

A NOTE ON AN INEQUALITY FOR THE GAMMA FUNCTION

CHRISTOPHER OLUTUNDE IMORU

Department of Mathematics
University of Ife
Ile-Ife, Oyo State, Nigeria

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ABSTRACT. Some inequalities for the Wallis functions are proved. The results of this paper are consequences of some characterization of convex functions. A generalization of a result of Boyd (1) and an extension of an inequality of Gantschi (3) are obtained.

KEY WORDS AND PHRASES. Gamma functions, characterization of convex functions, Inequalities for Gamma functions.

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The aim of this note is to show that some inequalities for the Wallis function

$$W(\xi, \theta) = \frac{\Gamma(\xi + 1)}{\Gamma(\xi + \theta)}, \quad (\xi, \theta) \in \mathbb{R}_+ \times (0, 1), \quad (1)$$

are natural consequences of the property of convex functions or of differentiable functions. Indeed, our results are, to some extent, consequences

of the following characterization of convex functions.

THEOREM 1. A real-valued function ϕ is convex on a closed interval $\bar{I} \subseteq \mathbb{R}$ if and only if for every point $x_0 \in \bar{I}$, the function

$$x \longrightarrow \frac{\phi(x) - \phi(x_0)}{x - x_0}, \quad x \in \bar{I}, \quad (2)$$

is non-decreasing on \bar{I} . In particular, if ϕ is convex on \bar{I} , $u \neq v$, $x \neq y$, $u \leq x$, $v \leq y$, for all $u, v, x, y \in \bar{I}$, then

$$\frac{\phi(v) - \phi(u)}{v - u} \leq \frac{\phi(y) - \phi(x)}{y - x}. \quad (3)$$

The proof of the theorem is well known; see for example, ([3], pp. 15-18). It is, therefore, omitted.

THEOREM 2. Let u, v, x, y, w and z be positive real-numbers satisfying $u \neq v$, $w \neq z$, $u \leq x \leq w$, $x < y \leq z$ and $v \leq y$.

Then the following inequality is valid

$$\left[\frac{\Gamma(v)}{\Gamma(u)} \right]^{\frac{y-x}{v-u}} \leq \frac{\Gamma(y)}{\Gamma(x)} \leq \left[\frac{\Gamma(z)}{\Gamma(w)} \right]^{\frac{y-x}{z-w}} \quad (4)$$

PROOF. Since the function $\eta \rightarrow \log \Gamma(\eta)$, $\eta \in \mathbb{R}_+$, is convex, it follows from inequality (3) that

$$\frac{\log \Gamma(v) - \log \Gamma(u)}{v - u} \leq \frac{\log \Gamma(y) - \log \Gamma(x)}{y - x} \leq \frac{\log \Gamma(z) - \log \Gamma(w)}{z - w}, \quad (5)$$

provided u, v, x, y, w and z satisfy the hypothesis of the theorem. Since inequality (5) is equivalent to inequality (4), the proof of the theorem is complete.

COROLLARY 1. For $(\xi, \theta) \in \mathbb{R}_+ \times [0, 1]$, we have

$$(m + \xi)^{1-\theta} \leq \frac{\Gamma(m + \xi + 1)}{\Gamma(m + \xi + \theta)} \leq (m + \xi + \theta)^{1-\theta}, \quad m \in \mathbb{Z}. \quad (6)$$

PROOF. Set $u = m + \xi$, $v = m + \xi + 1$, $x = m + \xi + \theta$, $y = m + \xi + 1$, $w = m + \xi + \theta$ and $z = m + \xi + 1 + \theta$.

Then inequalities (5) reduce to inequalities (6).

The case $\xi = 0$ and $0 < \theta < 1$ is due to Gautschi ([3], § 3. 6. 51).

Inequalities (6) in the form

$$\frac{1}{(m + \xi + \theta)^{1-\theta}} < \frac{\Gamma(m + \xi + \theta)}{\Gamma(m + \xi + 1)} < \frac{1}{(m + \xi)^{1-\theta}},$$

were obtained by Lazarević and Lupas [2] who made use of the fact that the Gamma function is logarithmic convex and an unpublished result of Lupas on inequalities involving the Gamma function.

We now prove a more general result which contains, as a special case, an improved version of Boyd's result [1], namely,

$$\left\{m + \frac{1}{4} + \frac{1}{32m + 32}\right\}^{\frac{1}{2}} < \frac{\Gamma(m + 1)}{\Gamma(m + \frac{1}{2})} < \left\{\frac{(m + \frac{1}{2})^2}{m + \frac{3}{4} + \frac{1}{32m + 32}}\right\}^{\frac{1}{2}}. \tag{7}$$

We first obtain the following results on differentiable functions:

THEOREM 3. Let ϕ_1 and ϕ_2 be two differentiable real-valued functions on an open interval S in \mathbb{R} . Let $x, y, u, v \in S$, $x \neq y$, $u \neq v$. Then there exists $\eta \in (0, 1)$ such that for every positive real number α ,

$$\frac{\phi_1(y) - \phi_1(x)}{y - x} = \frac{\phi_2(v) - \phi_2(u)}{v - u} + \alpha \eta^{\alpha-1} [\phi_1'(x + \eta^\alpha(y - x)) - \phi_2'(u + \eta^\alpha(v - u))]. \tag{8}$$

PROOF. Consider the function

$$F(\lambda) = \frac{v - u}{\alpha} \phi_1(x + \lambda^\alpha(y - x)) - \frac{y - x}{\alpha} \phi_2(u + \lambda^\alpha(v - u)).$$

This function is differentiable on $[0, 1]$. By the usual Mean Value Theorem for differentiable functions, we obtain the desired conclusion.

THEOREM 4. Let ϕ be a differentiable real-valued function on an open interval S in \mathbb{R} and let ϕ' be non-decreasing on S .

Suppose $u, v, x, y \in S$, $u \neq v$, $x \neq y$ and either $x > u, v > y$ or $x < u, v < y$. Then, for some $\alpha_0 \in \mathbb{Z}_+$ (the set of positive integers) such that

$$(1 - \eta^\alpha)(x - u) + \eta^\alpha(y - v) \geq 0, \quad 0 < \eta < 1, \quad \alpha \geq \alpha_0, \quad (9)$$

we have

$$\frac{\phi(y) - \phi(x)}{y - x} \geq \frac{\phi(v) - \phi(u)}{v - u}. \quad (10)$$

We note, however, that inequality (10) is valid if $x \geq u, y \geq v$ and α is an arbitrary positive real number.

PROOF. Let $\phi_1 = \phi_2 = \phi$ in Theorem 3. The assumptions on x, y, u and v imply that $\frac{x - u}{x - u + v - y}$ is an arbitrary real number between 0 and 1.

Suppose $0 < \eta < \frac{x - u}{x - u + v - y} < 1$. Then, for all $\alpha \in \mathbb{Z}_+$, $\eta^\alpha < \frac{x - u}{x - u + v - y}$. If, however, $0 < \frac{x - u}{x - u + v - y} < \eta < 1$, there exists $\alpha_0 \in \mathbb{Z}_+$ such that for all $\alpha \geq \alpha_0, \alpha \in \mathbb{Z}_+$, $\eta^\alpha \leq \frac{x - u}{x - u + v - y}$. Hence, in either case, $(1 - \eta^\alpha)(x - u) + \eta^\alpha(y - v) \geq 0$, for all $\alpha \in \mathbb{Z}_+, \alpha \geq \alpha_0$. The conclusion follows by Theorem 3 and the non-decreasing character of ϕ' .

We remark on passing, that inequality (10) is strict unless ϕ is a constant or linear function. Furthermore, inequality (10) is reversed if ϕ is non-increasing.

COROLLARY 2. Let ϕ be a twice differentiable real-valued convex function on an open interval S in \mathbb{R} . Let x, y, u and v satisfy the conditions of Theorem 4. Then inequality (10) holds if inequality (9) is valid. The inequality is reversed if ϕ is concave.

PROOF. Since ϕ is convex on S , ϕ'' is non-negative on S . Hence ϕ' is non-decreasing on S . If, however, ϕ is concave, ϕ' is non-increasing on S . Consequently, the conclusion of the corollary follows from Theorem 4.

An immediate consequence of the above corollary can be obtained by specializing ϕ . For example, if we take $\phi(\alpha)$, $\alpha \in R_+$, as $\log \Gamma(\alpha)$, then this function satisfies the condition of Corollary 2. Consequently, if inequality (9) holds and x, y, u, v satisfy the conditions of Theorem 4, we have

$$\frac{\Gamma(y)}{\Gamma(x)} \geq \left\{ \frac{\Gamma(v)}{\Gamma(u)} \right\}^{\frac{y-x}{v-u}}. \tag{11}$$

For $m \geq -\frac{1}{2}$, let $\gamma \in R - \{0\}$ be such that $\eta = \frac{m}{\gamma}$, $0 < \eta < 1$. Put $x = m + \frac{1}{2}$, $y = m + 1$, $u = m + \theta(m)$ and $v = m + 1 + \theta(m)$ where $\frac{1}{4} \leq \theta(m) < \frac{1}{2}$.

Since $x - u > 0$, $y - v < 0$ and $\frac{1}{4} \leq \theta(m) < \frac{1}{2}$, inequality (11) holds if and only if for some positive integer α , $\frac{1 - \eta^\alpha}{\eta^\alpha} \geq \frac{v - y}{x - u} \geq 1$.

Hence

$$(m + \theta(m))^{\frac{1}{2}} \leq \frac{\Gamma(m + 1)}{\Gamma(m + \frac{1}{2})} \quad \text{if} \quad \frac{1}{4} \leq \theta(m) \leq \frac{1}{2} [1 - (\frac{m}{\gamma})^\alpha], \quad 0 < (\frac{m}{\gamma})^\alpha \leq \frac{1}{2}.$$

Letting $\alpha \rightarrow \infty$, we get

$$(m + \theta(m))^{\frac{1}{2}} \leq \frac{\Gamma(m + 1)}{\Gamma(m + \frac{1}{2})} \quad \text{if} \quad \frac{1}{4} \leq \theta(m) \leq \frac{1}{2}. \tag{12}$$

Now write $v = m + 1$, $u = m + \frac{1}{2}$, $y = m + 1 + \theta(m)$ and $x = m + \theta(m)$. Then $x - u < 0$ and $v - y < 0$. Consequently, inequality (11) holds if and only if $\frac{1 - \eta^\alpha}{\eta^\alpha} \leq 1 \leq \frac{v - y}{x - u}$. Equivalently,

$$(m + \theta(m))^{\frac{1}{2}} \geq \frac{\Gamma(m + 1)}{\Gamma(m + \frac{1}{2})}, \tag{13}$$

provided

$$\frac{1}{4} \leq \theta(m) \leq \frac{1}{2} [1 - (\frac{m}{\gamma})^\alpha], \quad \frac{1}{2} \leq (\frac{m}{\gamma})^\alpha < 1;$$

a condition which reduces to $\theta(m) = \frac{1}{4}$.

Combining inequalities (12) and (13), we obtain

$$(m + \theta(m))^{\frac{1}{2}} \leq \frac{\Gamma(m+1)}{\Gamma(m + \frac{1}{2})}, \quad \frac{1}{4} \leq \theta(m) \leq \frac{1}{2}. \quad (14)$$

The converse of this result was obtained by Watson [4], namely, if

$$\frac{\Gamma(m+1)}{\Gamma(m + \frac{1}{2})} = (m + \theta(m))^{\frac{1}{2}}, \quad \text{then } \frac{1}{4} \leq \theta(m) \leq \frac{1}{2} \quad \text{for } m \geq -\frac{1}{2} \text{ and}$$

$$\frac{1}{4} \leq \theta(m) \leq \frac{1}{\pi} \quad \text{for } m \geq 0.$$

For $m \geq -\frac{1}{2}$, $\frac{1}{4} < \theta(m) \leq \frac{1}{2}$, we obtain

$$\frac{\Gamma(m+1)}{\Gamma(m + \frac{1}{2})} = \frac{m + \frac{1}{2}}{\{\frac{\Gamma(m + \frac{1}{2})}{\Gamma(m+1)}\}} < \left\{ \frac{(m + \frac{1}{2})^2}{m + \frac{1}{2} + \theta(m + \frac{1}{2})} \right\}^{\frac{1}{2}},$$

Hence, this inequality and inequality (14) combined yield

$$\{m + \theta(m)\}^{\frac{1}{2}} < \frac{\Gamma(m+1)}{\Gamma(m + \frac{1}{2})} < \left\{ \frac{(m + \frac{1}{2})^2}{m + \frac{1}{2} + \theta(m + \frac{1}{2})} \right\}^{\frac{1}{2}}, \quad (15)$$

where $\frac{1}{4} < \theta(m) \leq \frac{1}{2}$.

Taking $\theta(m) = \frac{1}{4} + \frac{1}{32m + 32}$, $m = 1, 2, \dots$, we obtain inequality (7).

On putting $\theta(m) = \frac{1}{4} + \frac{1}{32m + 8 + \frac{36}{4m - 3}}$, we obtain an inequality due to

Slavić ([5], inequality (12)).

A result which is better than any one known, except for the formula (15) of Slavić's paper [5] is obtained by putting

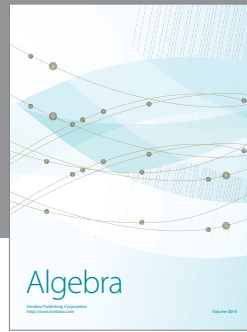
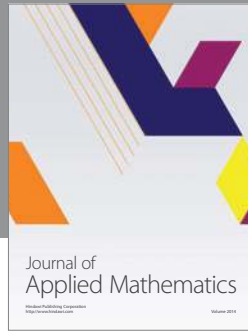
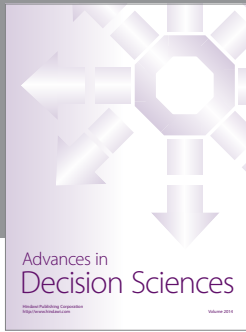
$$\theta(m) = \frac{1}{4} + \frac{1}{32m + 8 + \frac{36}{4m + 5}}.$$

It is our conjecture that formula (15) of Slavić's paper [5] can be obtained from our general result, namely inequality (15), by appropriate choice of

$$\theta = [-\frac{1}{2}, \infty] + [\frac{1}{4}, \frac{1}{2}].$$

REFERENCES

1. Boyd, A. V. Note on the Paper of Uppuluri, Pacific J. Math. 22 (1967) 9-10.
2. Lazarević, I. B. and A. Lupas. Functional Equations for Wallis and Gamma Functions. Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz., 461-497 (1974) 245-251.
3. Mitrinović, D. S. Analytic Inequalities. Grundlehren der Alath. Wiss., Band 165, Berlin - Heidelberg - New York, 1970, 400 pp.
4. Watson, G. N. A Note on Gamma Functions, Proc. Edinburg Math. Soc. 2 (1958/59) 11.
5. Slavić, D. V. On Inequality for $\Gamma(x + 1)/\Gamma(x + \frac{1}{2})$, Univ. Beograd Publ. Elektrotechn Fak. Ser. Mat. Fiz 499, 17-20.



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