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A Note on Bisexual Galton-Watson Branching Processes with Immigration *

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1. Introduction

Recently, from the branching model introduced in [1], new bisexual Galton-Watson branching processes allowing immigration have been developed in [2] and some probabilistical analysis about them has been obtained. In particular, for the bisexual Galton-Watson process allowing the immigration of females and males, it has been proved (see [3]) that, under certain conditions, the sequence representing the number of mating units per generation converges in distribution to a positive, finite and non-degenerate random variable. The aim of this paper is to provide, through a different methodology, an alternative proof of this limit result. In this new, and more technical proof, we make use of the underlying probability generating functions. In Section 2, a brief description of the probability model is considered and some basic definitions and results are given. Section 3 is devoted to prove the asymptotic result previously indicated.

2. The Probability Model

The bisexual Galton-Watson process with immigration of females and males (BGWPI) denoted by $\{(F_n^*, M_n^*),\ n = 1, 2, \dots\}$ is defined, see [2], in the form:

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$$Z_0^* = N\,, \quad (F_{n+1}^*, M_{n+1}^*) = \sum_{i=1}^{Z_n^*} (f_{ni}, m_{ni}) + (F_{n+1}^I, M_{n+1}^I)\,,$$
$$Z_{n+1}^* = L(F_{n+1}^*, M_{n+1}^*)\,, \quad n = 0, 1, \dots \tag{1}$$

where N is a positive integer and the empty sum is considered to be $(0,0)$. $\{(f_{ni}, m_{ni})\}$ and $\{(F_n^I, M_n^I)\}$ are independent sequences of i.i.d. non-negative integer-valued random variables with mean vectors $\mu = (\mu_1, \mu_2)$ and $\mu^I = (\mu_1^I, \mu_2^I)$, respectively. Intuitively f_{ni} (m_{ni}) represents the number of females (males) produced by the ith mating unit in the nth generation and F_n^I (M_n^I) may be viewed as the number of immigrating females (males) in the nth generation. The mating function $L : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing in each argument, integer-valued for integer-valued arguments and such that $L(x,y) \leq xy$. Consequently, from an intuitive outlook, F_n^* (M_n^*) will be the number of females (males) in the nth generation, which form $Z_n^* = L(F_n^*, M_n^*)$ mating units. These mating units reproduce independently through the same offspring distribution for each generation.

It can be shown that $\{Z_n^*\}$ and $\{(F_n^*, M_n^*)\}$ are Markov chains with stationary transition probabilities. We denote by $p_{kl} = P[(f_{01}, m_{01}) = (k,l)]$, $k, l = 0, 1, \dots$ and assume that μ and μ^I are finite.

Definition 2.1. A BGWPI is said to be superadditive if the mating function L is superadditive, i.e. satisfies, for every positive integer n, that

$$L\left(\sum_{i=1}^{n} (x_i, y_i)\right) \geq \sum_{i=1}^{n} L(x_i, y_i)\,, \quad x_i, y_i \in \mathbb{R}^+\,, \quad i = 1, \dots, n\,.$$

Definition 2.2. For a BGWPI and each positive integer k, we define the average reproduction rate per mating unit, denoted by r_k^*, as:

$$r_k^* = k^{-1} E[Z_{n+1}^* \mid Z_n^* = k]\,, \qquad k = 1, 2, \dots\,.$$

For a superadditive BGWPI with finite mean vector μ and mating function verifying that $L(x,y) \leq x + y$ it is derived (see [3]) that $\lim_{k\to\infty} r_k^* = r$ being r the named asymptotic growth rate (or growth rate).

Definition 2.3. A superadditive BGWPI is said to be subcritical, critical or supercritical if r is $<$, $=$ or $>$ 1, respectively.

3. A limit result for the sequence $\{Z_n^*\}$

In this section we consider a subcritical and superadditive BGWPI defined by (1) and, considering as a tool the underlying probability generating functions, we provide an alternative proof to theorem 3.2 in [3]. Previously it will be necessary to introduce the following lemma:

Lemma 3.1. *Let ψ a positive, non decreasing and continuous function on $[0,1]$ such that $\psi(1)=1$ and $\psi'(1^-)\in(0,\infty)$. Then for $\delta\in(0,1)$ it is verified that*

$$\sum_{k=1}^{\infty}(1-\psi(1-\delta^k))<\infty\,.$$

Proof. Consider $h(x)=1-\psi(1-\delta^x)$, $x\in\mathbb{R}^+$. It is clear that h is a positive, non increasing and continuous function. Moreover, it follows that $\lim_{x\to\infty}h(x)=0$. Then, from the integral criteria for convergence of series, it will be sufficient to prove that $\int_1^\infty h(x)dx<\infty$. Making use of the transformation $s=1-\delta^x$ we get that $\int_1^\infty h(x)dx$ is proportional to $\int_0^1(1-s)^{-1}(1-\psi(s))ds$ which is convergent taking into account that $\psi'(1^-)<\infty$ and $(1-s)^{-1}(1-\psi(s))$ is bounded on $[0,1]$. ∎

Theorem 3.1. *If $E[L(f_{01},m_{01})]>0$, $E[L(F_1^I,M_1^I)]>0$, $p_{00}>0$ and there exists $\alpha>0$ and $N_0\geq 1$ such that for $k>N_0$, $r_k^*\leq r+k^{-1}\alpha$ then $\{Z_n^*\}$ converges in distribution to a positive and finite random variable Z^* as $n\to\infty$.*

Proof. Under the considered assumptions, it can be proved in [2] that $\{Z_n^*\}$ is an irreducible Markov chain. If $k_0=\inf\{k: P[L(F_1^I,M_1^I)=k]>0\}$ then, using that L is non decreasing in each argument, it is derived that $P[Z_n^*\geq k_0]=1$, $n=1,2,\dots$, and therefore if k^* is an essential state it is obtained that $k^*\geq k_0$.

Thus, if f_n^* and h_k^* denote the probability generating functions associated with Z_n^* and with the kth row of the transition matrix of $\{Z_n^*\}$, respectively, i.e. $f_n^*(s)=E[s^{Z_n^*}]$ and $h_k^*(s)=E[s^{Z_{n+1}^*}\mid Z_n^*=k]$, $s\in[0,1]$, then it is followed that:

$$f_n^*(s)=\sum_{j=k_0}^{\infty}s^jP[Z_n^*=j]\quad\text{and}\quad h_k^*(s)=\sum_{j=k_0}^{\infty}s^jP[Z_{n+1}^*=j\mid Z_n^*=k]\,.$$

From Jensen's inequality we obtain:

$$(h_k^*(s))^{1/k} \geq \varphi_k(s)\,, \qquad s \in [0,1]\,, \tag{2}$$

where

$$\varphi_k(s) = E\left[s^{k^{-1}L(\sum_{i=1}^{k}(f_{ni},m_{ni})+(F_{n+1}^I,M_{n+1}^I))}\right].$$

Since, for some $\xi \in (s,1)$:

$$\varphi_k(s) = 1 - r_k^*(1-s) + \frac{\varphi_k''(\xi)}{2}(1-s)^2$$

we have for $k > N_0$, that

$$\varphi_k(s) \geq a(s)\left(1 - \frac{(1-s)\alpha}{ka(s)}\right) \tag{3}$$

being $a(s) = 1 - r(1-s)$. Now

$$0 \leq \frac{(1-s)\alpha}{ka(s)} \leq (1-r)^{-1}\alpha\,, \qquad s \in [0,1]\,.$$

Therefore for $k > N_1 > \max\{N_0, (1-r)^{-1}\alpha\}$, taking into account (2) and (3), it is deduced that:

$$h_k^*(s) \leq (a(s))^k \left(1 - \frac{(1-s)\alpha}{ka(s)}\right)^k \leq (a(s))^k A(s)\,, \quad s \in [0,1]$$

where $A(s) = \left(1 - \frac{(1-s)\alpha}{N_1 a(s)}\right)^{N_1}$.

It is clear that A is a positive, non decreasing and continuous function on $\mathbb{R}^+$ verifying that $A(1) = 1$ and $A'(1) = \alpha$. Let $u(s)$ be an arbitrary probability generating function such that $u'(1) < \alpha$ (for example the probability generating function of a Poisson distribution with mean $\lambda < \alpha$) and for $s \in [0,1]$ we define the function:

$$\widehat{h}_k(s) = \begin{cases} (a(s))^k u(s) & \text{if } k = 1, \dots, N_1\,, \\ h_k^*(s) & \text{if } k > N_1 + 1\,. \end{cases}$$

If $\psi(s) = \min\{u(s), A(s)\}$, it follows that

$$\widehat{h}_k(s) \geq (a(s))^k \psi(s)\,, \qquad s \in [0,1]\,, \quad k = 1, 2, \dots$$

and from the comparison theorem for Markov chains (see [4], p.45) it will be sufficient to prove that k_0 is a positive recurrent state for the Markov chain with transition matrix rows associated to $\widehat{h}_k(s)$. If we denote this Markov chain by $\{\widehat{Z}_n\}$ then, without loss of generality, it may be assumed that $k_0 = 0$.

Let

$$\widehat{f}_m(s) = E[s^{\widehat{Z}_{n+m}} \mid \widehat{Z}_n = 0]\,, \qquad m = 0, 1, \dots .$$

It is not difficult to verify that:

$$\widehat{f}_m(s) \geq \prod_{j=0}^{m-1} \psi(a_j(s))\,, \qquad s \in [0,1]\,, \tag{4}$$

where a_j denotes the j times composition of the function a and $a_0(s) = s$. Consequently, if $p_{00}^{(m)}$ represents the m step transition probability from 0 to 0, taking into account (4) we deduced that

$$\lim_{m\to\infty} p_{00}^{(m)} = \lim_{m\to\infty} \widehat{f}_m(0) \geq \prod_{j=0}^{\infty} \psi(1 - r^j)$$

and therefore 0 will be a positive recurrent state if the limit above is positive or, equivalently, if $\sum_{j=0}^{\infty}(1 - \psi(1 - r^j)) < \infty$ which holds as a consequence of Lemma 1. From Markov chains theory we deduce that $\{Z_n^*\}$ converges in distribution to a positive and finite random variable Z^* whose probability distribution will be the corresponding stationary distribution. ∎

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