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A note on bounded positive entire solutions of semilinear elliptic equations

Manabu NAITO

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In this note we are concerned with bounded positive entire solutions of the second order semilinear elliptic equation

$$\Delta u + a(x)f(u) = 0, \quad x \in R^n, \tag{1}$$

where $n \geqq 3$ and Δ is the n-dimensional Laplace operator. By an entire solution of (1) we mean a function $u \in C^2(R^n)$ which satisfies (1) at every point of R^n. We assume throughout that $a(x)$ is a locally Hölder continuous function on R^n and $f(u)$ is a locally Lipschitz continuous function on $(0, \infty)$ which is positive and nondecreasing for $u > 0$. As usual, $|x|$ denotes the Euclidean length of $x \in R^n$.

Our result is the following:

THEOREM. *Suppose that there exist locally Hölder continuous functions $a_*(t)$ and $a^*(t)$ on $[0, \infty)$ such that*

$$-a_*(|x|) \leqq a(x) \leqq a^*(|x|) \quad \text{for} \quad x \in R^n; \tag{2}$$

$$a_*(t) \text{ and } a^*(t) \text{ are nonnegative for } t \geqq 0; \tag{3}$$

$$\int_0^\infty t a_*(t)dt = A_* < \infty \quad \text{and} \quad \int_0^\infty t a^*(t)dt = A^* < \infty. \tag{4}$$

Define the sets L_* and L^* by

$$L_* = \{\ell \mid \ell > 0 \text{ and } \ell - f(\ell)A_*(n-2)^{-1} > 0\}, \tag{5}$$

$$L^* = \{\ell \mid \ell = c - f(c)A^*(n-2)^{-1} > 0 \text{ for some } c > 0\}, \tag{6}$$

and suppose that $L_* \cap L^*$ is nonempty.

Then, for any $\ell \in L_* \cap L^*$, there exists an entire solution $u(x)$ of (1) *which is positive for $x \in R^n$ and satisfies*

$$u(x) \longrightarrow \ell \quad \text{as} \quad |x| \longrightarrow \infty. \tag{7}$$

Observe that, in the case of $f(u) = u^\gamma$, if $A_* = A^* > 0$ then the set $L_* \cap L^*$ becomes the interval:

$$L_* \cap L^* = (0, (1-\gamma^{-1})((n-2)/\gamma A^*)^{1/(\gamma-1)}] \quad \text{for} \quad \gamma > 1;$$

$$L_* \cap L^* = (0, \infty) \quad \text{if } A_* = A^* < n-2 \qquad \text{for} \quad \gamma = 1;$$

$$L_* \cap L^* = (((n-2)/A_*)^{1/(\gamma-1)}, \infty) \qquad \text{for} \quad 0 < \gamma < 1.$$

In [4] Ni proved that, when $f(u)=u^\gamma$ with $\gamma>1$, if $|a(x)| \leqq \phi^*(|x|)$ for $x \in R^n$ and

$$\phi^*(t) = O(t^p) \qquad \text{for} \quad p < -2 \quad \text{as } t \longrightarrow \infty, \tag{8}$$

then (1) has infinitely many positive entire solutions which are bounded and bounded away from zero in R^n, and moreover that if in addition either $a(x) \geqq 0$ or $a(x) \leqq 0$ for all $x \in R^n$, then (1) has infinitely many positive entire solutions which tend to positive constants as $|x| \to \infty$.

Recently Kawano [2] improved Ni's result by showing that, when $f(u)=u^\gamma$ with arbitrary non-zero γ (allowed to be negative), the same conclusion as Ni's holds even if condition (8) is replaced by the weaker one:

$$\int_0^\infty t\phi^*(t)dt < \infty \qquad \text{for} \quad \gamma \neq 1, \tag{9}$$

$$\int_0^\infty t\phi^*(t)dt < n-2 \qquad \text{for} \quad \gamma = 1. \tag{10}$$

Our result asserts more strongly that, when $f(u)=u^\gamma$ with γ positive, if Kawano's condition (9) or (10) is satisfied then not only infinitely many positive entire solutions which are bounded and bounded away from zero in R^n can be obtained, but also the limit of a positive entire solution as $|x| \to \infty$ can be arbitrarily specified in the interval $L_* \cap L^*$ as above. Furthermore our result asserts that the sign condition of $a(x)$ is unnecessary in proving the existence of positive entire solutions which tend to positive constants as $|x| \to \infty$.

Related results are also contained in [3].

For the proof of Theorem we make use of the following lemma.

LEMMA. *Suppose that there exist bounded positive functions* $w, v \in C^{2+\lambda}_{loc}(R^n)$, $\lambda \in (0, 1)$, *such that*

$$\Delta w + a(x)f(w) \geqq 0, \quad x \in R^n,$$

$$\Delta v + a(x)f(v) \leqq 0, \quad x \in R^n,$$

and

$$w(x) \leqq v(x), \quad x \in R^n.$$

Then (1) *has an entire solution* $u(x)$ *satisfying*

$$w(x) \leqq u(x) \leqq v(x), \quad x \in R^n. \tag{11}$$

This lemma was first proved by Akô and Kusano [1] and was recently proved by Ni [4] without the assumption of boundedness of w and v.

PROOF OF THEOREM. Let $\ell \in L_* \cap L^*$. From the definition we have $\ell > 0$, $\ell - f(\ell)A_*(n-2)^{-1} > 0$ and $\ell = c - f(c)A^*(n-2)^{-1}$ for some $c > 0$. Define the function $z(t)$ on $(0, \infty)$ by

$$z(t) = \ell - \frac{f(\ell)}{t^{n-2}} \int_0^t s^{n-3}\left(\int_s^\infty r a_*(r) dr\right) ds \qquad (t > 0).$$

It is easily seen that $z'(t) \geqq 0$ for $t > 0$, $z(t) \to \ell$ as $t \to \infty$, $z(t) \to \ell - f(\ell)A_*(n-2)^{-1}$ as $t \to 0$, and $(t^{n-1}z'(t))' = f(\ell)t^{n-1}a_*(t)$ for $t > 0$. Therefore the function $w(x)$ on R^n defined by

$$w(x) = z(|x|) \quad \text{for } x \neq 0; \; w(x) = \ell - f(\ell)A_*(n-2)^{-1} \quad \text{for } x = 0$$

satisfies $0 < \ell - f(\ell)A_*(n-2)^{-1} \leqq w(x) \leqq \ell$ for $x \in R^n$, and $w(x) \to \ell$ as $|x| \to \infty$. Moreover it is immediately verified that $w(x)$ is twice continuously differentiable in the whole space R^n and satisfies $\Delta w(x) = f(\ell)a_*(|x|) \geqq -a(x)f(w(x))$ for every $x \in R^n$.

On the other hand, the function $y(t)$ on $(0, \infty)$ defined by

$$y(t) = c - \frac{f(c)}{t^{n-2}} \int_0^t s^{n-3}\left(\int_0^s r a^*(r) dr\right) ds \qquad (t > 0)$$

has the properties that: $y'(t) \leqq 0$ for $t > 0$, $y(t) \to c - f(c)A^*(n-2)^{-1} = \ell$ as $t \to \infty$, $y(t) \to c$ as $t \to 0$, and $(t^{n-1}y'(t))' = -f(c)t^{n-1}a^*(t)$ for $t > 0$. It follows that the function $v(x)$ on R^n defined by

$$v(x) = y(|x|) \quad \text{for } x \neq 0; \; v(x) = c \quad \text{for } x = 0$$

satisfies $0 < \ell \leqq v(x) \leqq c$ for $x \in R^n$, $v(x) \to \ell$ as $|x| \to \infty$, and $\Delta v(x) = -f(c)a^*(|x|) \leqq -a(x)f(v(x))$ for every $x \in R^n$.

Thus we see that $w(x)$ and $v(x)$ satisfy all of the required conditions in the above lemma, and so we conclude that equation (1) has an entire solution $u(x)$ satisfying (11). Since $\lim_{|x| \to \infty} w(x) = \lim_{|x| \to \infty} v(x) = \ell$, (7) is clear. This completes the proof.

References

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Department of Mathematics,
Faculty of Education,
Tokushima University