

A Note on Category of Multisets (MUL)

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Abstract

The paper briefly describes the concept of *categories*, *multisets*, and *category of multisets*. The main contribution of this paper is the formulation of *consistency* criteria to hold for the existence of mono, epi, and iso-morphisms in *Mul*.

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1 Introduction

Category theory was formulated by Samuel Eilenberg and Saunders MacLane, and it has occupied a central position in most of the branches of mathematics, some areas of theoretical computer science and mathematical physics.

Category is an algebraic structure consisting of a collection of *objects*, linked together by a collection of *arrows (morphisms)* satisfying two basic properties: the ability to compose the arrows associatively and the existence of an identity arrow for each object. Objects and arrows may be abstract entities of any kind. The theory

generalizes all of mathematics in terms of objects and arrows, independent of what the objects and arrows represent. It basically studies structures in terms of mappings between them. This as well applies to categories themselves taken as structures with *functors* as mappings between them and further, such mappings can be applied between functors, called *natural transformations* (see [1], for various details).

On the other hand, we invariably encounter with systems which contain objects with repeated elements or attributes (for example, groups of people, systems of elementary particles, etc., having two or more elements with the same property). A (formal) mathematical structure is required to model this kind of data. In the recent years, such mathematical structures have been developed which are, in general, called *multiset-based structures*.

Applications of multisets abound, especially in mathematics and computer science, (see [2], for details).

Recently, considering multitude of applications of multisets on one hand and that of categories on the other, studies investigating categories of multisets have started getting attention (see [3], [4]). In this paper, our discussion will be largely confined to [3].

2 The concept of a multiset

A Multiset (mset, for short) is an unordered collection of objects in which, unlike a standard (Cantorian) set, duplicates or multiples of objects are admitted. In other words, an mset is a collection in which objects may appear more than once and each individual occurrence of an object is called its *element*. All duplicates of an object in an mset are indistinguishable. The objects of an mset are the distinguishable or distinct elements of the mset. The distinction made between the terms *object* and *element* does enrich the multiset language. However, use of the term *element* alone may suffice if there does not arise any confusion.

The numeric view of multisets (called *multinumber* in [3]) is represented by a numeric function α from a universal set D to a numeric set T as follows:

$\alpha : D \rightarrow T$ is

$$\left\{ \begin{array}{l} \text{a set if } T = \{0, 1\}, \text{ a two-valued Boolean algebra;} \\ \text{a multiset if } T = \mathbf{N}, \text{ the set of all non-negative integers;} \\ \text{a signed multiset if } T = \mathbf{Z}, \text{ the set of all integers.} \end{array} \right.$$

In this paper, we consider multisets in which each element occur a finite number of times. Despite the fact that the concept of multiset is a generalization of the concept of set, there are some facts of set theory which do not hold in multiset theory, especially Cantor's theorem and De Morgan's laws (see [2], for details).

The use of square brackets to represent an mset is quasi-general. Thus, an mset containing one occurrence of a , two occurrences of b , and three occurrences of c is notationally written as $[[a, b, b, c, c, c]]$ or $[a, b, b, c, c, c]$ or $[a, b, c]_{1,2,3}$ or $[a, 2b, 3c]$ or $[a.1, b.2, c.3]$ or $[1/a, 2/b, 3/c]$ or $[a^1, b^2, c^3]$ or $[a^1b^2c^3]$. For convenience, the curly brackets are also used in place of the square brackets.

Monro [3] applies *equivalence relation approach* to explicate the concept of multisets.

In [3], a multiset $[a, a, b]$ is regarded as being really of the form $[a, a', b]$ where a and a' are different objects of the same *sort* and b is of a different *sort*. Notationally, elements of distinct sorts are to be denoted by distinct letters and elements of the same sort be denoted by the same letter with a sequence of dashes distinguishing different elements of that sort.

Note that it was Herman Weyl [5] who first formulated equivalence relation approach to multisets and applied it to a variety of problems in Physics, Chemistry and Genetics. In ([5], p. 239), an *aggregate (mset, in Monro's work)* is defined as *a set of elements with an equivalence relation*. In fact, Monro's explication of multisets is an adaptation from Weyl [5], though with an entirely different motivation viz., defining a category of multisets.

In [3], a multiset A is formally defined as a pair $\langle A_0, \rho \rangle$ where A_0 is a set and ρ an equivalence relation on A_0 . The set A_0 is called *the field of the multiset*. Elements of A_0 in the same equivalence class will be said to be of the same *sort* and elements in different equivalence classes will be said to be of different *sorts*. For example, an mset $[a^2, b, c^3, d]$ will be represented as $[a, a', b, c, c', c'', d]$ where a, a' are of the same sort and c, c', c'' are also of the same sort, while b and d are of two different sorts. In other words, various equivalence classes determine the sorts.

It needs to be emphasized that representing an mset, suggested as above, does not make it a *set* rather only a *set-like* in which multiplicity of an object is represented by so-called different objects using a sequence of dashes and called the *elements* of the same sort. In fact, the set of *objects* giving rise to the mset in the above example is $\{a, b, c, d\}$, called the *root set* (see [2], for example). In our opinion, Monro's

representation of an mset as a set-like is an excellent mathematical explication of its concept which made defining mset morphisms and *Mul* possible.

3 Category of multisets (*Mul*)

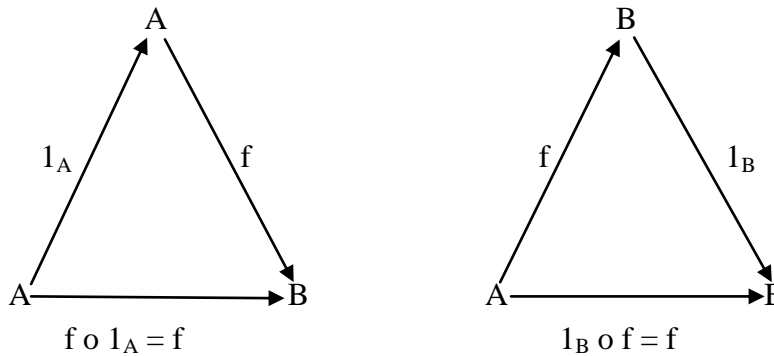
3.1 Multiset morphism and *Mul*

Monro [3] is a prototype work on *Mul*.

Definition 2.2 [3]: Let $X = \langle X_0, \rho \rangle$ and $Y = \langle Y_0, \sigma \rangle$ be multisets. A *morphism* of multisets is a function $f: X_0 \rightarrow Y_0$ which respects sorts; that is, if $x, x' \in X_0$ and $x\rho x'$, then $f(x)\sigma f(x')$. Usually, a multiset morphism is denoted $f: X \rightarrow Y$, suppressing explicit mention of X_0 and Y_0 . The category of multisets and multiset morphisms is denoted *Mul*.

In the same vein, an identity morphism of msets is defined as follows:

Let $A = \langle A_0, \rho \rangle$ be an mset. A morphism from A to A , defined $1_A(a) = a \forall a \in A_0$, is said to be the *identity morphism* for A and also, for every morphism $f: A \rightarrow B$, $f \circ 1_A = f$ and $1_B \circ f = f$, where 1_A and 1_B are the identity morphisms for A and B respectively illustrated below:



Typically, multisets can themselves be regarded as categories such that if A is a multiset and $a, b \in A$, there is an arrow from a to b if a and b are of the same sort, and no arrows from a to b if a and b are of different sorts.

3.2 Proposition 2.3 of [3]

Let $A = (A_0, \rho)$, $B = (B_0, \sigma)$ be msets. Let $f: A \rightarrow B$ be an mset morphism, and $f: A_0 \rightarrow B_0$ be the function between the fields. Then $f: A \rightarrow B$ is

- (i) a monomorphism in **Mul** iff $f: A_0 \rightarrow B_0$ is injective;
- (ii) an epimorphism in **Mul** iff $f: A_0 \rightarrow B_0$ is surjective;
- (iii) an isomorphism in **Mul** iff $f: A_0 \rightarrow B_0$ is bijective and also has the property that $a \rho a'$ iff $f(a) \sigma f(a')$, for all $a, a' \in A_0$.

Further, by taking an example $f: [a, b] \rightarrow [c, c']$, where a and b are of different sorts, c and c' of the same sort, $f(a) = c$ and $f(b) = c'$, it is shown that f is an mset morphism, and is both a monomorphism and an epimorphism, but not an isomorphism, which is unlike the case for Set isomorphism. This incongruous result shows that **Mul**, unlike **Set**, is not *balanced*.

In order to redress the aforesaid incongruousness, Monro [3] introduced an elegant notion of *strong monomorphism* and many other related notions and results.

Notes:

We observe that the propositions 2.3 of [3], mentioned in section 3.2 require some modifications as below:

Let $X = \langle X_0, \rho \rangle$ and $Y = \langle Y_0, \sigma \rangle$ be two msets. Then

- (i) $f: X \rightarrow Y$ is a monomorphism in **Mul** iff $f: X_0 \rightarrow Y_0$ is one-to-one, sort-preserving, and $|S_i| \leq |T_k|$ where S_i is a sort in X , T_k be the corresponding sort in Y , and $|S_i|$ is the cardinality of S_i , $|T_k|$ is the cardinality of T_k .

For example, $f: [a, a', a''] \rightarrow [b, b']$ can not be a monomorphism.

Note that objects of the same sort can not be mapped to objects of different sorts.

- (ii) $f: X \rightarrow Y$ is an epimorphism in **Mul** iff $f: X_0 \rightarrow Y_0$ is onto, sort-preserving, and $|S_i| \geq |T_k|$. For example, $f: [a, a'] \rightarrow [b, b', b'']$ can not be an epimorphism.
- (iii) $f: X \rightarrow Y$ is an isomorphism in **Mul** iff $f: X_0 \rightarrow Y_0$ is a bijection, sort-preserving, and $|S_i| = |T_k|$.

We choose to call the aforesaid conditions required to hold between the cardinalities of the sorts as the *consistency criteria* for the existence of mono, epi, and isomorphisms in **Mul**. The consistency criteria introduced in this paper are missing in [3].

4 Conclusions

Category, multiset, and **Mul** are briefly defined. The main contribution of the paper consists in introducing modifications in the definitions of mono, epi, and isomorphisms in **Mul**.

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