# A Note on Anti-Pluricanonical Maps for 5-Folds 

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#### Abstract

We prove that the anti-pluricanonical map $\Phi_{\left|-m K_{X}\right|}$ is birational when $m \geq 16$ for 5 -fold $X$ whose anticanonical divisor is nef and big.


## 1. Introduction

Throughout the ground field $k$ is always supposed to be algebraically closed of characteristic zero. Let $X$ be a non-singular $n$-fold over $k$ and assume its anticanonical divisor $-K_{X}$ is a nef and big divisor. It is an interesting problem to find an explicit lower bound $l(n)$ such that the rational map $\Phi_{\left|-m K_{X}\right|}$ associated with the complete linear system $\left|-m K_{X}\right|$ is a birational map onto its image for any $m \geq l(n)$. Ando ([1, Theorem 9]) first gave the bounds $l(2)=3, l(3)=5$ and $l(4)=12$. Fukuda [3] improved Ando's method and got the bounds $l(3)=4, l(4)=11$ and

$$
l(n)=2^{n-2} \cdot(n+4[n / 2]-1)-2[n / 2]-1
$$

for any $n \geq 5$. Chen [2] also used the similar ideas to improve Ando's partial results. Using the Key Lemma in [3], this note proves

Main Theorem. Let $X$ be a smooth 5 -fold whose anticanonical divisor $-K_{X}$ is nef and big. Then $\Phi_{\left|-m K_{X}\right|}$ is a birational map when $m \geq 16$.

## 2. Preparations

In this note we use the standard terminology as in [4, 5]. For example, $c_{i}:=c_{i}(T X)$ is the $i$-th Chern class of the tangential bundle; $H^{i}(X, \mathcal{F})$ denotes the $i$-th cohomology with coefficient in a coherent sheaf $\mathcal{F}$, and $h^{i}(X, \mathcal{F})=\operatorname{dim}_{k} H^{i}(X, \mathcal{F})$. We simply denote $H^{i}\left(X, \mathcal{O}_{X}(D)\right)$ by $H^{i}(X, D)$ if the sheaf is induced by a divisor $D$.

We will use Lemma 1, a special case of the Key Lemma in [3], which improved the Theorem 5 in [1].

[^0]Lemma 1 (Ando [1], Fukuda [3], Chen [2]). Let $X$ be a nonsingular projective variety of dimension $n$ and $-K_{X}$ is a nef and big divisor. We assume:
(i) For each $i$ with $1 \leq i \leq n-2$, there exists a natural number $r_{i}$ such that $\operatorname{dim} \Phi_{\left|-r_{i} K_{X}\right|}(X) \geq i$.
(ii) There exist an integer $r_{0} \geq 3$ such that $H^{0}\left(X,-r K_{X}\right) \neq 0$ for any $r \geq r_{0}$.

Then $\Phi_{\left|-m K_{X}\right|}$ is birational for all $m \geq r_{0}+\left(r_{1}+\cdots r_{n-2}\right)$.
Proof. In the Key Lemma in [3], we let the nef and big divisor $R=-K_{X}$ and the numerically trivial divisor $T=0$. By our assumptions we have $H^{0}\left(X,-r K_{X}\right)=$ $H^{0}\left(X,-(r+1) K_{X}+K_{X}\right) \neq 0$ for any $r+1 \geq r_{0}+1:=\hat{r}_{0} \geq 4$. So both (1) and (2) of the Key Lemma are satisfied. Hence $\Phi_{\left|-m K_{X}\right|}=\Phi_{\left|-(m+1) K_{X}+K_{X}\right|}$ is birational when $m+1 \geq \hat{r}_{0}+\left(r_{1}+\cdots r_{n-2}\right)$, thus $\Phi_{\left|-m K_{X}\right|}$ is birational for all $m \geq r_{0}+\left(r_{1}+\cdots r_{n-2}\right)$.

To use Lemma 1, we need the following Lemma, it is the Proposition 6 in [1], we refer to $[1,2,3]$ for reference of it.

Lemma 2 (Matsusaka \& Maehara). Let $D$ be a nef and big divisor and $\operatorname{dim} X=n$. If $h^{0}(X, m D)>m^{r} D^{n}+r$, then $\operatorname{dim} \Phi_{|m D|}>r$.

## 3. Proof of the Main Theorem

Let

$$
P(m):=\chi\left(\mathcal{O}_{X}\left(-m K_{X}\right)\right)=\sum_{i=0}^{5}(-1)^{i} h^{i}\left(X,-m K_{X}\right)=h^{0}\left(X,-m K_{X}\right),
$$

Change to:

$$
P(m):=\chi\left(\mathcal{O}_{X}\left(-m K_{X}\right)\right)=\sum_{i=0}^{5}(-1)^{i} h^{i}\left(X, \mathcal{O}_{X}\left(-m K_{X}\right)\right)=h^{0}\left(X, \mathcal{O}_{X}\left(-m K_{X}\right)\right)
$$

since $-K_{X}$ is nef and big, by the Kawamata-Viehweg vanishing theorem (cf. [5, Corollary 1-2-2]) we have $H^{i}\left(X,-m K_{X}\right)=0$ for $m \geq 0$ and $i>0$. Thus $\chi\left(\mathcal{O}_{X}\right)=1$. Note by definition, $c_{1}=-K_{X}$, and $c_{i}=0$ for $i>5$. Combine these facts with Hirzebruch-RiemannRoch formula ( $\left[4, P_{432}\right]$ ), we have

$$
\begin{aligned}
P(m)= & \int_{X} \operatorname{ch}\left(\mathcal{O}_{X}\left(-m K_{X}\right)\right) \operatorname{Td}(X) \\
= & m(m+1)(2 m+1)\left(3 m^{2}+3 m-1\right) \frac{\left(-K_{X}\right)^{5}}{720}+m(m+1)(2 m+1) \frac{\left(-K_{X}\right)^{3} \cdot c_{2}}{144} \\
& +(2 m+1) \\
= & (2 m+1)\left\{\left(m(m+1)\left[\left(3 m^{2}+3 m-1\right) a+b\right]+1\right\},\right.
\end{aligned}
$$

where $720 a=\left(-K_{X}\right)^{5}$ and $144 b=\left(-K_{X}\right)^{3} \cdot c_{2}$.

To use Lemma 1 and Lemma 2, we need priori estimates of $P(m)$ for $m \geq 0$.
PROposition 1.
(i) $P(1)=0, P(2) \geq 0 \Rightarrow P(3) \geq 35$;
(ii) $P(1)=1, P(2) \geq 1 \Rightarrow P(3) \geq 21$;
(iii) $P(1)=2, P(2) \geq 2 \Rightarrow P(3) \geq 7$;
(iv) $P(1)=3 \Rightarrow P(2) \geq 6$;
(v) $P(1)=3, P(2)=6 \Rightarrow P(3)=49$;
(vi) $\quad P(m+1)>P(m)$ when $m>3$ and $P(3) \geq 7$.

Proof. Assume $P(1)=3[2(5 a+b)+1]:=l \geq 0$, we have $b=\frac{1}{6}(l-3)-5 a$. By $P(2) \geq P(1)$ we get $a \geq \frac{1}{360}(10-4 l)$, so we have $P(3) \geq 35-14 l$. Hence we get (i)-(iii).

Now we assume $P(1)=3$ and $P(2)=5[6(17 a+b)+1]:=l$. Then $b=-5 a$ and $l=5(12 a+1)>5$, and hence $P(2) \geq 6$ and we have (iv). If $P(2)=2 P(1)=6$. Then we have $a=\frac{1}{60}$ and $b=-\frac{1}{12}$, So $P(3)=7[12(35 a+b)+1]=49>7$ we get $(\mathrm{v})$.

If $P(2)>6$, then $P(3) \geq P(2) \geq 7$. Combine with (i)-(v) we always have $P(3) \geq 7$. Since $P(1) \geq 0$, we have $b \geq-5 a-\frac{1}{2}$ and $[3 m(m+1)-1] a+b>0$ when $m \geq 3$. Thus $P(m+1)-P(m)>(m+1)\{[3 m(m+1)-1] a+b+1\}[(m+2)(2 m+3)-(m+1)(2 m+1)]>0$ when $m \geq 3$, we get (vi).

## PRoposition 2.

(i) $\operatorname{dim} \Phi_{\left|-m K_{X}\right|}(X) \geq 1$, for any $m \geq 3$;
(ii) $\operatorname{dim} \Phi_{\left|-m K_{X}\right|}(X) \geq 2$, for any $m \geq 4$;
(iii) $\operatorname{dim} \Phi_{\left|-m K_{X}\right|}(X) \geq 3$, for any $m \geq 6$.

Proof. By Proposition $1, P(3)=7[12(35 a+b)+1] \geq 7$, so we have (i) and $b \geq$ $-35 a$. Thus $P(4)=9[20(59 a+b)+1] \geq 180 \times 24 a+9>6\left(-K_{X}\right)^{5}+2$. By Lemma 2 we have (ii). $P(6)=13[42(125 a+b)+1] \geq \frac{13 \times 21}{4}\left(-K_{X}\right)^{5}+13>36\left(-K_{X}\right)^{5}+3$, we have (iii).

Proof of Main Theorem. By Proposition 1 we have $h^{0}\left(X,-3 K_{X}\right) \geq 7$, so we can put $r_{0}=3$. By Proposition 2, we can set $r_{1}=3, r_{2}=4, r_{3}=6$. By Lemma 1 when $m \geq r_{0}+r_{2}+r_{3}+r_{4}=16$, then $\Phi_{\left|-m K_{X}\right|}$ is a birational map.

## 4. An example

EXAMPLE 1. Let $\pi: E=\mathcal{O}_{\boldsymbol{P}^{1}} \oplus \mathcal{O}_{\boldsymbol{P}^{1}} \oplus \mathcal{O}_{\boldsymbol{P}^{1}} \oplus \mathcal{O}_{\boldsymbol{P}^{1}} \oplus \mathcal{O}_{\boldsymbol{P}^{1}}(1) \rightarrow \boldsymbol{P}^{1}$ be a rank 5 vector bundle. Let $X=\mathbb{P}(E)$. Then by calculations in the Exercise 8.4 of [4, $\left.P_{253}\right]$, $K_{X}=-5 L+\pi^{*}\left(\operatorname{det}(E)+K_{P^{1}}\right)=-5 L-H$, where $L \in\left|\mathcal{O}_{X}(1)\right|, H \in\left|\pi^{*} \mathcal{O}_{\boldsymbol{P}^{1}}(1)\right|$. Clearly $X$ is a Fano manifold and $-K_{X}$ is a nef and big divisor. Note $L^{5}=H \cdot L^{4}=1$,
so $\left(-K_{X}\right)^{5}=2 \times 5^{5}$. By Leray spectral sequence and the fact that $R^{i} \pi_{*}(\mathcal{O}(l))=0$ for any $i>0$ and $l>-5$, we have $H^{i}\left(X,-m K_{X}\right)=0$ when $i>0$ and $H^{0}\left(X,-m K_{X}\right)=$ $H^{0}\left(X, \mathcal{O}_{X}(5 m) \otimes \pi^{*} \mathcal{O}_{\boldsymbol{P}^{1}}(m)\right)=H^{0}\left(\boldsymbol{P}^{1}, S^{5 m}(E) \otimes \mathcal{O}_{\boldsymbol{P}^{1}}(m)\right)$ for any $m>0$. Note that

$$
\begin{aligned}
S^{5 m}(E) & =\bigoplus_{i=0}^{5 m} S^{5 m-i}\left(\mathcal{O}_{\boldsymbol{P}^{1}} \oplus \mathcal{O}_{\boldsymbol{P}^{1}} \oplus \mathcal{O}_{\boldsymbol{P}^{1}} \oplus \mathcal{O}_{\boldsymbol{P}^{1}}\right) \otimes \mathcal{O}_{\boldsymbol{P}^{1}}(i) \\
& =\bigoplus_{i=0}^{5 m}\left(\mathcal{O}_{\boldsymbol{P}^{1}}(i) \oplus \mathcal{O}_{\boldsymbol{P}^{1}}(i) \oplus \cdots \oplus \mathcal{O}_{\boldsymbol{P}^{1}}(i)\right),
\end{aligned}
$$

it is a bundle of $\operatorname{rank} \frac{1}{6}(5 m-i-1)(5 m-i)(5 m-i+1)$ in the last bracket of above summation. So,

$$
\begin{aligned}
h^{0}\left(\boldsymbol{P}^{1}, S^{5 m}\right. & \left.(E) \otimes \mathcal{O}_{\boldsymbol{P}^{1}}(m)\right) \\
& =\frac{1}{6} \sum_{i=0}^{5 m}(5 m-i-1)(5 m-i)(5 m-i+1) h^{0}\left(\boldsymbol{P}^{1}, \mathcal{O}_{\boldsymbol{P}^{1}}(m+i)\right) \\
& =\frac{1}{6} \sum_{i=0}^{5 m}(m+i+1)(5 m-i-1)(5 m-i)(5 m-i+1) \\
& =\frac{1}{24} m(5 m-1)(5 m+1)(5 m+2)(10 m+3) .
\end{aligned}
$$

It is easy to check that $h^{0}\left(\operatorname{IP}(E),-K_{X}\right)=91$, we can take $r_{0}=r_{1}=3$; and $h^{0}(\operatorname{IP}(E)$, $\left.-4 K_{X}\right)=62909>10\left(-K_{X}\right)^{5}+2$, we take $r_{2}=4$, and $h^{0}\left(\mathbb{P}(E),-5 K_{X}\right)=186030>$ $5^{2}\left(-K_{X}\right)^{5}+3$ we can take $r_{3}=5$. So $\Phi_{\left|-m K_{X}\right|}$ is a birational map when $m \geq 15$.

QUESTION. Find out the lowest bound $l(n)$ such that $\Phi_{\left|-m K_{X}\right|}$ is birational when $m \geq$ $l(n)$.

We also don't know how to improve the bounds $l(n)$ given in [3] for $n>5$ since the Hirzebruch-Riemann-Roch formula is more complicate in these cases.

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## References

[ 1 ] T. Ando, Pluricanonical systems of algebraic varieties of general type of dimension $\leq 5$, Adv. Stud. in Pure Math., 10 (1987), [Algebraic Geometry, Sendi, 1985], 1-10.
[ 2 ] M. CHEN, A note on pluricanonical maps for varieties of dimension 4 and 5, J. Math. Kyoto Univ. 37 (1997), 513-517.
[3] S. FUKUDA, A note on Ando's paper "pluricanonical systems of algebraic varieties of general type of dimension $\leq 5$, Tokyo J. Math. 14 (1991), 479-487.
[ 4 ] R. Hartshorne, Algebraic Geometry, GTM 52 (1977), Springer.
[5] Y. Kawamata, K. Matsuda and K. Matsuki, Introduction to minimal model problem, Adv. Stud. in Pure Math. 10, (1987), [Algebraic Geometry, Sendi, 1985], 283-360.

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