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A note on Arhangelskii's inequality

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As we know, the famous Arhangelskii's inequality, "$X\in\mathcal{T}_2$, $|X|\leq 2^{L(X)\cdot\chi(X)}$" has been generalized to "$|X|\leq 2^{L(X)\cdot\psi(X)t(X)}$" ([**1**]) and the latter has not been improved so far. In this paper, we will do this, see Theorem below.

Let X be a topological space and k, λ infinite cardinal, ω_0 the smallest infinite ordinal and the smallest infinite cardinal. $[X]^{\leq k}$ will denote the collection of subsets A of X with $|A|\leq k$. Let us write ([**3**]) $qL(X)=\omega_0\cdot\min\{k\mid$ there exists an $A\in[X]^{\leq 2^k}$, such that (*) for each open cover $\mathcal{U}$ of X, there exists a $\mathcal{V}\in[\mathcal{U}]^{\leq k}$, and $B\in[A]^{\leq k}$ satisfying $\bigcup\mathcal{V}\cup\bar{B}=X\}$.

Clearly, $d(X)\geq qL(X)$, $L(X)\geq qL(X)$, $s(X)\geq qL(X)$, and all the inequalities are strict.

It is only slightly less trivial to show that $s(X)\geq qL(X)$, where $s(X)=\sup\{|A|: A$ is a discrete subspace of $X\}$. To do this, we have to apply the following lemmas.

LEMMA 1 (Šapirovskii [**4**]). *Let* $X\in\mathcal{T}_2$, *if* $s(X)=k$, *then there is a set* S *of* X *with* $|S|\leq 2^k$ *such that* $X=\cup\{\bar{A}: A\subseteq S,\ |A|\leq k\}$.

LEMMA 2 (Šapirovskii). *Let* $\mathcal{U}$ *be an open cover of a topological space* X, *let* $s(X)=k$, *then there is a subset* A *of* X *with* $|A|\leq k$ *and a subcollection* $\mathcal{V}$ *of* $\mathcal{U}$ *with* $|\mathcal{V}|\leq k$ *such that* $X=\bar{A}\cup(\bigcup\mathcal{V})$.

By virtue of Lemma 1, we only show that the set S in Lemma 1 satisfies (*) in the definition of $qL(X)$. In fact, for each open cover $\mathcal{U}$ of X, let subset B satisfy the condition in Lemma 2, for each $b\in B$, there is an $A_b\subset S$ with $|A_b|\leq k$ such that $b\in\bar{A}_b$. Let $A=\bigcup_{b\in B}A_b$, then $|A|\leq k\cdot k=k$, $A\subset S$, $\bar{B}\subset\bar{A}$ and $X=\bar{B}\cup(\bigcup\mathcal{V})\subseteq\bar{A}\cup(\bigcup\mathfrak{B})$. This completes the proof of the inequality $s(X)\geq qL(X)$.

Again let us write $S\psi(X)=\omega_0\cdot\min\{k\mid$ for each $x\in X$, there exists a family of open neighborhoods $\{U_\alpha(x)\}_{\alpha<k}$, such that $\{x\}=\bigcap_{\alpha<k}\overline{U_\alpha(x)}\}$.

LEMMA. *Let* X *be a space with* $t(X)\cdot S\psi(X)\leq k$, *then for each* $A\in[X]^{\leq a^k}$, *we have* $|\bar{A}|\leq a^k$, *where* $a\geq 2$.

PROOF. For each $x\in\bar{A}$, by $S\psi(X)\leq k$, there exists a family of open neighborhoods $\{U_\alpha(x)\}_{\alpha<k}$, such that $\{x\}=\bigcap_{\alpha<k}\overline{U_\alpha(x)}$. Thus $\{x\}=\bigcap_{\alpha<k}\overline{U_\alpha(x)\cap A}$. By

$t(X)\leq k$, $\forall\alpha<k$, there exists $A_\alpha\subset U_\alpha(x)\cap A\subset A$ satisfying $|A_\alpha|\leq k$ and $x\in\bar{A}_\alpha$, hence, $\{x\}=\bigcap_{\alpha<k}\bar{A}_\alpha$ and $\{A_\alpha\}_{\alpha<\lambda}\in[[A]^{\leq k}]^{\leq k}$. Therefore, $|\bar{A}|\leq|[[A]^{\leq k}]^{\leq k}|=a^k$.

THEOREM. *If* $X\in\mathcal{T}_2$, *then* $|X|\leq 2^{qL(X)\cdot t(X)\cdot S\psi(X)}$.

PROOF. Let $qL(X)\cdot t(X)\cdot S\psi(X)=k$, then there exists a subset $A\subset X$ with $|A|\leq 2^k$ satisfying (*). For each $x\in X$, let $\mathfrak{V}_x$ denote a family of open neighborhoods $\{U_\alpha(x)\}_{\alpha<k}$ such that $\{x\}=\bigcap_{\alpha<k}\overline{U_\alpha(x)}$. Construct an increasing sequence $\{H_\alpha: 0\leq\alpha<k^+\}$ of closed sets of X and a sequence $\{\mathfrak{V}_\alpha: 0\leq\alpha<k^+\}$ of open collections in X such that

(1) $|H_\alpha|\leq 2^k$, $0\leq\alpha<k^+$,

(2) $\mathfrak{V}_\alpha=\{V: V\in\mathfrak{V}_p,\ p\in\bigcup_{\beta<\alpha}H_\beta\}$, $0<\alpha<k^+$,

(3) if W is the union of $\leq k$ elements of $\mathfrak{V}_\alpha$, $B\in[A]^{\leq k}$ and $X\setminus(W\cup\bar{B})\neq\emptyset$, then $H_\alpha\setminus(W\cup\bar{B})\neq\emptyset$.

The construction is by transfinite induction. Let $0<\alpha<k^+$ and assume that $\{H_\beta: \beta<\alpha\}$ have been constructed. Note that $\mathfrak{V}_\alpha$ is defined by (2), so $|\mathfrak{V}_\alpha|\leq 2^k$. For each set W which is the union of $\leq k$ elements of $\mathfrak{V}_\alpha$, and for each $B\in[A]^{\leq k}$ which $X\setminus(W\cup\bar{B})\neq\emptyset$. Choose one point $P_{W,B}$ of $X\setminus(W\cup\bar{B})$, let A_α be the set of points chosen in this way. Since $|\mathfrak{V}_\alpha|\leq 2^k$, $|A|\leq 2^k$, thus $|A_\alpha|\leq 2^k$. Let $H_\alpha=\overline{A_\alpha\cup\bigcup_{\beta<\alpha}H_\beta}$, by lemma, $|H_\alpha|\leq 2^k$ and H_α is closed, $H_\alpha\supset H_\beta$, for $\beta<\alpha$. This completes the construction of $\{H_\alpha: 0\leq\alpha<k^+\}$.

Now let $H=\bigcup_{\alpha<k^+}H_\alpha$, it is closed (since $t(X)\leq k$ and $\forall\alpha<k^+$, H_α is closed). We will show that $H\cup\bar{A}=X$, i.e., $X\setminus H\subset\bar{A}$. Let $q\in X\setminus H$, for each $p\in H$, take $V_p\in\mathfrak{V}_p$ such that $q\notin\bar{V}_p$. So $\bigcup\{V_p: p\in H\}\supset H$, by $qL(X)\leq k$, there exist $\mathcal{V}\in[\{V_p: p\in H\}]^{\leq k}$, $B\in[A]^{\leq k}$ such that $H\subset\bigcup\mathcal{V}\cup\bar{B}$. If $q\in\bar{B}\subset\bar{A}$, then the proof is complete. If $q\notin\bar{B}$, then $q\notin\bigcup\mathcal{V}\cup\bar{B}$, i.e., $\bigcup\mathcal{V}\cup\bar{B}\neq X$. Thus, we have $P_{\cup\mathcal{V},B}\in X\setminus(\bigcup\mathcal{V}\cup\bar{B})\subset X\setminus H$. On the other hand, there exists $\beta<k^+$ with $\mathcal{V}\in[\mathfrak{V}_\beta]^{\leq k}$, so $P_{\cup\mathcal{V},B}\in H_{\beta+1}\subset H$, a contradiction with the fact $P_{\cup\mathcal{V},B}\in X\setminus H$.

COROLLARY 1. *If* $X\in\mathcal{T}_2$, *then* $|X|\leq 2^{L(X)\cdot t(X)\cdot\psi(X)}$.

PROOF. Since $S\psi(X)\leq L(X)\cdot\psi(X)$ (it is easy). Thus $L(X)\cdot\psi(X)=L(X)\cdot S\psi(X)$. We have $|X|\leq 2^{qL(X)\cdot t(X)\cdot S\psi(X)}\leq 2^{L(X)\cdot t(X)\cdot S\psi(X)}=2^{L(X)\cdot t(X)\cdot\psi(X)}$.

EXAMPLE. Let X be Niemytzki plane [**2**, Example 1.2.4], then $d(x)=qL(X)=\chi(X)=t(X)=S\psi(X)=\omega_0$. But $L(X)\cdot t(X)\cdot\psi(X)\geq L(X)>\omega_0$.

COROLLARY 2 ([**3**]). *If* $X\in\mathcal{T}_3$, *then* $|X|\leq 2^{qL(X)\cdot t(X)\cdot\psi(X)}$.

PROOF. Since $X\in\mathcal{T}_3$, we have $S\psi(X)=\psi(X)$.

COROLLARY 3 ([**3**]). *If* $X\in\mathcal{T}_2$, *then* $|X|\leq 2^{qL(X)\cdot\chi(X)}$.

REMARK 1. In the theorem, $S\psi(X)$ cannot be replaced by $\psi(X)$, since there exists a Hausdorff space X of cardinality 2^c that contains a countable dense subset A consisting of isolated points of X such that the subspace $X \setminus A$ is discrete. ([**2**, Ex. 3.1. F.(d) Hint]). It is easy to check $qL(X)=d(X)=\omega_0=\psi(X)=t(X)$. But $|X|>c=2^{qL(X)\cdot t(X)\cdot \psi(X)}$.

For example, "Assume ($2^{\omega_0}<2^{\omega_1}$), let X be a normal space with $qL(X)=\omega_0$, then $e(X)=\omega_0$, where $e(X)$, called the extent of X, is the smallest cardinal number $\mathfrak{m}\geq\aleph_0$ such that every closed subset of X consisting exclusively of isolated points has cardinality$\leq\mathfrak{m}$." which generalized Jones' theorem that if $2^{\omega_0}<2^{\omega_1}$, then every separable normal Moore space is metrizable.

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