A NOTE ON BLOW-UPS OF TORIC SURFACES
AND CSC KÄHLER METRICS

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Abstract. Let X be a compact toric surface. Then there exists a sequence of torus equivariant blow-ups of X such that the blown-up toric surface admits a cscK metric.

1. Introduction. The problem of finding constant scalar curvature Kähler metrics (cscK for short) on compact Kähler manifolds is of main interest in Kähler geometry since the works of Yau [24], Tian [22] and Donaldson [5]. The Yau-Tian-Donaldson conjecture asserts that there exists a cscK metric on a polarized Kähler manifold (X, L) in the Kähler class $c_1(L)$ if and only if the pair (X, L) is stable in a suitable GIT sense. At that time, we know that the existence of a cscK metric implies the K-polystability by a result of Stoppa [20]. Stoppa's argument relies on an important result of Arezzo and Pacard on blow-ups of cscK manifolds [1], [2]. Arezzo and Pacard managed to show that if (X, ω) is cscK, then the blow-up of X at finitely many well-chosen points admits a cscK metric. In order to prove their result, they use a gluing method. They build the blown-up manifold by gluing a local model, the blow-up of C^n at 0, endowed with the Burns-Simanca metric [19], which is ALE and scalar flat.

Another result on blow-ups and special metrics is the one of Taubes [21] that asserts that there are anti-self-dual metrics on the blow-up at sufficiently many points of any smooth oriented 4-manifold. In the Kähler case, a metric is anti-self-dual if and only if it is scalar-flat.

These two results suggest the following:

CONJECTURE 1. *Let X be a compact Kähler manifold. Then there exists a finite sequence of blow-ups of X such that the blown-up manifold obtained admits a cscK metric.*

In the case of ruled surfaces, this conjecture has been proved by Le Brun and Singer [12] when the base curve is of genus greater than or equal to 2 and then in general by Kim, Le Brun and Pontecorvo [11]. However, in their argument, even if the starting surface is toric, the resulting blown-up surface does not admit the torus action anymore.

On the other hand, Rollin and Singer found another method to built cscK metrics on the blown-up surfaces, based on parabolic structures [17], from which we can deduce a lot of examples of toric surfaces with cscK metrics.

In this paper, we show that if the initial surface is toric, then we can prove the conjecture within the toric world.

THEOREM 1.   *Let X be a compact complex smooth toric surface. Then there exists a finite sequence of blow-ups of X at torus fixed points such that the blown-up toric surface admits a cscK metric.*

REMARK 1.   By a remark of Wang and Zhou [23], using the theorem of Arezzo, Pacard and Singer on blow-ups of extremal metrics [3], we know that each toric surface admits an extremal metric. However, from Matsushima and Lichnerowicz obstruction, we know from Calabi [4] that the Hirzebruch surfaces admit no extremal metric of constant scalar curvature. Thus we need to blow-up toric surfaces if we want them to admit cscK metrics.

REMARK 2.   About the existence of Kähler-Einstein metrics on toric manifolds, a recent result of Legendre [13] shows that each toric orbifold admits a unique singular Kähler-Einstein metric.

The proof of this result relies on simple considerations on the classification of toric surfaces and the result of Arezzo and Pacard on blow-ups of cscK metrics. We hope that it will outline a stabilization process of blowing up a manifold at suitable points and its link with the existence of cscK metrics. Basically, the idea is to blow up the toric surface until it has a good shape. We first build an infinite sequence of toric surfaces

$$\cdots \to X_{j+1} \to X_j \to \cdots \to X_1 \to X_0$$

with $X_0 = \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and X_{j+1} obtained from X_j by blowing up all the torus fixed points. By the theorem of Arezzo and Pacard, each of these X_j admits a cscK metric.

Then, using the classification of toric surfaces, we prove that for each smooth toric surface X, there exists a sequence of blow-ups at torus fixed points such that the blown-up surface is one of the $X_j's$. From the fan description point of view, we add rays to the fan describing X until it becomes very symmetric, isomorphic to the fan of an X_j. In fact it turns out that the X_j endowed with a suitable polarization are K-polystable and asymptotically Chow polystable.

We will deduce from our result the following corollary:

COROLLARY 2.   *Let X be a smooth compact toric surface. Then there is a sequence of toric blow-ups of X* (*see Section 2*)

$$\mathrm{Bl}_{(p_m,\ldots,p_1)}(X) \to X$$

and a polarization $L_X \to \mathrm{Bl}_{(p_m,\ldots,p_1)}(X)$ *such that* $(\mathrm{Bl}_{(p_m,\ldots,p_1)}(X), L_X)$ *is K-polystable and asymptotically Chow-polystable.*

REMARK 3.   Donaldson has proved that a polarized toric surface (X, L) admits a cscK metric in the class $c_1(L)$ if and only if (X, L) is K-polystable [6]. However, it is a hard problem, even for toric surfaces, to check K-polystability. It contrasts with our explicit construction. We can compute from the fan describing the toric surface the points to blow-up in order to reach one of the $X_j's$.

Corollary 2 lies on deep results of Mabuchi [14] and Stoppa [20] and computation of the Lie algebra characters associated to Mabuchi's obstruction. This computation is simplified by the symmetries of the surfaces.

REMARK 4.   The idea of using symmetries on toric polytopes has also been used by Donaldson [5], in order to construct a 9 points blow-up of CP^2 that is not K-polystable for the associated polarization while the Futaki invariant vanishes. In particular, this polarized toric surface admits no cscK metric in the corresponding Kähler class.

We will also compute a bound on the number of blow-ups necessary to get a cscK metric with our construction. We will see by an example that this bound is far from being optimal, raising the natural question:

QUESTION 3.   What is the minimal number of blow-ups necessary to obtain a cscK metric on an iterated blow-up of the Hirzebruch surface F_n ?

We point out that, by Wang and Zhou [23], each toric surface admits an extremal metric. This metric is cscK if and only if the Futaki invariant vanishes [8], and an explicit formula for this invariant described by the polytope is known by the work of Donaldson [5]. However, this is far from answering the question since the Kähler classes of the extremal metrics and the Kähler classes where the Futaki invariant vanishes might be far from each other.

1.1.   Plan of the paper.   In Section 2, we recall results on blow-ups of toric surfaces. Then the proof of Theorem 1 is given in Section 3. The last section deals with stability issues related to our construction.

2.   Blow-ups of toric CSCK surfaces.

2.1.   Toric surfaces.   A toric surface X is a complex algebraic surface endowed with an effective action of a complex torus T^C of dimension 2, such that there exists an open and dense orbit in X. Thus X is a compactification of T^C and is obtained by gluing divisors to this torus. The structure of X is then entirely described by the way of glueing these divisors to the torus, and this procedure of gluing is encoded by a fan $\Sigma = \Sigma(X)$ in a $\mathbf{Z}$-lattice N of rank 2 [15]. A fan Σ is a set of strongly convex rational polyhedral cones σ in $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$ satisfying the gluing conditions:

- If $\sigma \in \Sigma$ and τ is a face of σ then $\tau \in \Sigma$.
- If $\sigma_1, \sigma_2 \in \Sigma$ then $\sigma_1 \cap \sigma_2$ is a face of σ_1 and σ_2.

To each cone σ in Σ, one can associate an affine toric variety $U_\sigma = \mathrm{Spec}(\mathbf{C}[\sigma^\vee \cap M])$, where M is the dual lattice of N and

$$\sigma^\vee = \{u \in M_{\mathbf{R}}; \langle u, x \rangle \geq 0 \text{ for all } x \in \sigma\}.$$

Then the conditions of the fan ensure that these affine charts are glued together and form a toric variety X.

This gives a simple description of these surfaces as each of their algebraic properties admits a combinatorial counterpart in terms of the fan.

We will denote by $\Sigma^{(i)}(X)$ the set of cones of dimension i in $\Sigma(X)$. We will identify the rays in $\Sigma^{(1)}(X)$ and primitive generators in N of these rays.

EXAMPLE 2.1.1.   Let $\boldsymbol{F}_n$ be the n-th Hirzebruch surface, that is the total space of the fibration

$$\boldsymbol{P}(\mathcal{O} \oplus \mathcal{O}(n)) \to \boldsymbol{CP}^1.$$

Note that $\boldsymbol{F}_0 = \boldsymbol{CP}^1 \times \boldsymbol{CP}^1$. For a suitable coordinate $N \simeq \boldsymbol{Z}^2$, we have $\Sigma^{(1)}(\boldsymbol{F}_n) = \{(0, -1), (1, 0), (0, 1), (-1, -n)\}$. As an example, $\boldsymbol{F}_2$ admits the fan description of Figure 1.
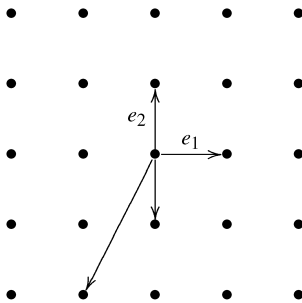

FIGURE 1. $\boldsymbol{F}_2$.

Here we only represent the elements of $\Sigma^{(1)}(\boldsymbol{F}_2)$ by arrows. The other cones of Σ are $\{0\}$ and the two dimensional cones delimited by the rays of $\Sigma^{(1)}(\boldsymbol{F}_n)$. We will use this description of a fan in the remaining of this paper.

2.2.   Blow-ups.   The blow-up of a toric surface $X(\Sigma)$ along a torus invariant orbit closure is described in term of the fan Σ. To each cone $\sigma \in \Sigma$, one can associate a torus invariant closed subvariety V_σ as the closure of $\mathrm{Spec}(\boldsymbol{C}[\sigma^\perp \cap M])$ in X, where $\sigma^\perp = \{u \in M_{\boldsymbol{R}}; \langle u, x \rangle = 0 \text{ for all } x \in \sigma\}$. Assume that σ is of dimension 2, then V_σ is a torus fixed point. Let τ be the ray in $N_{\boldsymbol{R}}$ generated by the sum of the primitive generators of one-dimensional faces of σ. Then τ induces a subdivision of σ into rational strictly convex polyhedral cones, $\sigma = \sigma' \cup \sigma''$. Then the blow-up of X at V_σ is the toric surface described by the fan $(\Sigma \setminus \{\sigma\}) \cup \{\sigma', \sigma'', \gamma\}$, where $\gamma = \sigma' \cap \sigma''$.

EXAMPLE 2.2.1.   Consider the cone $\sigma_1 \in \Sigma(\boldsymbol{F}_2)$ generated by $(0, 1)$ and $(-1, -n)$ in the lattice $N = \boldsymbol{Z}^2$. The blow-up of $\boldsymbol{F}_2$ at the point $p = V_{\sigma_1}$ is pictured on Figure 2.

We will say that a blow-up is *toric* if it is the blow-up at torus fixed points, so that the action of the torus lifts to the blown-up manifold. We introduce the following notation: If X is the blow-up of Y at a point $p \in Y$, we denote $X = \mathrm{Bl}_p(Y)$. Then by Proposition 2.2.2, as $\boldsymbol{F}_1 \simeq \mathrm{Bl}_p(\boldsymbol{CP}^2)$ for p a torus fixed point, any compact toric surface X different from $\boldsymbol{CP}^2$ can be written as $\mathrm{Bl}_{p_m}(\mathrm{Bl}_{p_{m-1}} \cdots (\mathrm{Bl}_{p_1}(\boldsymbol{F}_n)) \cdots)$ for some $n \geq 0$ and $p_{j+1} \in$
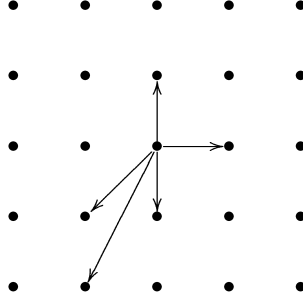
FIGURE 2. $\mathrm{Bl}_p(\boldsymbol{F}_2)$.

$\mathrm{Bl}_{p_j}(\mathrm{Bl}_{p_{j-1}} \cdots (\mathrm{Bl}_{p_1}(\boldsymbol{F}_n)) \cdots)$. We will use the notation $\mathrm{Bl}_{(p_m,\dots,p_1)}(\boldsymbol{F}_n) = \mathrm{Bl}_{p_m}(\mathrm{Bl}_{p_{m-1}} \cdots (\mathrm{Bl}_{p_1}(\boldsymbol{F}_n)) \cdots)$ and call it an *iterated blow-up*.

The combinatorial description of singular points and completness for toric surfaces lead to the following result [7, page 43]:

PROPOSITION 2.2.2.   *Each smooth compact toric surface is obtained from an iterated toric blow-up of* $\boldsymbol{CP}^2$ *or one of the Hirzebruch surfaces* $\boldsymbol{F}_n$.

2.3.   The Arezzo and Pacard theorem.   In the proof of Theorem 1, we will build a sequence of toric surfaces, each admitting a cscK metric. This sequence is built by successive blow-ups. We recall the following theorem of Arezzo and Pacard [1] on blow-ups of cscK manifolds:

THEOREM 2.3.1.   *Let* (X, ω) *be a compact cscK manifold and let* $(p_j)_{j=1,\dots,m}$ *be m distinct points of X. Assume that X admits no non-zero holomorphic vector field. Consider the blow-up of X at* (p_j):

$$\pi : \mathrm{Bl}_{(p_j)}(X) \to X \,.$$

Then for each strictly positive real numbers (a_j), *there exists* $\varepsilon_0 > 0$ *such that for each* $\varepsilon \in (0, \varepsilon_0)$, *there is a cscK metric on* $\mathrm{Bl}_{(p_j)}(X)$ *in the Kähler class*

$$\pi^*[\omega] - \varepsilon^2 \sum_j a_j \mathrm{PD}(E_j)$$

where $\mathrm{PD}(E_j)$ *denotes the Poincaré dual of the exceptional divisor* $E_j = \pi^{-1}(p_j)$.

In the second paper [2], Arezzo and Pacard proved a similar result when X admits holomorphic vector fields. However, in this case the blow-up points need to satisfy some stability and genericity conditions [2, Theorem 1.3]. Another way to use their theorem in this case is to perform the analysis modulo a finite group of automorphisms that preserves no holomorphic vector field. We will use a special case of [2, Theorem 1.4].

THEOREM 2.3.2.   *Let* (X, ω) *be a compact cscK manifold. Let* $G \subset \mathrm{Aut}(X)$ *be a finite group of isometries of* ω, *and* $G \cdot p^1 = \{p_1^1, \dots, p_{m_1}^1\}, \dots, G \cdot p^r = \{p_1^r, \dots, p_{m_r}^r\}$ *distinct*

orbits of points $(p^1, \ldots, p^r) \in X^r$. *Assume moreover that no non-zero holomorphic vector field is G-invariant. Consider the blow-up of X at* (p_j^i):

$$\pi : \mathrm{Bl}_{(p_j^i)}(X) \to X \,.$$

Then for any strictly positive real numbers $(a_i)_{i=1,\ldots,r}$, *there exists* $\varepsilon_0 > 0$ *such that, for each* $\varepsilon \in (0, \varepsilon_0)$, *there is a cscK metric* ω_ε *on* $\mathrm{Bl}_{(p_j^i)}(X)$ *in the Kähler class*

$$\pi^*[\omega] - \varepsilon^2 \sum_{i,j} a_i \mathrm{PD}(E_j^i) \,,$$

where $\mathrm{PD}(E_j^i)$ *denotes the Poincaré dual of the exceptional divisor* $E_j^i = \pi^{-1}(p_j^i)$. *Moreover*, *G lifts to a finite subgroup* $\tilde{G} \subset \mathrm{Aut}(\mathrm{Bl}_{(p_j^i)}(X))$ *of isometries of* ω_ε.

REMARK 2.3.3.   The result of Theorem 2.3.2 is slightly different from the result [2, Theorem 1.4]. We can choose the weights (a_i) of the cscK metric on the blown-up manifold to be the same as the initial weights (a_i). This is because under the hypothesis we've chosen, there is no non-zero holomorphic vector field, and thus the analysis of [2] reduces to the analysis in [1]. In this situation, the weights (a_i) need not to be perturbed for the result to hold.

3.   Proof of the main theorem.   Let X be a compact toric surface. We want to show that a torus equivariant blow-up of X admits a cscK metric. We will first build a sequence of toric surfaces X_j endowed with cscK metrics and then show that we can reach one of these X_j by an iterated blow-up of X.

3.1.   The surfaces X_j.   The X_j are built inductively. Let $X_0 = \boldsymbol{CP}^1 \times \boldsymbol{CP}^1$. Then X_{j+1} is the blow-up of X_j at its 2^{j+2} torus fixed points. Using the fan description of the toric blow-up process, the fans describing $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$, X_1 and X_2 are represented on Figures 3, 4 and 5.
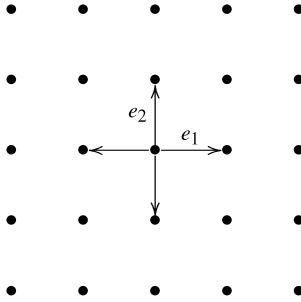

FIGURE 3. $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$.

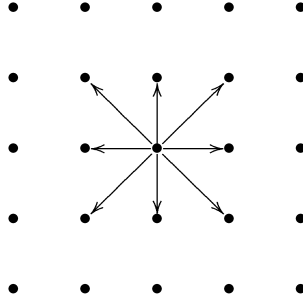PROPOSITION 3.1.1.   *Each of the X_j admits a cscK metric.*
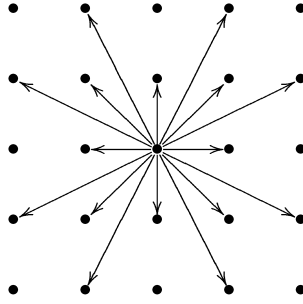
FIGURE 4. X_1.

FIGURE 5. X_2.

PROOF. We proceed by induction. First consider a cscK product metric ω_0 on $X_0 = \mathbf{CP}^1 \times \mathbf{CP}^1$ with the same volume on each factor, so that the associated polytope Δ_0 of the Kähler manifold is a square (see [10] for the description of a toric Kähler manifold in term of a polytope). This polytope is represented in Figure 6. Consider the subgroup G of $\mathrm{Aut}(X_0) = \mathbf{PGL}_2(\mathbf{C}) \times \mathbf{PGL}_2(\mathbf{C})$ generated by the following elements, described in terms of their action in homogeneous coordinates:

$$
\begin{aligned}
([u_1, u_2], [v_1, v_2]) &\longmapsto ([u_2, u_1], [v_1, v_2]), \\
([u_1, u_2], [v_1, v_2]) &\longmapsto ([u_1, u_2], [v_2, v_1]), \\
([u_1, u_2], [v_1, v_2]) &\longmapsto ([-u_1, u_2], [v_1, v_2]), \\
([u_1, u_2], [v_1, v_2]) &\longmapsto ([u_1, u_2], [-v_1, v_2]).
\end{aligned}
$$

By construction, G is a finite group of isometries of ω_0 as it is isomorphic to a finite group of symmetries of the associated polytope. Then the G-orbit of $([1, 0], [1, 0])$ consists of the four fixed points $\{p_1^0, \ldots, p_4^0\}$ under the torus action on X_0 induced by the $\mathbf{C}^*$-action on each $\mathbf{CP}^1$. No non-zero holomorphic vector field of X_0 is invariant under the G action, thus by Theorem 2.3.2, $X_1 = \mathrm{Bl}_{(p_1^0, \ldots, p_4^0)}(X_0)$ admits a cscK metric ω_1. Note that the weights of the exceptional divisors in X_1 are the same and have to be small by construction. A representation of the polytope Δ_1 associated to ω_1 is pictured in Figure 7. From Theorem 2.3.2, the metric
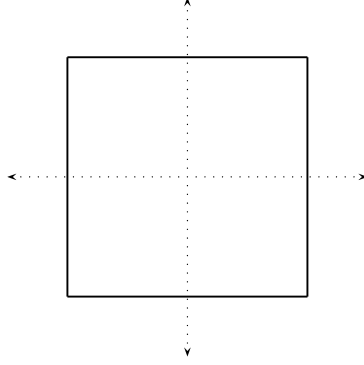
FIGURE 6.   Polytope Δ_0 associated to (X_0, ω_0).
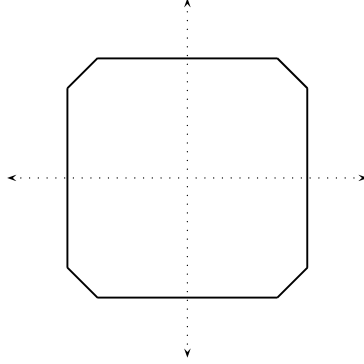
FIGURE 7.   Polytope Δ_1 associated to (X_1, ω_1).

ω_1 admits the lift of G as a subgroup of isometries. Let denote G_1 the lift of the subgroup of G generated by

$$([u_1, u_2], [v_1, v_2]) \quad \longmapsto \quad ([u_2, u_1], [v_1, v_2]),$$
$$([u_1, u_2], [v_1, v_2]) \quad \longmapsto \quad ([u_1, u_2], [v_2, v_1]).$$

This group corresponds to the symmetries of Δ_1 with respect to the two coordinate axes. The identity component of the group $\mathrm{Aut}(X_1)$ is the complex torus that defines the toric structure of X_1. Then no non-zero holomorphic vector field of X_1 is G_1-invariant. We apply Theorem 2.3.2 to (X_1, ω_1) blowing up the G_1-orbits of the fixed points under the torus action and choosing the weights (a_i) to be all the same. The surface obtained is exactly X_2, endowed with a cscK metric ω_2. Once again G_1 lifts to a group G_2 of isometries of ω_2 and no non-zero holomorphic vector field on X_2 is G_2-invariant. By induction, we build from a cscK metric ω_j on X_j, $j \geq 1$, a cscK metric ω_{j+1} on X_{j+1} such that the finite subgroup G_j of isometries of ω_j lifts to a finite subgroup of isometries G_{j+1} of ω_{j+1}. Note that for each $j \geq 1$, the Lie algebra of $\mathrm{Aut}(X_j)$ is that of the complex two torus that defines the toric

structure, and no non-zero holomorphic vector field is G_j-invariant. By iteration of Arezzo and Pacard theorem, i.e., Theorem 2.3.2, working modulo G_j at each step and blowing up the G_j-orbits of each torus fixed points, giving the same weight to each new exceptional divisor, we obtain the metrics ω_j. □

3.2. From X to X_j. To end the proof of the theorem, we will use the following two lemmas:

LEMMA 3.2.1. *For any sequence of m toric blow-ups of $\boldsymbol{F}_n$, $\mathrm{Bl}_{(p_m,...,p_1)}(\boldsymbol{F}_n)$, one of the following holds:*

- *$n = 0$,*
- *there is a sequence of m toric blow-ups of $\boldsymbol{F}_{n-1}$ such that $\mathrm{Bl}_{(p_m,...,p_1)}(\boldsymbol{F}_n) = \mathrm{Bl}_{(q_m,...,q_1)}(\boldsymbol{F}_{n-1})$, or*
- *there is a point $p_{m+1} \in \mathrm{Bl}_{(p_m,...,p_1)}(\boldsymbol{F}_n)$ such that there is a sequence of $m + 1$ toric blow-ups of $\boldsymbol{F}_{n-1}$ such that $\mathrm{Bl}_{(p_{m+1},...,p_1)}(\boldsymbol{F}_n) = \mathrm{Bl}_{(q_{m+1},...,q_1)}(\boldsymbol{F}_{n-1})$.*

LEMMA 3.2.2. *For any iterated toric blow-up $Y = \mathrm{Bl}_{(q_m,...,q_1)}(X_0)$ of X_0, there is an iterated toric blow-up of Y such that*

$$\mathrm{Bl}_{(q'_r,...,q'_1)}(Y) = X_j$$

for some j.

The classification of toric surfaces is the key for the proof of the theorem:

PROOF OF THEOREM 1. By Proposition 2.2.2, there exists a sequence of toric blow-ups such that $X = \mathrm{Bl}_{(p_m,...,p_1)}(\boldsymbol{F}_n)$. By iterating Lemma 3.2.1, there are toric blow-ups $\mathrm{Bl}_{(q_m,...,q_1)}(X)$ of X such that $\mathrm{Bl}_{(q_m,...,q_1)}(X)$ is an iterated blow-up of $\boldsymbol{F}_0 = X_0$. Then by Lemma 3.2.2 we can blow-up $\mathrm{Bl}_{(q_m,...,q_1)}(X) = \mathrm{Bl}_{(q'_r,...,q'_1)}(X_0)$ to reach one of the X_j. By Proposition 3.1.1, this blow-up of X has a cscK metric.

□

We end this section by the proof of the two lemmas:

PROOF OF LEMMA 3.2.1. Suppose $n \neq 0$. Consider the fan describing $\boldsymbol{F}_n$ on Figure 8.

Then a blow-up of $\boldsymbol{F}_n$ is described by adding a ray generated by the sum of two adjacent primitive vectors in $\Sigma^{(1)}(\boldsymbol{F}_n)$ to this fan. Each further blow-up is described by the same process so we can separate the blow-ups of $\boldsymbol{F}_n$ into two types, those arising by adding a ray in the cone $\sigma_1 = \boldsymbol{R}^+(0, 1) + \boldsymbol{R}^+(-1, -n)$, called type one blow-ups, and the other. These two types of blow-ups commute, we can perform every blow-up of type 1 before the other blow-ups to obtain $\mathrm{Bl}_{(p_m,...,p_1)}(\boldsymbol{F}_n)$. Then there are two possibilities. If there is no blow-up of type 1, we blow-up V_{σ_1} and we are in the third case of Lemma 3.2.1. Indeed, the blow-up of $\boldsymbol{F}_n$ at V_{σ_1} is a one-point blow-up of $\boldsymbol{F}_{n-1}$, (see Figure 9).

In the other case, the first blow-up of type one is necessarily at V_{σ_1} and we are in the second case of Lemma 3.2.2. □

C. TIPLER
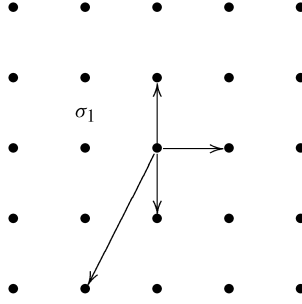
FIGURE 8. $\boldsymbol{F}_n$.
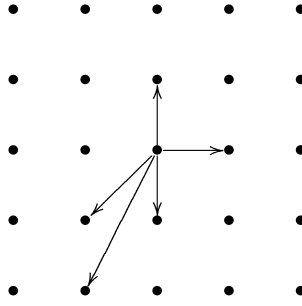
FIGURE 9. $\mathrm{Bl}_p(\boldsymbol{F}_n) = \mathrm{Bl}_q(\boldsymbol{F}_{n-1})$.

We need to introduce some definitions. We say that an iterated toric blow-up $\mathrm{Bl}_{(p_m,\dots,p_1)}(X)$ is a *purely iterated blow-up* if for each $i \geq 2$, p_i lies on the exceptional divisor E_{i-1} coming from the blow-up at p_{i-1}. In that case, we call m the *length* of this iterated blow-up. Let X be a toric surface and n be the minimal integer such that X is obtained from toric blow-ups of $\boldsymbol{F}_n$. Then we define the integer $l(X)$ to be the maximal length of purely iterated blow-ups arising in the description of X from $\boldsymbol{F}_n$.

PROOF OF LEMMA 3.2.2. Let $\Sigma^{(1)}(Y)$ be the set of rays of the fan of Y, identified with primitive generators of these rays. Then the fan description of a toric blow-up implies that there is a sequence of fans Σ_j such that $\Sigma_0 = \Sigma(X_0)$, $\Sigma_m = \Sigma(Y)$ and $\Sigma_{j+1}^{(1)}$ is obtained from $\Sigma_j^{(1)}$ by adding a ray generated by $v_i^j + v_{i+1}^j$, with v_i^j and v_{i+1}^j two adjacent elements of $\Sigma_j^{(1)}$. Consider the set of all $(m+1)$-tuples $(v_0, \dots, v_m)$ of $\Sigma_m^{(1)}$ such that $v_i \in \Sigma_i^{(1)}$, and for each $i \geq 1$, $v_i = v_{i-1} + w_{i-1}$ with $w_{i-1} \in \Sigma_{i-1}^{(1)}$ adjacent to v_{i-1} or $w_{i-1} = 0$. The case when $w_{i-1} \neq 0$ corresponds to the blow-ups. Denote $l(v_0, \dots, v_m)$ the number of distinct elements in $(v_0, \dots, v_m)$ and let $l_0(Y)$ be the maximum of the $l(v_0, \dots, v_m)$. Then $l_0(Y) = l(Y) + 1$. Then we can blow-up Y in order to make its fan symmetrical until we obtain the fan describing $X_{l(Y)}$. It is enough to add the rays that are missing so that we can

get every $(m+1)$-tuple $(v_0, \ldots, v_m)$ with $l_0(v_0, \ldots, v_m) = l_0(Y)$ which are coming from the set of rays of $X_{l(Y)}$. An example is pictured on Figure 10.
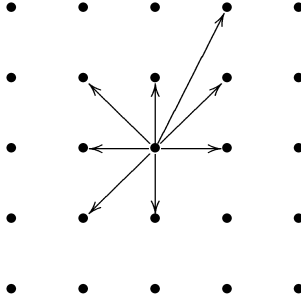

FIGURE 10. Y.

EXAMPLE 3.2.3.   On this example $l(Y) = 2$ because of the sequence $\{(0,1), (1,1), (1,2)\}$. We blow-up 4 times to add symmetries, as represented on Figure 11. We blow-up further to reach X_2.
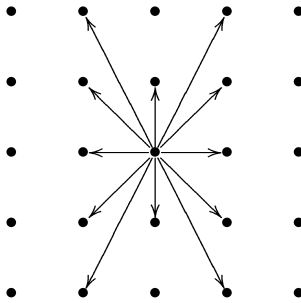
$\square$

FIGURE 11.   adding symmetries.

3.3.   Bound on the number of blow-ups.   We prove the following upper bound on the minimal number of blow-ups that are needed to get a cscK metric with our method:

PROPOSITION 3.3.1.   *Let X be a toric surface. Let n be the minimal integer such that X is obtained from toric blow-ups of $\boldsymbol{F}_n$. Then there is a sequence of at most*

$$2^{n+l_0(X)+1} - \sharp(\Sigma^{(1)}(X))$$

blow-ups of X such that the blown-up surface admits a cscK metric.

REMARK 3.3.2. Note that for $X = \boldsymbol{F}_n$ we have $l_0(X) = 1$ and $2^{n+2} - 4$ is the number of blow-ups necessary to go from $\boldsymbol{F}_n$ to X_n and in that case this estimate is sharp for our method.

PROOF. We need to estimate the number of blow-ups necessary to go from X to the nearest X_j. As X is a blow-up of $\boldsymbol{F}_n$ for n minimal, there is an n times blow-ups of X such that $\mathrm{Bl}_{(q_n,\dots,q_1)}(X)$ is a blow-up of $X_0 = \boldsymbol{F}_0$. Then from the proof of Lemma 3.2.2, we know that there exists an iterated blow-up from $\mathrm{Bl}_{(q_n,\dots,q_1)}(X)$ to $X_{n+l_0(X)-1}$. So we can go from X to $X_{n+l_0(X)-1}$ by an iterated blow-up. Then the number of blow-ups necessary is bounded by the number of rays in $\Sigma^{(1)}(X_{n+l_0(X)-1})$ minus the number of rays in $\Sigma^{(1)}(X)$, which gives the result. $\square$

This result on the minimal number of points to be blown-up in order to get a cscK metric is not sharp. Indeed, from [18, Section 4.1], we know that the special blow-up of $\boldsymbol{F}_2$ pictured on Figure 12 admits a cscK metric.

Here we do not need to blow-up further in order to reach X_2 (Figure 5). It suggests the following problem:

QUESTION 3.3.3. What is the minimal number of blow-ups of $\boldsymbol{F}_n$ necessary to ensure the existence of a cscK metric on the blown-up surface?

4. Blown-up surfaces and stabilities. In GIT, several notions of stability for polarized varieties are studied. It is in general a hard problem to check that a polarized variety is stable or not. However, the existence of a cscK metric enables us to conclude for K-stability and asymptotic Chow stability in some cases. In this section, we deduce from Theorem 1 that each toric surface can be blown up to satisfy K-stability and asymptotic Chow stability for some polarization.
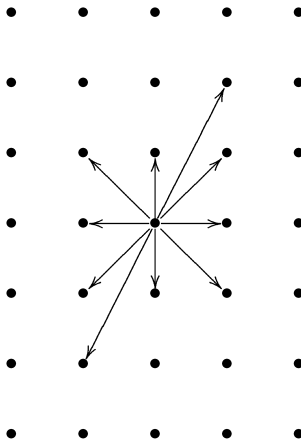

FIGURE 12. A special blow-up of $\boldsymbol{F}_2$.

4.1. X_j and K-stability. We refer the reader to [20] for the definition of K-stability. We can assume that the cscK metrics ω_j built on the X_j in Proposition 3.1.1 lie in rational classes. Up to scaling, we can assume that these metrics represent a polarization L_j of X_j, that is $[\omega_j] = c_1(L_j)$. By a result of Stoppa [20], the (X_j, L_j) are K-polystable. Together with Theorem 1 we have the following corollary.

COROLLARY 4.1.1. *Let X be a smooth compact toric surface. Then there is a sequence of toric blow-ups*

$$\mathrm{Bl}_{(p_m,\ldots,p_1)}(X) \to X$$

and a polarization $L_X \to \mathrm{Bl}_{(p_m,\ldots,p_1)}(X)$ such that $(\mathrm{Bl}_{(p_m,\ldots,p_1)}(X), L_X)$ is K-polystable.

4.2. Asymptotic Chow-polystability of X_j. We refer to [9] for the definition of asymptotic Chow-stability. By a theorem of Mabuchi [14], if (X, L) is a polarized Kähler manifold with cscK metric in the class $c_1(L)$ and if Mabuchi's obstruction vanishes, then (X, L) is asymptotically Chow polystable. From the work of Ono [16] and Futaki [9], Mabuchi's obstruction admits a simple description for toric varieties. Assume that (X, L) is a polarized toric manifold of complex dimension n, with L a torus equivariant ample line bundle. Suppose that the lattice N is identified with $\mathbf{Z}^n$. Then this polarization defines a polytope $\Delta \subset \mathbf{R}^n$ that encodes the symplectic structure of X [15]. Then set for each $k \in N$

$$\mathbf{s}_\Delta(k) = \sum_{\mathbf{a} \in k\Delta \cap \mathbf{Z}^n} \mathbf{e}(\mathbf{a})$$

and

$$E_\Delta(k) = \sharp(k\Delta \cap \mathbf{Z}^n).$$

Then

$$k \mapsto \mathrm{Vol}(\Delta)\mathbf{s}_\Delta(k) - kE_\Delta(k) \int_\Delta \mathbf{x}\, dv = \sum_i k^i \mathcal{F}_{\Delta,i}$$

is a polynomial in k with coefficients $\mathcal{F}_{\Delta,i}$, with dv the euclidian volume of $\mathbf{R}^n$. Then Futaki has shown that the vanishing of Mabuchi's obstruction is equivalent to the vanishing of the $\mathcal{F}_{\Delta,i}$.

LEMMA 4.2.1. *Let Δ_j be the polytope associated to the toric polarized surface (X_j, L_j) described in Section 4.1. Then $\mathcal{F}_{\Delta_j,1} = \mathcal{F}_{\Delta_j,2} = 0$.*

PROOF. This is a straightforward computation, using the symmetries of Δ_j with respect to the two coordinate axes. Note that $\mathcal{F}_{\Delta,1}$ is the Futaki invariant and thus necessarily vanishes by the existence of the cscK metric. □

From this lemma and Mabuchi's theorem we deduce the following corollary.

COROLLARY 4.2.2. *Let X be a smooth compact toric surface. Then there is a sequence of toric blow-ups of X*

$$\mathrm{Bl}_{(p_m,\ldots,p_1)}(X) \to X$$

and a polarization $L_X \to \mathrm{Bl}_{(p_m,\ldots,p_1)}(X)$ such that $(\mathrm{Bl}_{(p_m,\ldots,p_1)}(X), L_X)$ is asymptotically Chow-polystable.

REFERENCES

[1] C. AREZZO AND F. PACARD, Blowing up and desingularising constant scalar curvature Kähler manifolds, Acta Math. 196 (2006), no. 2, 179–228.

[2] C. AREZZO AND F. PACARD, Blowing up constant scalar curvature Kähler manifolds II, Ann. of Math. (2) 170 (2009), no. 2, 685–738.

[3] C. AREZZO, F. PACARD AND M. SINGER, Extremal metrics on blowups, Duke Math. J. 157 (2011), 1–51.

[4] E. CALABI, Extremal Kähler metrics, Seminars on Differential Geometry (S. T. Yau Ed.) pp. 259–290, Ann. of Math. Stud. 102, Princeton University Press, 1982.

[5] S. K. DONALDSON, Scalar curvature and stability of toric varieties, J. Differential Geom. 62 (2002), 289–349.

[6] S. K. DONALDSON, Constant scalar curvature metrics on toric surfaces, Geom. Func. Anal. 19 (2009), 83–136.

[7] W. FULTON, Toric varieties, Princeton University Press, 1993.

[8] A. FUTAKI, Kähler-Einstein metrics and integral invariants, Lecture Notes in Math. 1314, Springer-Verlag, Berlin, 1988.

[9] A. FUTAKI, Asymptotic Chow polystability in Kähler geometry, in Fifth International Congress of Chinese Mathematicians Part 1, pp. 139–153, AMS/IP Stud. Adv. Math. 51, Amer. Math. Soc., Providence, RI, 2012.

[10] V. GUILLEMIN, Kähler structures on toric varieties, J. Differential Geom. 40 (1994), 285–309.

[11] J.-S. KIM, C. LE BRUN AND M. PONTECORVO, Scalar-flat Kähler surfaces of all genera, J. Reine Angew. Math. 486 (1997), 69–95.

[12] C. LE BRUN AND M. SINGER, Existence and deformation theory for scalar-flat Kähler metrics on compact complex surfaces, Invent. Math. 112 (1993), 273–313.

[13] E. LEGENDRE, Toric Kähler-Einstein metrics and convex compact polytopes, preprint ArXiv 1112.3239.

[14] T. MABUCHI, An energy-theoretic approach to the Hitchin-Kobayashi correspondance for manifolds, I, Invent. Math. 159 (2005), 225–243.

[15] T. ODA, Convex bodies and algebraic geometry, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 15, Springer-Verlag Berlin, 1988.

[16] H. ONO, A necessary condition for Chow semistability of polarized toric manifolds, J. Math. Soc. Japan 63 (2011), no. 4, 1377–1389.

[17] Y. ROLLIN AND M. SINGER, Constant scalar curvature Kähler surfaces and parabolic polystability, J. Geom. Anal. 19 (2009), no. 1, 107–136.

[18] Y. ROLLIN AND C. TIPLER, Deformations of extremal toric manifolds, preprint ArXiv 1201.4137.

[19] S. R. SIMANCA, Kähler metrics of constant scalar curvature on bundles over CP_{n-1}, Math. Ann. 291 (1991), no.2, 239–246.

[20] J. STOPPA, K-stability of constant scalar curvature Kähler manifolds, Adv. Math. 221 (2009), no. 4, 1397–1408.

[21] C. H. TAUBES, The existence of anti-self-dual conformal structures, J. Differential Geom. 36 (1992), no. 1, 163–253.

[22] G. TIAN, Kähler-Einstein metrics with positive scalar curvature, Invent. Math. 137 (1997), 1–37.

[23] X. WANG AND B. ZHOU, On the existence and nonexistence of extremal metrics on toric Kähler surfaces, Adv. Math. 226 (2011), no. 5, 4429–4455.

[24] S.-T. YAU, Open problems in geometry, Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), 1–28, Proc. Sympos. Pure Math. 54, Part 1, Amer. Math. Soc., Providence, RI, 1993.

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