} or $\neg E_{2^m}$. If for some m, L contains E_{2^m}, then $\mathsf{L} \supseteq \mathsf{S5\Pi}_{2^m}$, which implies $\mathsf{L} = \mathsf{S5\Pi}_{2^m}$ since $\mathsf{S5\Pi}_{2^m}$ is negation-complete. If for all m, L contains $\neg E_{2^m}$, then $\mathsf{L} \supseteq \mathsf{S5\Pi}^+_\infty$, which implies $\mathsf{L} = \mathsf{S5\Pi}^+_\infty$ since $\mathsf{S5\Pi}^+_\infty$ is negation-complete.

3 Conclusion

Having proved Theorems 1-3, let us return to Question 1. Some specific cases of the question include: Is the Π-logic of all complete modal algebras recursively axiomatizable? Is the Π-logic of all complete S4 algebras (interior algebras) recursively axiomatizable? As shown by Harding and Bezhanishvili [2007], a number of standard varieties of modal algebras, including the varieties of all modal algebras and of S4 algebras, are closed under (upper) MacNeille completion. But without a quantifier elimination argument, the obstacle described in Section 1 for completion returns. The known negative results are cautionary: for modal algebras or S4 algebras that are not only complete but also atomic and with a completely multiplicative $\Box$, there can be no recursive axiomatizations. This follows from results of Fine [1970, Proposition 7] for Kripke frames and the well-known duality between Kripke frames and complete and atomic modal algebras with a completely multiplicative $\Box$ [Thomason, 1975]. Moreover, by a result of Kremer [1997b], there can be no recursive axiomatization of the Π-logic of all S4 algebras that are complete and atomic. What happens if we require only that the algebras be complete is an open question.

Acknowledgements. I thank Guram Bezhanishvili, Yifeng Ding, and Peter Fritz for helpful discussion, as well as an anonymous reviewer for helpful suggestions.

References

G. Aldo Antonelli and Richmond H. Thomason. Representability in second-order propositional poly-modal logic. *The Journal of Symbolic Logic*, 67(3):1039–1054, 2002.

Ruth C. Barcan. The identity of individuals in a strict functional calculus of second order. *The Journal of Symbolic Logic*, 12(1):12–15, 1947.

Francesco Belardinelli and Wiebe van der Hoek. Epistemic quantified Boolean logic: expressiveness and completeness results. In *Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence (IJCAI 2015)*, pages 2748–2754, 2015.

Francesco Belardinelli and Wiebe van der Hoek. A semantical analysis of second-order propositional modal logic. In *Proceedings of the Thirtieth AAAI Conference on Artificial Intelligence (AAA-16)*, pages 886–892, 2016.

[14]Thanks to Peter Fritz (personal communication) for this conjecture.

Guram Bezhanishvili and Nick Bezhanishvili. An algebraic approach to canonical formulas: Modal case. *Studia Logica*, 99(1/3):93–125, 2011.

R. A. Bull. On modal logic with propositional quantifiers. *The Journal of Symbolic Logic*, 34(2):257–263, 1969.

S. N. Burris and H. P. Sankappanavar. *A Course in Universal Algebra*. Springer-Verlag, New York, 1981. URL `https://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html`.

Balder ten Cate. Expressivity of second order propositional modal logic. *Journal of Philosophical Logic*, 35(2):209–223, 2006.

Brian F. Chellas. *Modal logic: an introduction*. Cambridge University Press, New York, 1980.

J. Michael Dunn and Gary M. Hardegree. *Algebraic Methods in Philosophical Logic*. Clarendon Press, Oxford, 2001.

Kit Fine. *For Some Proposition and So Many Possible Worlds*. PhD thesis, University of Warwick, 1969. URL `http://wrap.warwick.ac.uk/72219/`.

Kit Fine. Propositional quantifiers in modal logic. *Theoria*, 36(3):336–346, 1970.

Peter Fritz. Propositional quantification in bimodal **S5**. Manuscript, 2017.

Dov M. Gabbay. Montague type semantics for modal logics with propositional quantifiers. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 17:245–249, 1971.

Dov M. Gabbay. On 2nd order intuitionistic propositional calculus with full comprehension. *Archiv für mathematische Logik und Grundlagenforschung*, 16:177–186, 1974.

Herman Geuvers. Conservativity between logics and typed λ calculi. In H. Barendregt and T. Nipkow, editors, *Types for Proofs and Programs*, volume 806 of *Lecture Notes in Computer Science*, pages 131–154. Springer-Verlag, Berlin, 1994.

Steven Givant and Paul Halmos. *Introduction to Boolean Algebras*. Springer, New York, 2009.

Paul Halmos and Steven Givant. *Logic as Algebra*. The Mathematical Association of America, 1998.

Paul R. Halmos. Algebraic logic, I. Monadic Boolean algebras. *Compositio Mathematica*, 12:217–249, 1955.

Paul R. Halmos. *Algebraic Logic*. Chelsea Publishing Company, New York, 1962.

Paul R. Halmos. *Lectures on Boolean Algebras*. D. Van Nostrand, Princeton, 1963.

John Harding and Guram Bezhanishvili. MacNeille completions of modal algebras. *Houston Journal of Mathematics*, 33(2):355–384, 2007.

Michael Kaminski and Michael Tiomkin. The expressive power of second-order propositional modal logic. *Notre Dame Journal of Formal Logic*, 37(1):35–43, 1996.

David Kaplan. S5 with quantifiable propositional variables. *The Journal of Symbolic Logic*, 35(2):355, 1970.

Philip Kremer. Quantifying over propositions in relevance logic: nonaxiomatizability of primary interpretations of $\forall p$ and $\exists p$. *The Journal of Symbolic Logic*, 58(1):334–349, 1993.

Philip Kremer. On the complexity of propositional quantification in intuitionistic logic. *The Journal of Symbolic Logic*, 62(2):529–544, 1997a.

Philip Kremer. Propositional quantification in the topological semantics for **S4**. *Notre Dame Journal of Formal Logic*, 38(2):295–313, 1997b.

Saul A. Kripke. A completeness theorem in modal logic. *The Journal of Symbolic Logic*, 24 (1):1–14, 1959.

Antti Kuusisto. Second-order propositional modal logic and monadic alternation hierarchies. *Annals of Pure and Applied Logic*, 166:1–28, 2015.

Bruno Poizat. *A Course in Model Theory: An Introduction to Contemporary Mathematical Logic*. Springer, New York, 2000.

Tomasz Połacik. Pitts' quantifiers are not topological quantification. *Notre Dame Journal of Formal Logic*, 39(4):531–545, 1998.

H. Rasiowa and R. Sikorski. A proof of the completeness theorem of Gödel. *Fundamenta Mathematicae*, 37(1):193–200, 1950.

H. Rasiowa and R. Sikorski. *The Mathematics of Metamathematics*. Monografie Matematyczne, Tom 41. Państwowe Wydawnictwo Naukowe, Warsaw, 1963.

Dana S. Scott. The algebraic interpretation of quantifiers: Intuitionistic and classical. In A. Ehrenfeucht, V. W. Marek, and M. Srebrny, editors, *Andrzej Mostowski and Foundational Studies*, pages 289–312. IOS Press, Amsterdam, 2008.

Schiller Joe Scroggs. Extensions of the Lewis system S5. *The Journal of Symbolic Logic*, 16 (2):112–120, 1951.

D. Skvortsov. Non-axiomatizable second order intuitionistic propositional logic. *Annals of Pure and Applied Logic*, 86:33–46, 1997.

S. K. Sobolev. The intuitionistic propositional calculus with quantifiers. *Mathematical Notes of the Academy of Sciences of the USSR*, 22:528–532, 1977.

Yoshihito Tanaka. Kripke completeness of infinitary predicate multimodal logic. *Notre Dame Journal of Formal Logic*, 40(3):326–341, 1999.

S. K. Thomason. Categories of frames for modal logic. *The Journal of Symbolic Logic*, 40 (3):439–442, 1975.

Richard Zach. Decidability of quantified propositional intuitionistic logic and S4 on trees of height and arity $\leq \omega$. *Journal of Philosophical Logic*, 33(2):155–164, 2004.

Konrad Zdanowski. On second order intuitionistic propositional logic without a universal quantifier. *The Journal of Symbolic Logic*, 74(1):157–167, 2009.

Appendix

Below we outline the proof of Lemma 3 from Fine 1969, with thanks to Yifeng Ding for a helpful presentation of this quantifier elimination (personal communication).

As abbreviations, define for $\varphi \in \mathcal{L}\Pi\mathrm{Mg}$:

$$\begin{aligned} \mathrm{Q}_0\varphi &:= \neg \mathrm{M}_1\varphi; \\ \mathrm{Q}_i\varphi &:= \mathrm{M}_i\varphi \wedge \neg \mathrm{M}_{i+1}\varphi \quad \text{for } i \geq 1; \\ \mathrm{N}\varphi &:= \Diamond(\varphi \wedge \neg \mathrm{g}). \end{aligned}$$

Definition 9. Let P be a finite set of propositional variables that includes the constant g. A *state description* s over P is a conjunction, each conjunct of which is an element of P or its negation, with each element of P occurring in s. Let 2^P be the set of all state descriptions over P. A *model description of degree* n *over* P is a conjunction of:

1. a state description $a \in 2^P$;
2. for each $s \in 2^P$, either $\mathrm{M}_n s$ or $\mathrm{Q}_i s$ for some $i < n$;
3. for each $s \in 2^P$, either $\mathrm{N}s$ or $\neg\mathrm{N}s$.

Recall that the *modal depth* of a formula in the basic propositional modal language is defined by $md(p) = 0$, $md(\neg\varphi) = md(\varphi)$, $md(\varphi \wedge \psi) = \max\{md(\varphi), md(\psi)\}$, and $md(\Box\varphi) = md(\varphi) + 1$. As is well known, in $\mathsf{S5}$ (indeed, in $\mathsf{KD45}$), every modal formula is provably equivalent to a disjunction of conjunctions, each conjunct of which is either a literal (propositional variable or negation thereof) or $\Diamond\alpha$ or $\neg\Diamond\alpha$ for a conjunction α of literals. For the extended language with M and g, in which $\Diamond$ can be defined by $\Diamond\varphi \leftrightarrow (\mathrm{M}_1\varphi \vee \mathrm{N}\varphi)$, the analogous result is the following result from Fine 1969, Theorem 2, p. 49.

Lemma 13. Every quantifier-free formula of degree n is provably equivalent in $\mathsf{S5}\Pi\mathrm{Mg}$ to $\bot$ or to a disjunction of model descriptions of degree n.

Since the existential quantifier $\exists p$ distributes over disjunction, it only remains to show how to eliminate $\exists p$ from formulas of the form $\exists p\varphi$ where φ is a model description. This is facilitated by the following lemma (cf. Fine 1969, Lemma 5, p. 33).

Lemma 14. Where φ is a model description over P with $q \in P$, $\mathsf{S5}\Pi\mathrm{Mg}$ proves

$$\exists q\varphi \leftrightarrow \bigwedge_{s \in 2^{P \setminus \{q\}}} \exists q\varphi_s, \tag{23}$$

where φ_s is the conjunction of those conjuncts of φ as in Definition 9 that contain the state description $s \wedge q$ or the state description $s \wedge \neg q$.

The φ_s formulas can then be simplified, depending on whether s contains g or $\neg$g.

Lemma 15. If g is a conjunct of s, then φ_s in Lemma 14 is provably equivalent in $\mathsf{S5\Pi Mg}$ to $\bot$ or a formula of one of the following forms, where $[s \wedge \pm q]$ may be empty, and $\pm$ is $\neg$ (in which case $j \geq 1$) or empty (in which case $i \geq 1$):

1. $[s \wedge \pm q] \wedge \mathrm{Q}_i(s \wedge q) \wedge \mathrm{Q}_j(s \wedge \neg q)$;
2. $[s \wedge \pm q] \wedge \mathrm{Q}_i(s \wedge q) \wedge \mathrm{M}_n(s \wedge \neg q)$;
3. $[s \wedge \pm q] \wedge \mathrm{M}_n(s \wedge q) \wedge \mathrm{Q}_j(s \wedge \neg q)$;
4. $[s \wedge \pm q] \wedge \mathrm{M}_n(s \wedge q) \wedge \mathrm{M}_m(s \wedge \neg q)$.

If $\neg$g is a conjunct of s, then φ_s is provably equivalent to $\bot$ or a formula of the form:

5. $[s \wedge \pm q] \wedge \pm\mathrm{N}(s \wedge q) \wedge \pm\mathrm{N}(s \wedge \neg q)$.

Proof. If g is a conjunct of s, eliminate the N conjuncts from φ_s using $\mathrm{N}(\mathrm{g} \wedge \psi) \leftrightarrow \bot$.

If $\neg$g is a conjunct of s, eliminate the M_n conjuncts from φ_s using $\mathrm{M}_n(\neg\mathrm{g} \wedge \psi) \leftrightarrow \bot$; eliminate the Q_i conjuncts for $i \geq 1$ using $\mathrm{Q}_i(\neg\mathrm{g} \wedge \psi) \leftrightarrow \bot$; and eliminate the Q_0 conjuncts using $\mathrm{Q}_0(\neg\mathrm{g} \wedge \psi) \leftrightarrow \top$. □

Now it only remains to show how to eliminate $\exists q$ prefixed to a formula of one of the forms given by 1–5 in Lemma 15. This is done as follows [Fine, 1969, Lemmas 4–5, pp. 51–52].

Lemma 16. For any propositional formula ψ and $k, \ell \in \mathbb{N}$, the following equivalences are provable in $\mathsf{S5\Pi Mg}$ (where $j \geq 1$ when $\pm$ is $\neg$, and $i \geq 1$ when $\pm$ is empty):

$$\begin{aligned}
\exists q\big(\mathrm{g} \wedge (\psi \wedge \pm q) \wedge \mathrm{Q}_i(\psi \wedge q) \wedge \mathrm{Q}_j(\psi \wedge \neg q)\big) &\quad\leftrightarrow\quad (\mathrm{g} \wedge \psi \wedge \mathrm{Q}_{i+j}\psi);\\
\exists q\big(\mathrm{g} \wedge (\psi \wedge \pm q) \wedge \mathrm{Q}_i(\psi \wedge q) \wedge \mathrm{M}_\ell(\psi \wedge \neg q)\big) &\quad\leftrightarrow\quad (\mathrm{g} \wedge \psi \wedge \mathrm{M}_{i+\ell}\psi);\\
\exists q\big(\mathrm{g} \wedge (\psi \wedge \pm q) \wedge \mathrm{M}_k(\psi \wedge q) \wedge \mathrm{Q}_j(\psi \wedge \neg q)\big) &\quad\leftrightarrow\quad (\mathrm{g} \wedge \psi \wedge \mathrm{M}_{k+j}\psi);\\
\exists q\big(\mathrm{g} \wedge (\psi \wedge \pm q) \wedge \mathrm{M}_k(\psi \wedge q) \wedge \mathrm{M}_\ell(\psi \wedge \neg q)\big) &\quad\leftrightarrow\quad (\mathrm{g} \wedge \psi \wedge \mathrm{M}_{k+\ell}\psi);\\
\exists q\big(\neg\mathrm{g} \wedge (\psi \wedge \pm q) \wedge \mathrm{N}(\psi \wedge q) \wedge \mathrm{N}(\psi \wedge \neg q)\big) &\quad\leftrightarrow\quad (\neg\mathrm{g} \wedge \psi \wedge \mathrm{N}\psi);\\
\exists q\big(\neg\mathrm{g} \wedge (\psi \wedge \pm q) \wedge \mathrm{N}(\psi \wedge q) \wedge \neg\mathrm{N}(\psi \wedge \neg q)\big) &\quad\leftrightarrow\quad (\neg\mathrm{g} \wedge \psi \wedge \mathrm{N}\psi);\\
\exists q\big(\neg\mathrm{g} \wedge (\psi \wedge \pm q) \wedge \neg\mathrm{N}(\psi \wedge q) \wedge \neg\mathrm{N}(\psi \wedge \neg q)\big) &\quad\leftrightarrow\quad \bot.
\end{aligned}$$