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A NOTE ON CHARACTERIZATIONS OF MULTIVARIATE STABLE DISTRIBUTIONS*

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Abstract. Several characterizations of multivariate stable distributions together with a characterization of multivariate normal distributions and multivariate stable distributions with Cauchy marginals are given. These are related to some standard characterizations of Marcinkiewicz.

Key words and phrases: Multivariate stable distributions, Cauchy distribution, normal distribution, Marcinkiewicz theorem.

1. Introduction

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two independent random vectors and let f be a function defined on an interval of R. The distribution of $\boldsymbol{X}$ and $\boldsymbol{Y}$ can be characterized according to the form of f such that the distribution of the random vector $\lambda\boldsymbol{X} + f(\lambda)\boldsymbol{Y}$ does not depend on λ, where λ takes on values in the domain of f. These results complement previously obtained characterizations where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are required to be *identically* distributed and for one value λ^*, $\lambda^*\boldsymbol{X} + f(\lambda^*)\boldsymbol{Y}$ has the same distribution as $\boldsymbol{X}$. Kagan *et al.* ((1973), Section 13.7) discuss such characterizations. All these results are related, in spirit, to the Marcinkiewicz theorem (see Kagan *et al.* (1973)), which simply says that under suitable conditions if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent and identically distributed, and $\lambda_1\boldsymbol{X} + \tau_1\boldsymbol{Y}$ and $\lambda_2\boldsymbol{X} + \tau_2\boldsymbol{Y}$ have the same distribution, then $\boldsymbol{X}$ and $\boldsymbol{Y}$ have a normal distribution. Note that, in fact, τ_i, $i = 1, 2$, cannot be arbitrary constants, but must satisfy $\tau_i = (1 - \lambda_i^2)^{1/2}$, $i = 1, 2$. One interesting question is if one relaxes the condition that $\boldsymbol{X}$ and $\boldsymbol{Y}$ be identically distributed, how much more is required about equidistribution of linear forms to characterize normality. One of our basic results shows that under suitable conditions if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent and $\lambda\boldsymbol{X} + f(\lambda)\boldsymbol{Y}$

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has the same distribution for all λ, then $\boldsymbol{X}$ and $\boldsymbol{Y}$ must be normal and $f(\lambda)$ has, up to scaling, the form $(1-\lambda^2)^{1/2}$. For a discussion of and references concerning the statistical motivation of the Marcinkiewicz result, see Kagan *et al.* ((1973), Section 2.1).

An additional application of our simplest result, Theorem 3.1, occurs in the context of the one-parameter fixed marginal problem (see Kimeldorf and Sampson (1975)). Suppose that $\boldsymbol{X} = (X_1, \ldots, X_p)'$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_p)'$ are independent random vectors with support R^p. Suppose we want to construct a one-parameter fixed marginal family of p-dimensional distributions of the form $\lambda \boldsymbol{X} + f(\lambda)\boldsymbol{Y}$, for some function f. Theorem 3.1 basically says that the only one-dimensional marginals possible are normal marginals and $f(\lambda)$ must be of the form $(1-\lambda^2)^{1/2}$.

Our remarks are also related to the problems of the distribution of the projection of a random vector and the stability of the distribution of that vector. It is known that if $\boldsymbol{X}$ has a multivariate stable distribution, then for any direction $a \neq 0$ of R^n, $a'\boldsymbol{X}$ has a univariate stable distribution, but the converse problem is still not completely solved. Press (1972) studies this problem and his paper motivates several other papers, e.g., Paulauskas (1976) and Marcus (1983). Paulauskas gives a counterexample that if all projections of $\boldsymbol{X}$ are stable-distributed, $\boldsymbol{X}$ is not necessarily multivariate stable-distributed in the case that the order is 1. Marcus gives a counterexample for the order less than 1, and considers the projections on subspaces of R^n such that the stable distributed projection conditions become necessary and sufficient.

The characterizations provided in this note can be interpreted in the context of characterizations of stable distributions by projections. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two independent n-dimensional random vectors. If $\boldsymbol{X}$ and $\boldsymbol{Y}$ both have multivariate stable distributions with the same order, then $\lambda\boldsymbol{X} + f(\lambda)\boldsymbol{Y}$ has a multivariate stable distribution for all constants λ and $f(\lambda)$. If, for all λ and $f(\lambda)$ and for any direction $a \neq 0$ of $\boldsymbol{R}^n$, $a'(\lambda\boldsymbol{X} + f(\lambda)\boldsymbol{Y})$ has a stable distribution, by the preceding discussion, $\boldsymbol{X}$ and $\boldsymbol{Y}$ are not necessarily multivariate stable-distributed. We would like to consider some conditions on $f(\lambda)$ such that the stable projection conditions become necessary and sufficient. The stable distribution of $\boldsymbol{X}$ and $\boldsymbol{Y}$ can be characterized according to the form of f such that the distribution of the random vector $\lambda\boldsymbol{X} + f(\lambda)\boldsymbol{Y}$ does not depend on λ, where λ takes values on the domain of f. These results then complement previously obtained characterizations of stability.

2. Multivariate stable distributions

A distribution function $\boldsymbol{F}$ is said to be a univariate stable distribution if for every $b_1, b_2 > 0$, and $-\infty < c_1, c_2 < \infty$, there corresponds $b > 0$, and $-\infty < c < +\infty$ such that for every y, $-\infty < y < +\infty$,

$$\boldsymbol{F}\left(\frac{y-c_1}{b_1}\right) * \boldsymbol{F}\left(\frac{y-c_2}{b_2}\right) = \boldsymbol{F}\left(\frac{y-c}{b}\right),$$

where * denotes the convolution operator. Lévy (1924) showed that a univariate stable distribution has a characteristic function (ch.f.) ϕ given by

$$\ln\phi(t) = i\mu t - \gamma|t|^\alpha \left[1 + i\beta\frac{t}{|t|}\omega(t,\alpha)\right], \tag{2.1}$$

where $-\infty < t < \infty$, with given $-\infty < \mu < \infty$, $-1 \leq \beta \leq 1$, $0 < \alpha < 2$, $t/|t| = 0$ at $t = 0$, and for all t, $\omega(t, \alpha) = \tan(\pi\alpha/2)$ if $\alpha \neq 1$, and $= (2/\pi)\ln|t|$ if $\alpha = 1$.

A random variable with a stable distribution can be characterized by the identical distribution of that random variable and a linear combination of n independent copies of that random variable, depending on the interrelationships of the coefficients of the linear form (for details, see Kagan *et al.* ((1973), Theorem 13.7.2)). Eaton (1966) characterizes a stable distribution by using the solution of a functional equation.

Let $\boldsymbol{x}$ denote a $n \times 1$ vector over real numbers. Analogous to the univariate case, an n-dimensional multivariate distribution function $\boldsymbol{G}$ is said to be an n-dimensional multivariate stable distribution function if for every pair of scalars $b > 0$ and real vectors c_1, c_2 of R^n, there correspond a scalar $b > 0$ and real vector c of R^n such that for every $x \in R^n$,

$$\boldsymbol{G}\left(\frac{x - c_1}{b_1}\right) * \boldsymbol{G}\left(\frac{x - c_2}{b_2}\right) = \boldsymbol{G}\left(\frac{x - c}{b}\right).$$

Lévy (1937) and Feldheim (1937) have given the general form for the ch.f. of a multivariate stable distribution under an integral form. The results of Press (1972) and then Paulauskas (1976) indicate that closed expression for the ch.f. of a multivariate stable distribution analogous to the univariate case are still unknown.

Gupta *et al.* (1989*a*) give conditions for stability of a multivariate distribution with all projections stable and Gupta *et al.* (1986*b*) give a characterization of multivariate stable distributions which can be viewed as a multivariate version of a result of Eaton (1966). In this note, we study other characterizations of multivariate stable distributions, and some related problems.

3. Main results

We first consider characterizations of the univariate normal distribution and Cauchy distribution; afterwards, multivariate versions are considered.

THEOREM 3.1. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two independent nondegenerate random variables with finite second moment, and let f be a given non-negative function defined on some interval of $\boldsymbol{R}$. If the distribution of $\lambda\boldsymbol{X} + f(\lambda)\boldsymbol{Y}$ does not depend on λ for all λ in the domain of f, then* (i) *$f(\lambda) = \sqrt{a - b\lambda^2}$ where $a, b > 0$, and* (ii) *$\boldsymbol{X}$ and $\boldsymbol{Y}$ have normal distributions with means zero, and with variances $\sigma_{\boldsymbol{X}}^2$ and $\sigma_{\boldsymbol{Y}}^2$, respectively, where $\sigma_{\boldsymbol{X}}^2 = b\sigma_{\boldsymbol{Y}}^2$.*

PROOF. Let $\mu_{\boldsymbol{X}}$ and $\mu_{\boldsymbol{Y}}$ be the means and $\sigma_{\boldsymbol{X}}^2$ and $\sigma_{\boldsymbol{Y}}^2$ be the variances of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively. Then,

$$E[\lambda\boldsymbol{X} + f(\lambda)\boldsymbol{Y}] = \lambda\mu_{\boldsymbol{X}} + f(\lambda)\mu_{\boldsymbol{Y}} = h, \tag{3.1}$$

and

$$\mathrm{Var}[\lambda\boldsymbol{X} + f^2(\lambda)\boldsymbol{Y}] = \lambda^2\sigma^2 + f^2(\lambda)\sigma_{\boldsymbol{Y}}^2 = \sigma_0^2, \tag{3.2}$$

where h and σ_0^2 are two constants not depending on λ, since the distribution of $\lambda\boldsymbol{X} + f(\lambda)\boldsymbol{Y}$ does not depend on λ. By (3.1), $\mu_{\boldsymbol{X}} = 0$, if and only if $\mu_{\boldsymbol{Y}} = 0$, and then $h = 0$. Suppose $\mu_Y \neq 0$ (and then $\mu_X \neq 0$). By (3.1), $f(\lambda) = (h/\mu_Y) - (\mu_X/\mu_Y)\lambda$. Substitute in (3.2) to obtain that

$$\lambda^2\sigma_X^2 + \left(\frac{h}{\mu_Y} - \frac{\mu_X}{\mu}\lambda\right)^2 \sigma_Y^2 = \left(-\sigma_X^2 + \frac{\mu_X^2}{\mu_Y^2}\sigma_Y^2\right)\lambda^2 - 2h\frac{\mu_X}{\mu_Y}\sigma_Y^2\lambda + \frac{h^2}{\mu_Y^2}$$

depends on λ. This contradiction yields that, $\mu_X = \mu_Y = 0$, and, therefore, $f(\lambda)$ is defined by (3.2), giving

$$f(\lambda) = \sqrt{\frac{\sigma_0^2}{\sigma_Y^2} - \frac{\sigma_X^2}{\sigma_Y^2}\lambda^2} = \sqrt{a - b\lambda^2} \quad \text{where} \quad a = \frac{\sigma_0^2}{\sigma_Y^2}, \quad b = \frac{\sigma_X^2}{\sigma_Y^2}.$$

The domain of $f(\lambda)$ is the interval $[-\sqrt{a/b}, \sqrt{a/b}]$.

For $\lambda = 0$ or $\lambda = \sqrt{a/b}$, the two independent r.v.'s $\sqrt{a}\boldsymbol{Y}$ and $\sqrt{a/b}\boldsymbol{X}$ are identically distributed according to

$$\lambda \boldsymbol{X} + \sqrt{a - b\lambda^2}\,\boldsymbol{Y} = \left(\lambda\sqrt{\frac{b}{a}}\right)\left(\sqrt{\frac{a}{b}}\boldsymbol{X}\right) + \left(\sqrt{1 - \frac{b}{a}\lambda^2}\right)(\sqrt{a}\boldsymbol{Y}).$$

Hence, by Kagan *et al.* ((1973), Theorem 13.7.2(ia)), $\sqrt{a/b}\boldsymbol{X}$ and $\sqrt{a}\boldsymbol{Y}$ have normal distributions, and then $\boldsymbol{X}$ and $\boldsymbol{Y}$ have normal distributions with means zero and with variances σ_X^2 and σ_Y^2, respectively, where $\sigma_X^2 = b\sigma_Y^2$.

A multivariate version of Theorem 3.1 is given by Theorem 3.2.

THEOREM 3.2. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two independent nondegenerate n-dimensional random vectors with finite covariance matrices and let f be a non-negative function with domain an interval of R. If the distribution of $\lambda\boldsymbol{X} + f(\lambda)\boldsymbol{Y}$ does not depend on λ for all λ in the domain of f, then* (i) $f(\lambda) = (a - b\lambda^2)^{1/2}$ *for some $a, b > 0$, and* (ii) *$\boldsymbol{X}$ and $\boldsymbol{Y}$ have multivariate normal distributions with mean vectors zero vector, and covariance matrices Σ_X and Σ_Y, respectively, where $\Sigma_X = b\Sigma_Y$.*

PROOF. Let $c \neq 0$ be an arbitrary vector of R^n. The distribution of $c'(\lambda\boldsymbol{X} + f(\lambda)\boldsymbol{Y}) = \lambda c'\boldsymbol{X} + f(\lambda)c'\boldsymbol{Y}$ then does not depend on λ for all λ in the domain of f. By Theorem 3.1, $f(\lambda) = \sqrt{a - b\lambda^2}$, where $b = \sigma_{c'X}^2/\sigma_{c'Y} = c'\Sigma_X c/c'\Sigma_Y c$, for all $c \neq 0$ of R^n. Hence, $\Sigma_X = b\Sigma_Y$. Also by Theorem 3.2, $c'\boldsymbol{X}$ and $c'\boldsymbol{Y}$ have normal distributions with means $c'\mu_X = c'\mu_Y = 0$ for every $c \neq 0$ of R^n. Then $\boldsymbol{X}$ and $\boldsymbol{Y}$ have multivariate normal distributions with mean vectors being the zero vector and with covariance matrices Σ_X and Σ_Y, respectively, where $\Sigma_X = b\Sigma_Y$.

A Cauchy (μ, δ) distribution, where $-\infty < \mu < \infty$, $\delta > 0$, is a distribution with p.d.f. $g(x) = (\delta/\pi)(1/(\delta^2 + (x - \mu)^2))$ and having ch.f. $\phi(t) - e^{i\mu t - \delta|t|}$. From (2.1), it is seen that a Cauchy distribution is a stable distribution with $\alpha = 1$ and with $\beta = 0$.

The following result provides a characterization for univariate Cauchy distributions.

THEOREM 3.3. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be independent nondegenerate random variables with ch.f.'s ϕ_X, ϕ_Y, respectively, satisfying* (a) *$\phi'_{\boldsymbol{X}}$, $\phi'_{\boldsymbol{Y}}$ exist and are continuous on $(0,\infty)$, but discontinuous at 0, and* (b) *the limits of $\phi'_{\boldsymbol{X}}(t)$ and $\phi'_{\boldsymbol{Y}}(t)$ exist and are different from zero when $t \to 0+$. Let f be a non-negative function defined on an interval of $[0,\infty)$, containing zero, with $f'(\lambda)$ existing for every λ in the domain of f. If the distribution of $\lambda \boldsymbol{X} + f(\lambda)\boldsymbol{Y}$ does not depend on λ for all λ in the domain of f, then* (i) *$f(\lambda) = a - b\lambda$ for $0 \le \lambda \le a/b$ for some $a, b > 0$, and* (ii) *$\boldsymbol{X}$ and $\boldsymbol{Y}$ have Cauchy distributions with parameters (μ_1, δ_1) and (μ_2, δ_2), repectively, where $\mu_1 = b\mu_2$ and $\delta_1 = b\delta_2$.*

PROOF. By hypothesis, the ch.f. $\phi_{\lambda \boldsymbol{X}+f(\lambda)\boldsymbol{Y}}(t) = \phi_X(\lambda t)\phi_Y(f(\lambda)t)$ does not depend on λ. For every $t > 0$ and for every $\lambda > 0$ in the domain of f, take the derivative of $\phi_{\lambda \boldsymbol{X}+f(\lambda)\boldsymbol{Y}}(t)$ with respect to λ, to obtain that

$$\phi'_X(\lambda t)\phi_{\boldsymbol{Y}}(f(\lambda)t) + f'(\lambda)\phi_{\boldsymbol{X}}(\lambda t)\phi'_Y(f(\lambda)t) = 0. \tag{3.3}$$

Let $t \to 0+$ and set $\lim_{t\to 0+} \phi'_{\boldsymbol{X}}(t) = i\mu_1 - \delta_1$, $\lim_{t\to 0+} \phi'_{\boldsymbol{Y}}(t) = i\mu_2 - \delta_2$. Then the limit of (3.3) is a differential equation in $f(\lambda)$, namely, $(i\mu_1 - \delta_1) + f'(\lambda)(i\mu_2 - \delta_2) = 0$, or equivalently, $f'(\lambda) = -(i\mu_1 - \delta_1)/(i\mu_2 - \delta_2) = -b$, where b is a real number. Hence, combining with the given conditions of f, $f(\lambda) = a - b\lambda$, for $0 \le \lambda \le a/b$, $a > 0$, $b > 0$.

To find the distributions of $\boldsymbol{X}$ and $\boldsymbol{Y}$, we observe that by setting $\lambda = 0$ and $\lambda = a/b$, we obtain that the independent random variables $a\boldsymbol{Y}$ and $(a/b)\boldsymbol{X}$ have the same distribution as $(\lambda(b/a))((a/b)X) + (1 - (b/a)\lambda)(aY)$. Hence, by Kagan *et al.* ((1973), Theorem 13.7.2(ib)), $a\boldsymbol{Y}$ and $(a/b)\boldsymbol{X}$ have Cauchy distributions, so that $\boldsymbol{X}$ and $\boldsymbol{Y}$ have Cauchy distributions with parameters (μ_1, δ_1) and (μ_2, δ_2), respectively, where $\mu_1/\mu_2 = \delta_1/\delta_2 = b$.

Remark 3.1. In order to find the distributions of $\boldsymbol{X}$ and $\boldsymbol{Y}$, we can directly use (3.3) instead of using the result in Kagan *et al.* (1973). Substitute $f(\lambda) = a - b\lambda$ in (3.3), and let $\lambda \to 0+$, to get a differential equation in $\phi_{\boldsymbol{Y}}(t)$, $(i\mu_1 - \delta_1)\phi_{\boldsymbol{Y}}(\text{at}) - b\phi'_{\boldsymbol{Y}}(\text{at}) = 0$. Hence, $\phi_{\boldsymbol{Y}}(t) = \exp[(i\mu_1 t/b) - (\delta_1/b)t]$ for $t \ge 0$. Substituting the value of $\phi_{\boldsymbol{Y}}(t)$ again back into (3.3), we obtain a differential equation in $\phi_X(t)$. The result follows by solving for $\phi_{\boldsymbol{X}}(t)$, for $t > 0$. Similar techniques can be used in the second part of the proof of Theorem 3.1 to find the ch.f. of the distribution of $\boldsymbol{X}$.

THEOREM 3.4. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be independent n-dimensional random vectors with ch.f.'s $\phi_{\boldsymbol{X}}(t_1,\dots,t_n)$, $\phi_{\boldsymbol{Y}}(t_1,\dots,t_n)$, respectively, satisfying* (a) *$(\partial/\partial t_i)\cdot \phi_{\boldsymbol{X}}(t_1,\dots,t_n)$, $(\partial/\partial t_i)\phi_{\boldsymbol{Y}}(t_1,\dots,t_n)$ exist for all $t_i \ne 0$ and are continuous, except at $t_i = 0$, for $i = 1,\dots,n$; and* (b) *$\lim_{t_i\to 0+}(\partial/\partial t_i)\phi_{\boldsymbol{X}}(0,\dots,t_i,0,\dots,0)$ and $\lim_{t_i\to 0+}(\partial/\partial t_i)\phi_{\boldsymbol{Y}}(0,\dots,t_i,0,\dots,0)$ exist and are different from zero. Let f be a given non-negative function defined on an interval of $[0,\infty)$, containing*

0, and suppose that $f'(\lambda)$ exists for every λ in the domain of f. If the distribution of $\lambda \boldsymbol{X} + f(\lambda)\boldsymbol{Y}$ does not depend on λ for all λ in the domain of f, then (i) *$f(\lambda) = a - b\lambda$, for $0 \le \lambda \le a/b$, for some $a, b > 0$, and* (ii) *$\boldsymbol{X}$ and $\boldsymbol{Y}$ have multivariate stable distributions with Cauchy marginals.*

PROOF. For an arbitrary nonzero vector c of R^n, by hypothesis, the distribution of $c'(\lambda\boldsymbol{X} + f(\lambda)\boldsymbol{Y}) = \lambda(c'\boldsymbol{X}) + f(\lambda)(c'\boldsymbol{Y})$ does not depend on λ, for all λ in the domain of f. Then by Theorem 3.3, $f(\lambda) = a - b\lambda$, $0 \le \lambda \le a/b$ for some $a, b > 0$, and $c'\boldsymbol{X}$ and $c'\boldsymbol{Y}$ have Cauchy distributions with ch.f.'s

$$\phi_{c'\boldsymbol{X}}(t) = \phi_{\|c\|(c'/\|c\|)\boldsymbol{X}}(t) = e^{i\mu_1(c/\|c\|)\|c\|t - \delta_1(c/\|c\|)\|c\||t|}$$

and

$$\phi_{c'\boldsymbol{Y}}(t) = e^{i\mu_2(c/\|c\|)\|c\|t - \delta_2(c/\|c\|)\|c\||t|},$$

where $\mu_1(c/\|c\|) = b\mu_2(c/\|c\|)$, and $\delta_1(c/\|c\|) = b\delta_2(c/\|c\|)$ for every c of $\boldsymbol{R}^n$, and b does not depend on c. Hence, for every $t = (t_1, \ldots, t_n)'$ of $\boldsymbol{R}^n$.

$$\phi_{\boldsymbol{X}}(t) = \phi_{t'\boldsymbol{X}}(1) = e^{i\mu_1(t/\|t\|)\|t\| - \delta_1(t/\|t\|)\|t\|} \quad \text{and}$$
$$\phi_{\boldsymbol{Y}}(t) = \phi_{t'\boldsymbol{Y}}(1) = e^{i\mu_2(t/\|t\|)\|t\| - \delta_2(t/\|t\|)\|t\|},$$

where μ_1, μ_2, δ_1, δ_2 are continuous functions on the unit ball $\boldsymbol{B} = \{t : \|t\| = 1\}$ of R^n, $\delta_1, \delta_2 \ge 0$ and $\mu_1 = b\mu_2$, $\delta_1 = b\delta_2$, μ_1, μ_2 are odd functions and δ_1, δ_2 are even functions of t.

Since $\boldsymbol{X}$ and $\boldsymbol{Y}$ have a ch.f. under the form $\phi(t) = e^{i\mu(t/\|t\|)\|t\| - \delta(t/\|t\|)\|t\|}$, then for any pairs of scalars $b_1, b_2 > 0$ and real vectors c_1, c_2 of $\boldsymbol{R}^n$, and $\boldsymbol{t}$ of $\boldsymbol{R}^n$,

$$\begin{aligned} & e^{i(c_1+c_2)'t}\phi\left(\frac{t/b_1}{\| t/b_1 \|}\right)\phi\left(\frac{t/b_2}{\| t/b_2 \|}\right) \\ &= e^{i(c_1+c_2)'t} e^{i\mu(t/\|t\|)(1/b_1+1/b_2)\|t\|} e^{-\delta(t/\|t\|)(1/b_1+1/b_2)\|t\|} \\ &= e^{ic't}\phi\left(\frac{t/b}{\| t/b \|}\right), \quad \text{where} \quad c = c_1 + c_2 \quad \text{and} \quad \frac{1}{b} = \frac{1}{b_1} + \frac{1}{b_2}. \end{aligned}$$

Thus, $\boldsymbol{X}$ and $\boldsymbol{Y}$ have multivariate stable distributions with Cauchy marginals.

Remark 3.2. In the case when the projection of $\boldsymbol{X}$ on any direction $a \ne 0$ has a stable distribution with $\alpha \le 1$, Marcus (1983) points out that it is not necessary that the distribution of $\boldsymbol{X}$ be a multivariate stable distribution. Gupta *et al.* (1989*a*) give necessary and sufficient conditions for the stability of $\boldsymbol{X}$ based on projections of $\boldsymbol{X}$; they show that if all projections of $\boldsymbol{X}$ are Cauchy distributed, then the distribution is multivariate stable.

The following two results provides characterizations, respectively, for univariate and multivariate stable distributions.

THEOREM 3.5. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two independent random variables and let $f(\lambda)$ be a non-negative function defined on an interval of $\boldsymbol{R}$. Suppose that the* *distribution of* $\lambda \boldsymbol{X} + f(\lambda)\boldsymbol{Y}$ *does not depend on* λ. *Then* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *have stable distributions with order* α, $0 < \alpha \leq 2$ *and with parameters* $(0, \gamma_1, \beta)$ *and* $(0, \gamma_2, \beta)$, *respectively, for* $\alpha \neq 1$, *or* $(\mu_1, \gamma_1, 0)$ *and* $(\mu_2, \gamma_2, 0)$, *respectively, for* $\alpha = 1$, *if and only if* $f(\lambda) = (a - b|\lambda|^\alpha)^{1/\alpha}$ *for some* $a, b > 0$ *and* $b = \gamma_1/\gamma_2$ *for* $\alpha \neq 1$, *and* $b = \mu_1/\mu_2 = \gamma_1/\gamma_2$, *for* $\alpha = 1$.

PROOF. Suppose $\boldsymbol{X}$ and $\boldsymbol{Y}$ have stable distributions with degree $\alpha \neq 1$, with parameters $0, \gamma_1, \beta$ and $0, \gamma_2, \beta$, respectively. Then their respective ch.f.'s are $\phi_{\boldsymbol{X}}(t) = e^{\gamma_1 |t|^\alpha (1+i\beta \tan(\pi\alpha/2))}$ and $\phi_{\boldsymbol{Y}}(t) = e^{\gamma_2 |t|^\alpha (1+i\beta \tan(\pi\alpha/2))}$. Since the distribution of $\lambda \boldsymbol{X} + f(\lambda)\boldsymbol{Y}$ does not depend on λ, its ch.f.

$$\phi_{\lambda \boldsymbol{X}+f(\lambda)\boldsymbol{Y}}(t) = \phi_{\boldsymbol{X}}(\lambda t)\phi_{\boldsymbol{Y}}(f(\lambda)t) = e^{(\gamma_1\lambda^\alpha + \gamma_2 f^\alpha(\lambda))|t|^\alpha(1+i\beta\tan(\pi\alpha/2))}$$

does not depend on λ. Hence, $\gamma_1\lambda^\alpha + \gamma_2 f^\alpha(\lambda) = c$, where c is a constant not depending on λ. Then by the given conditions of f, $f(\lambda) = (c/\gamma_2 - (\gamma_1/\gamma_2)|\lambda|^\alpha)^{1/\alpha} = (a - b|\lambda|^\alpha)^{1/\alpha}$, where $a = c/\gamma_2 > 0$, $b = \gamma_1/\gamma_2 > 0$, for $|\lambda| \leq (a/b)^{1/\alpha}$.

Conversely, let $f(\lambda) = (a - b|\lambda|^\alpha)^{1/\alpha}$ where $a, b > 0$, for $|\lambda| \leq (a/b)^{1/\alpha}$, and we have that the distribution of $\lambda\boldsymbol{X} + (a - b\lambda^\alpha)^{1/\alpha}\boldsymbol{Y}$ does not depend on λ. For $\lambda = 0$ or $\lambda = (a/b)^{1/\alpha}$, the corresponding independent r.v.'s $a^{1/\alpha}\boldsymbol{Y}$ and $(a/b)^{1/\alpha}\boldsymbol{X}$ are identically distributed with

$$\lambda\boldsymbol{X} + (a - b\lambda^\alpha)^{1/\alpha}\boldsymbol{Y} = \lambda\left(\frac{b}{a}\right)^{1/\alpha}\left(\left(\frac{a}{b}\right)^{1/\alpha}\boldsymbol{X}\right) + \left(1 - \frac{b}{a}\lambda^\alpha\right)^{1/\alpha}(a^{1/\alpha}\boldsymbol{Y}).$$

Then by Kagan *et al.* ((1973), Theorem 13.7.2(c)) the distributions of $\boldsymbol{X}$ and $\boldsymbol{Y}$ are stable distributions with parameters $0, \gamma_1, \beta$ and $0, \gamma_2, \beta$. In the case $\alpha = 2$, $\boldsymbol{X}$ and $\boldsymbol{Y}$ have normal distributions with means zero and variances $2\gamma_1$, and $2\gamma_2$, where $2\gamma_1 = b(2\gamma_2)$.

For the case $\alpha = 1$, the proof follows exactly similar as in the case $\alpha \neq 1$, with $f(\lambda) = a - b|\lambda|$, $|\lambda| \leq b/a$ and $\boldsymbol{X}$ and $\boldsymbol{Y}$ now having stable distributions with parameters $(\mu_1, \gamma_1, 0)$ and $(\mu_2, \gamma_2, 0)$, that is, with Cauchy distributions with parameters (μ_1, γ_1) and (μ_2, γ_2), where $\mu_1 = b\mu_2$ and $\gamma_1 = b\gamma_2$.

THEOREM 3.6. *Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be two independent random vectors and let* $f(\lambda)$ *be a non-negative function defined on an interval of* $\boldsymbol{R}$. *Suppose that the distribution of* $\lambda\boldsymbol{X} + f(\lambda)\boldsymbol{Y}$ *does not depend on* λ. *Then* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *have multivariate stable distributions of order* α *if and only if* $f(\lambda) = (a - b|\lambda|^\alpha)^{1/\alpha}$ *for some* $a, b > 0$.

PROOF. If X and Y have multivariate stable distributions of order α, for an arbitrary nonzero vector c of R^n, $c'\boldsymbol{X}$ and $c'\boldsymbol{Y}$ have stable distributions of order α, and by hypothesis, the distribution of $c'(\lambda\boldsymbol{X} + f(\lambda)\boldsymbol{Y}) = \lambda(c'\boldsymbol{X}) + f(\lambda)(c'\boldsymbol{Y})$ does not depend on λ for all λ in the domain of f. By Theorem 3.5, under the condition that f is a non-negative function defined on an interval of R, $f(\lambda) = (a - b|\lambda|^\alpha)^{1/\alpha}$ for some $a, b > 0$.

Conversely, if $f(\lambda) = (a - b|\lambda|^\alpha)^{1/\alpha}$ for some $a, b > 0$, by Theorem 3.5, for any nonzero c of R^n, the distribution of $c'\boldsymbol{X}$ and $c'\boldsymbol{Y}$ are stable with ch.f.

$$\phi_{c'\boldsymbol{X}}(t) = \phi_{\|c\|(c'/\|c\|)\boldsymbol{X}}(t) = e^{-\gamma_1(c/\|c\|)(\|c\||t|)^\alpha(1+i\beta(c/\|c\|)\tan(\pi\alpha/2))},$$
$$\phi_{c'\boldsymbol{Y}}(t) = \phi_{\|c\|(c'/\|c\|)\boldsymbol{Y}}(t) = e^{-\gamma_2(c/\|c\|)(\|c\||t|)^\alpha(1+i\beta(c/\|c\|)\tan(\pi\alpha/2))},$$

if $0 < \alpha < 2$, $\alpha \neq 1$; and

$$\phi_{c'\boldsymbol{X}}(t) = \phi_{\|c\|(c'/\|c\|)\boldsymbol{X}}(t) = e^{i\mu_1(c/\|c\|)\|c\||t| - \gamma_1(c/\|c\|)\|c\||t|},$$
$$\phi_{c'\boldsymbol{Y}}(t) = \phi_{\|c\|(c'/\|c\|)\boldsymbol{Y}}(t) = e^{i\mu_2(c/\|c\|)\|c\||t| - \gamma_2(c/\|c\|)\|c\||t|},$$

if $\alpha = 1$, where $\gamma_1(t)$ and $\gamma_2(t)$ are even functions of t, $\beta(t)$ is an odd function of t, and $\mu_1(t)$, $\mu_2(t)$ are odd functions of t. Hence, $\boldsymbol{X}$ and $\boldsymbol{Y}$ have respective ch.f.'s given by

$$\phi_{\boldsymbol{X}}(w) = e^{-\gamma_1(w/\|w\|)\|w\|^\alpha(1+i\beta(w/\|w\|)\tan(\pi\alpha/2))},$$
$$\phi_{\boldsymbol{Y}}(w) = e^{-\gamma_2(w/\|w\|)\|w\|^\alpha(1+i\beta(w/\|w\|)\tan(\pi\alpha/2))},$$

and $\gamma_1(t)/\gamma_2(t) = b$, and b does not depend on t, if $0 < \alpha \leq 2$, $\alpha \neq 1$; and

$$\phi_{\boldsymbol{X}}(w) = e^{i\mu_1(w/\|w\|)\|w\| - \gamma_1(w/\|w\|)\|w\|},$$
$$\phi_{\boldsymbol{Y}}(w) = e^{i\mu_2(w/\|w\|)\|w\| - \gamma_2(w/\|w\|)\|w\|}$$

and $\mu_1(t)/\mu_2(t) = \gamma_1(t)/\gamma_2(t) = b$, b independent of t, if $\alpha = 1$.

By a similar proof as in Theorem 3.4, $\boldsymbol{X}$ and $\boldsymbol{Y}$ with such ch.f.'s will have multivariate stable distributions with order α in the case $0 < \alpha \leq 2$, $\alpha \neq 1$, and a multivariate stable distribution with Cauchy marginals, if $\alpha = 1$.

The following result, which characterizes the multivariate normal distributions, is somewhat related to a result of Eaton (1966).

THEOREM 3.7. *Suppose $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent n-dimensional random vectors with finite covariance matrices. If the distributions of $\boldsymbol{AX} + (a\boldsymbol{I} - b\boldsymbol{AA}')^{1/2}\boldsymbol{Y}$ does not depend on $\boldsymbol{A}$, for some $a, b > 0$ and for all $n \times n$ matrics $\boldsymbol{A}$ such that $a\boldsymbol{I} - b\boldsymbol{AA}'$ is non-negative definite, then $\boldsymbol{X}$ and $\boldsymbol{Y}$ have, respectively, $\boldsymbol{N}(0, \sigma_{\boldsymbol{X}}^2\boldsymbol{I})$ and $\boldsymbol{N}(0, \sigma_{\boldsymbol{Y}}^2\boldsymbol{I})$ distributions, where $\sigma_{\boldsymbol{X}}^2 = b\sigma_{\boldsymbol{Y}}^2$.*

PROOF. Let $\boldsymbol{A} = \lambda\boldsymbol{I}$, where $\lambda \leq \sqrt{a/b}$, so that $a\boldsymbol{I} - b\boldsymbol{A}\cdot\boldsymbol{A}' = a\boldsymbol{I} - b\lambda^2\boldsymbol{I} = (a - b\lambda^2)\boldsymbol{I}$, which is then a non-negative definite matrix, and by assumption the distribution of $\lambda\boldsymbol{X} + \sqrt{a - b\lambda^2}\,\boldsymbol{Y}$ does not depend on λ, for all $|\lambda| \leq \sqrt{a/b}$. It follows from Theorem 3.2, that $\boldsymbol{X}$ and $\boldsymbol{Y}$ have multivariate normal distributions, with mean vectors 0 and respective convariance matrices $\Sigma_{\boldsymbol{X}}$ and $\Sigma_{\boldsymbol{Y}}$, with $\Sigma_{\boldsymbol{X}} = b\Sigma_{\boldsymbol{Y}}$. Now observe that for every $n \times n$ orthogonal matrix $\boldsymbol{P}$, $a\boldsymbol{I} - b(\lambda\boldsymbol{P})(\lambda\boldsymbol{P})'$ is non-negative definite for $|\lambda| \leq \sqrt{a/b}$. Take $\lambda = \sqrt{a/b}$, and $\boldsymbol{A} = \lambda\boldsymbol{P}$, so that $\boldsymbol{AX} + (aI - b\boldsymbol{AA}')\boldsymbol{Y} = \sqrt{a/b}P\boldsymbol{X} + [aI - b(\sqrt{a/b}\boldsymbol{P})(\sqrt{a/b}\boldsymbol{P}')]$, $\boldsymbol{Y} = \sqrt{a/b}P\boldsymbol{X}$. By assumption $\boldsymbol{PX}$ has the same distribution for all orthogonal $\boldsymbol{P}$ and hence $\Sigma_{\boldsymbol{X}} = \sigma_{\boldsymbol{X}}^2\boldsymbol{I}$. Similarly $\Sigma_{\boldsymbol{Y}} = \sigma_{\boldsymbol{Y}}^2\boldsymbol{I}$, and $\sigma_{\boldsymbol{X}}^2 = b\sigma_{\boldsymbol{Y}}^2$.

4. Comment

The consideration of the class of characterizations leads to the following type of more general question. Let $g_\lambda(u, v) = \lambda u + f(\lambda)v$ be viewed as a parametric family of functions from $\boldsymbol{R}^p \times \boldsymbol{R}^p$ to $\boldsymbol{R}^p$. Our results then concern the nondependence of the distribution of $g_\lambda(\boldsymbol{X}, \boldsymbol{Y})$ upon λ. The question then arises as to which more general possible forms of the parametric family $g_\lambda(u, v)$ provide meaningful characterizations of the distributions of $\boldsymbol{X}$ and $\boldsymbol{Y}$, and of the form $g_\lambda(u, v)$.

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