A Note on Competitive Bidding with Asymmetric Information — **Source link** ⧉

Rafael Rob

Institutions: University of Pennsylvania

Related papers:

- A Game Theory Approach to Competitive Bidding

- A Survey on Discrete Bidding Games with Asymmetric Information

- A Game Theoretic Bidding Agent for the Ad Auction Game

- A Decision Analysis Approach to Solving the Signaling Game

- Games and Decision Making

A NOTE ON COMPETITIVE BIDDING WITH ASYMMETRIC INFORMATION

Rafael Rob

Institute for Mathematics and its Applications
University of Minnesota
Minneapolis, Minnesota 55455

A Note on Competitive Bidding with Asymmetric Information

Rafael Rob

University of Minnesota
and
University of Pennsylvania

A NOTE ON COMPETITIVE BIDDING WITH ASYMMETRIC INFORMATION

Rafael Rob

Abstract

This note reexamines a problem of competitive bidding under asymmetrical information about the value of the object which was originally formulated and studied by R. Wilson [5]. We derive the equilibrium conditions by means of two alternative approaches and correct thereby a technical error in Wilson's paper. The relative gains of the seller and the two buyers are computed for a specific numerical example.

(Games/ Group Decisions - Bidding, Games/Group Decisions/Gambling)

Institute for Mathematics and its Applications

University of Minnesota

Minneapolis, Minnesota 55455

1. Introduction

An interesting case of competitive bidding with an asymmetrical knowledge about the true value of the auctioned object is examined by R. Wilson [5]. The primary motivation for his study is the insight it provides about the value of information, or, more specifically, about the relative gains of the informed bidder vs. the uninformed bidder. As a by-product one can learn about the ability of the seller to appropriate or realize the value of the item he offers for sale and about the identity of the buyer. In short, his analysis tells us about allocation and imputation under conditions of imperfect and asymmetric information—a fundamental concern of economic theory.

The purpose of this paper is to expand Wilson's analysis. My objective is to examine the bidding game formulated by him, under alternative behavior assumptions about the sequencing of moves. That is, I will consider the outcomes that result by letting the two bidders assume (alternatively) the role of a leader. A few flaws in Wilson's analysis will be corrected thereby. It will be shown, in particular, that his optimality condition (8) (which pertains to the case where the uninformed is a Stackelberg leader) is incorrect. Consequently, it is possible to exhibit bidding strategies which dominate the optimal policy he derives for an "easy example." The correct conditions will be obtained by means of two different approaches. Examples illustrating the results will be offered. I shall start out by presenting the real-life situation we wish to investigate.

Two companies are competing for the rights to drill oil on an offshore parcel. They will independently submit sealed tenders stating the amount of money they are willing to pay for the rights. It is commonly understood that the party submitting the higher bid wins the object (the rights) and pays the amount offered. The net profits of the winner are therefore the difference between the actual discounted market value of the deposits (net of extraction costs) and what he pays for them. This description makes it clear that a poorly informed bidder might win the auction but lose money on the rights. That happens each time his bid exceeds the value of the deposits. Such overpayment is, in turn, due to his ignorance

1

regarding the true value. Superior information is, in this sense, an "asset" the value of which we wish to explore (see also [2] for a related problem).

To focus on this issue, consider the extreme case where one party (party 2, henceforth) is perfectly informed about the market worth, v of the rights. For instance, party 2 may already possess the rights to a nearby parcel. The other party (party 1), in contrast, is a newcomer and has only fragmentary information about v, which is derived perhaps from geological experiments. Let $F(v)$, $f(v)$ be the c.d.f. and the density function respectively of v, considered from party 1's viewpoint.

Early investigation of this problem by Woods [6] led to the conclusion that the uninformed will altogether refrain from bidding. Under the assumption that the competitor (party 2) merely bids for normal profits, i.e., submits $v - \epsilon$, party 1 could win the auction only by losing money on the obtained object, i.e., by submitting a bid higher than $v - \epsilon$. Realizing that, party 2 can ensure winning by submitting a marginal bid which, in turn, confirms party 1's conjecture and drives it out of the bidding game. This reasoning is, however, faulty: the informed is interested in obtaining the rights at the lowest achievable price. His objective, in other words, is expected profits and not just the probability of winning the auction, as Woods has implicitly assumed. Secondly, as Wilson notes, the variable sum nature of the game is ignored. By lowering one's bid one can induce the other party to follow suit. Both parties could thereby gain at the expense of the seller (a governmental agency, presumably). How this maneuver could be carried out in practice will become apparent as we proceed. Another shortcoming of Woods' analysis is the predicted success of the seller to appropriate the full value of the parcel in spite of the extremely weak form of competition. This empirical implication is in contrast to one's intuition regarding the ability of the buyers to extract some of the gains from trade. With only two competitors, situated at highly asymmetrical positions, one would expect that at least some of the rent is captured by the buyer.

A more recent paper by Weverbergh [4] reanalyzed Wilson's work from a different standpoint. Weverbergh observes that the uninformed acts as a dominant

2

player in Wilson's formulation. Party 1, in other words, construes the reaction of party 2 to his own move and proceeds to choose a strategy inducing what from his perspective seems a best response. This, in particular, assumes a special sequencing of moves: namely, that party 1 moves first while party 2 takes party 1's strategy as the "last word" and optimally reacts to it. Weverbergh replaces this leader-follower structure by the symmetric situation and studies the game under the conventional Nash-Cournot behavioral assumption. In keeping with the original formulation, I will maintain Wilson's assumption that the uninformed moves first. Thereafter I will analyze the model under the alternative supposition, namely that the informed is the dominant player.

2. Analysis

A. The uninformed as a stackelberg leader

Following the footsteps of Wilson, it is first clear that the uninformed will will employ a randomized strategy, for otherwise the informed will "exploit him to death." Let $G(p)$, $g(p)$ be the c.d.f. and the p.d.f. of party 1's strategy. The informed, on the other hand, is quite happy with a pure strategy inasmuch as the uncertainty faced by the uninformed already introduces an automatic randomization for him. The v uncertainty, in other words, sufficiently conceals the actions of party 2 and he accordingly selects a pure strategy which we denote by $q(v)$.

Given $G(\cdot)$, it is easy to determine the best response of a player of type v, $q(v)$, to it. If he bids q, his probability of obtaining the rights is just $G(q)$ ($= \Pr(p \leqslant q | G)$) while he will gain $v - q$ provided he wins the auction. Being risk-neutral, his objective is to maximize expected profits, i.e., $\underset{q}{\text{Max}}\, G(q)(v-q)$ ($= \pi_2(q)$). The first order conditions for this problem are given by:

$$v = \frac{G(q)}{g(q)} + q = \psi(q) + q = h(q) \tag{1}$$

where we define

$$\psi(q) = \frac{G(q)}{g(q)} \quad , \qquad h(q) = \psi(q) + q \quad . \tag{2}$$

3

The bid strategy q(v) is simply the root to equation (1), the right hand side of which we assume to be monotonically increasing. In the appendix we show that the last assumption as well as the second order conditions are satisfied provided G is log-concave. Equation (1) summarizes then the response of party 2 to party 1's strategy and we diagram one such hypothetical response in Figure 1.
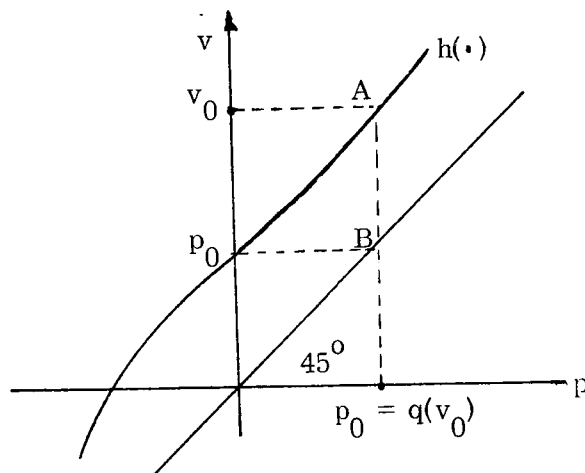
Figure 1

Under the behavioral assumption of the Stackleberg equilibrium concept, party 1 fully understands how party 2's strategy is being formed. That is, he knows how his own choice of strategy, $G(\cdot)$, affects the entire shape of the h curve and how the latter, in turn, determines 2's strategy. Party 1 is, in effect, selecting an h function to optimally manipulate party 2's behavior, i.e., to maximize his own gain. For a given bid p, in the support of his strategy's c.d.f. G, his expected profits are given by

$$\int_{-\infty}^{h(p)} (v-p)f(v)dv + \int_{h(p)}^{\infty} 0 \cdot f(v)dv \tag{3}$$

In terms of Figure 1 we can explicate equation (3) as follows (assuming that player 1 bids p_0).

4

1) If $v > h(p_0)$ (i.e., if $q(v) > p_0$), player 2 wins the auction and player 1's profits are zero (this is the second term in (3)).

2) If $h(p_0) > v > p_0$, i.e., if v is in the strip AB, player 1 wins the auction and his profits are positive.

3) If $v < p_0 < h(p_0)$, player 1 wins the auction but overpays for the rights. His losses are just $p_0 - v$.

Expression (3), accordingly, may or may not be positive and it is clearly to player 1's advantage to have a uniformly higher h curve. Figure 1 tells us also why a pure strategy for player 1 is inferior—it simply eliminates the whole area between the h curve and the 45° ray. Intuitively, if player 1 were to bid p, all v players with $v > p$ would bid marginally to win by submitting $p + \epsilon$ while all v players with $v < p$ would bid to lose.

Since different p are generated by the random device employed by party 1, we must average over these p's to obtain 1's objective—the overall expected return:

$$\pi_1 = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{h(p)} (v - p) f(v) dv \right] g(p) dp \tag{4}$$

It is this functional which party 1 wishes to maximize by selecting an appropriate g satisfying the restraints

$$g(p) \geqslant 0 \tag{5a}$$

$$\int_{-\infty}^{\infty} g(p) dp = 1 \tag{5b}$$

Let us form now the Lagrangian expression

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{h(p)} (v - p) f(v) dv - \lambda \right] g(p) dp , \tag{6}$$

to which we apply the standard Euler conditions, suitably modified to account for the nonnegativity constraint, $g \geqslant 0$ (see Zahl [7]). Write first the function

5

$$H(G, g, p) = \left[\int_{-\infty}^{\frac{G}{g}+p} (v-p)f(v)dv - \lambda \right] g \tag{7}$$

The Euler conditions, in terms of this function, are

$$H_G - \frac{d}{dp}(H_g) \leqslant 0 \tag{8a}$$

$$= 0 , \quad \text{if } g(p) > 0 \tag{8b}$$

Straightforward computation shows that

$$H_G = \frac{G}{g} f(\frac{G}{g} + p) = \left[h(p) - p \right] f(h(p))$$

$$H_g = \int_{-\infty}^{h(p)} (v-p)f(v)dv - \lambda - \left[h(p) - p \right]^2 f(h(p)) \quad .$$

From this the desired first order condition can be derived:

$$\phi(\psi + p) + \psi\psi' + \frac{f'(\psi+p)}{f(\psi+p)} (\psi'+1)\psi^2 = 0 \tag{9}$$

where $\phi(V) = \frac{F(v)}{f(v)}$. This nonlinear differential equation together with the appropriate boundary conditions $G(\underline{p}) = 0$, $G(\bar{p}) = 1$ could, in principle, be solved for an optimal strategy G.* The second order conditions, in turn, are given by

$$f(\frac{G}{g}+p) + \frac{G}{g} f'(\frac{G}{g}+p) < 0 , \quad \text{all p in the support of G} . \tag{10}$$

Let us consider now Wilson's easy example where $F(v) = v$ on $[0,1]$. He suggests $G(p) = kp^{1/\alpha}$ as a solution, where $k = (1+\alpha)^{1/\alpha}$, $\alpha = \frac{1+\sqrt{5}}{2}$ and $p \in \left[0, \frac{1}{\alpha+1}\right]$. If we try, however, a different distribution, $G_\beta(p) = k_\beta p^\beta$, with the same support, $\left[0, \frac{1}{\alpha+1}\right]$, but with $\beta < 1/\alpha$, we see that the corresponding profits for party 1 are given by

$$\pi(\beta) = \frac{1}{2\alpha^4} \frac{1-\beta^2}{\beta(\beta+2)}$$

---

* Note the difference between this and equation (8) in $[5]$.

6

Since $\pi(\,\cdot\,)$ is a decreasing function of β, party 1 can do better by selecting a distribution strategy which is more biased towards small bids (note that the G_β's are stochastically ordered); G is therefore not an equilibrium strategy.

Moreover, since no lower bound restriction of the form $p \geqslant 0$, for example, is imposed, this problem has, in fact, no solution at all. Consider indeed the following family of strategies, parameterized by $\theta > 1$:

$$g(p\,|\,\theta) \;=\; \frac{1}{\theta}\, 1_{(-\theta,\,0)}$$

$$\frac{G(p)}{g(p)} \;=\; p+\theta \;\;,\quad \text{on } (-\theta,0) \;\;.$$

Straightforward computation yields

$$q_\theta(v) \;=\; \frac{v-\theta}{2}\;\;,\qquad \pi_1^* \;=\; -\frac{1}{8}+\frac{\theta}{4}\;\;,\qquad \pi_2^* \;=\; \frac{(v+\theta)^2}{4\theta}\;\;.$$

We observe that $\pi_1^* \rightarrow \infty$ as $\theta \rightarrow \infty$ and that party 1 can obtain arbitrarily large profits by submitting sufficiently small bids. The idea behind this construction is simple. Inasmuch as the uninformed assumes in this formulation the role of a leader, he can "lure" the informed to bid nonaggressively by employing low bids also. He and the informed thereby benefit at the expense of the seller. In practice, the seller will protect himself against this maneuver by announcing a submission fee and/or a minimum price below which the rights will not be sold. The correct optimal strategy for this problem when a lower bound constraint (of the form $p \geqslant 0$) is added is obtained by solving the differential equation $\psi + p + \psi\psi' = 0$ with appropriate boundary conditions to account for the fact that G is a c.d.f. A solution technique for this problem is outlined in the appendix. Shortly I will present a simpler example.

Before engaging in this let us first suggest an alternative solution method involving the concept of the variational derivative (see Gelfand and Fomin [1, p. 29] and Ryder and Heal [3], for an application) which is of interest in its own right. For the class of positive g curves define the functional J($\,\cdot\,$) by

7

$$J(g) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\frac{G}{g}+p} (v-p)f(v)dv \right] g(p)dp \tag{11}$$

and consider a variation

$$\delta g(p) = \begin{cases} \beta & p \in I \\ 0 & p \notin I \end{cases} \tag{12}$$

where $I = (c, c+\alpha)$. The perturbed curve $g^{\Delta} \equiv g + \delta g$ gives rise to the new functions

$$G^{\Delta}(p) = \begin{cases} G(p) & p < c \\ G(p) + (p-c)\beta & c < p < c+\alpha \\ G(p) + \alpha\beta & c+\alpha < p \end{cases} \tag{13a}$$

$$h^{\Delta}(p) = \begin{cases} h(p) & p < c \\ \dfrac{G(p) + (p-c)\beta}{g(p) + \beta} + p & p \in I \\ h(p) + \dfrac{\alpha\beta}{g(p)} & c+\alpha < p \end{cases} \tag{13b}$$

We can now compute the value of the functional J at g^{Δ} and the limit of the differential quotients,

$$\lim_{\alpha,\beta \to 0} \frac{J(g^{\Delta}) - J(g)}{\alpha\beta} = \frac{\delta J}{\delta g} \quad .$$

The value of this derivative, evaluated at the optimal g, must not exceed the shadow price λ. In the appendix we show that this first order condition is expressed by

$$\int_{-\infty}^{h(c)} (v-c)f(v)dv + \int_{c}^{\infty} [h(p) - p]f(h(p))dp - [h(c) - c]^2 f(h(c)) \leqslant \lambda \tag{14}$$

$$= \lambda \ , \ \text{if } g(c) > 0 \ .$$

If we differentiate this condition with respect to c on intervals where g does not vanish we get equation (9) again.

Let us finally consider a solvable example. Suppose that v is distributed over the interval $\left[1, \frac{\pi}{2}+1\right]$ with a c.d.f. given by $F(v) = \sin(v-1)$. From $F(\psi+p) + \psi\psi'f(\psi+p) + f'(\psi+p)(\psi'+1)\psi^2 = 0$ we see that $\psi \equiv 1$ solves this problem and that $G(p) = g(p) = \exp(p - \frac{\pi}{2})$ is the corresponding distribution on the interval $\left[0, \frac{\pi}{2}\right]$ (there is a positive mass at the origin). Equation (1) gives us $h(p) = 1+p$ and $q(v) = v-1$. Party 2 therefore bids to gain fixed profits, should he win the auction and his expected profits are given by $G(q(v)) = \exp(v-1-\frac{\pi}{2})$. Averaged over all types v, we obtain $\frac{1}{2}\left[1-e^{-\pi/2}\right]$ (≈ 0.40). Some more computation shows that the profits for the uninformed are given by $e^{-\pi/2}$ (≈ 0.21) so that the return to the informed is approximately 1.9 times the return to the uninformed. The gains from trade to the informed, uninformed and the seller respectively are divided in terms of percentage points as $(\bar{\pi}_2, \bar{\pi}_1, \bar{\pi}_s) = (25.5, 12.0, 61.6)$.

B. The informed as a Stackelberg leader

Reversing the roles of the two bidders we now assume that the informed can commit himself to a bid strategy $q(v)$ whereas the uninformed chooses a best response $b([q])$ to it. As before, the choice of a policy, $q(v)$, is governed by the reactions of party 1 to it. Fix now a bid function $q(v)$ for party 2 and consider the problem faced by party 1.

$$\underset{p}{\text{Max}} \int_0^{h(p)} (v-p)f(v)dv \tag{15}$$

(We are employing the same notation as before, i.e., $h(p) = q^{-1}(p)$, etc.) The solution, p_0, to (15) is what we denote by $b([q])$. Let $v_0 = h(p_0)$. By definition we have:

$$\int_0^{h(p_0)} (v-p_0)f(v)dv \geqslant \int_0^{h(p)} (v-p)f(v)df \tag{16}$$

where $p_0 = b([q])$.

9

Consider now Figure 2:
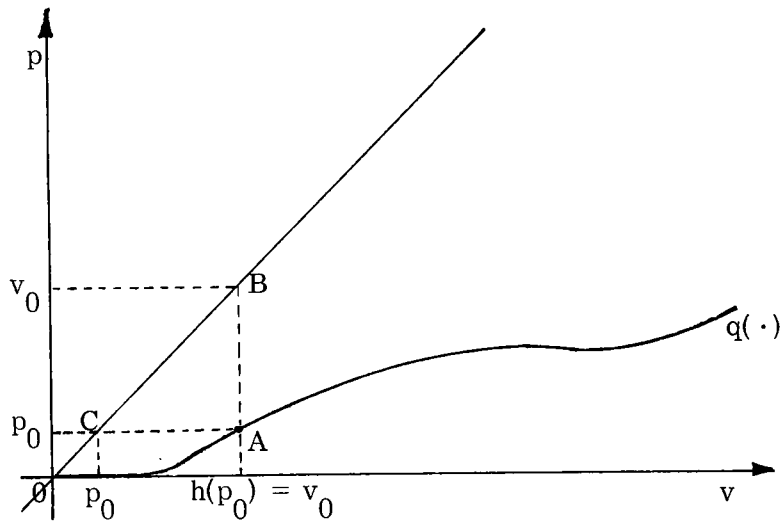
Figure 2

We see that party 2 will win all tracts with value $0 \leqslant v \leqslant v_0 \equiv h(p_0)$. When the value exceeds his bid, i.e., when $p_0 < v < h(p_0)$ his profits are positive. The triangle ABC is then associated with his profits. On the remaining tracts, namely, those with $0 < v < p_0$ he will lose money. His losses are, similarly, associated with the triangle $0p_0C$.

Turning to party 2 we observe that he wins all tracts with $v > v_0$ and that his profits are given by

$$\int_{v_0}^{\infty} \left[v - q(v)\right] f(v)dv \tag{17}$$

where $v_0 = h(b[q])$ is determined by (16). Looking at (17) we note that for a given v_0, party 2 will attempt to make the solution curve $q(\cdot)$ as low as possible for all $v \geqslant v_0$. That is, he will try to minimize his bids for all tracts he wins. The only

10

restraint it faces in this regard is the incentive condition (16), namely, that party 1 will not deviate from the bid p_0 and thereby win more tracts. From these considerations it is clear that the incentive constraints (16) are satisfied with equality for all $p \geqslant p_0$.

Assuming sufficient differentiability we characterize the optimal bid policy of party 2 as the solution of the following differential equation with a "free initial condition" (p_0, v_0):

$$\left[h(p) - p\right] f\big(h(p)\big) h'(p) - F\big(h(p)\big) = 0 \tag{18a}$$

$$h(p_0) = v_0 \tag{18b}$$

Using the transformation $h(p) = v$, $q(v) = p$ we can equivalently write:

$$q'(v) = \frac{\left[v - q(v)\right] f(v)}{F(v)} \tag{19a}$$

$$q(v_0) = p_0 \tag{19b}$$

where (p_0, v_0) is yet to be determined. The solution to this linear equation is easily established:

$$q(v) = \frac{1}{F(v)} \left\{ \int_{v_0}^{v} r f(r) dr + p_0 F(v_0) \right\} \tag{20}$$

Turning to the choice of boundary (p_0, v_0) it is first clear that p_0 is set equal to 0. (Note only that the family of solutions never crosses and that party 2's objective is to minimize $q(v)$ beyond v_0.) It remains to determine v_0 as the maximizer of

$$\int_{v_0}^{\infty} \left[q(v; v_0) - v\right] f(v) dv \tag{21}$$

The first-order conditions are given by:

11

$$0 = -\left[q(v_0; v_0) - v_0\right]f(v_0) + \int_{v_0}^{\infty} \frac{\partial}{\partial v_0} q(v; v_0)f(v)dv$$

$$= v_0 f(v_0) - \int \frac{v_0 f(v_0)}{F(v)} f(v)dv$$

where the last equality follows from (20). Continuing, we have

$$v_0 f(v_0)\left[1 - \int_{v_0}^{\infty} \frac{f(v)}{F(v)} dv\right] = v_0 f(v_0)\left[1 + \log F(v_0)\right] = 0 \quad , \tag{22}$$

that is, v_0 is the root of $F(v_0) = 1/e$. Differentiating (22) one can easily verify that the second-order conditions are satisfied as well. Let us summarize all this in the following

<u>Claim</u>: Given the above sequencing of moves, the informed will bid to win the top $(1 - \frac{1}{e})$ fractile of the tracts, leaving the botton $1/e$ fractile to the uninformed. His bid function is given by

$$q^*(v) = \frac{\int_{v_0}^{v} rf(r)dr}{F(v)} \tag{23}$$

where

$$F(v_0) = 1/e \quad . \tag{23}$$

Using this result we can actually derive closed-form expressions for the relative gains of the two parties and the seller in terms of the primitive datum, i.e., the distribution $F(v)$.

The uninformed wins all tracts $0 \leqslant v \leqslant v_0$ and pays $p_0 = 0$. Thus:

$$\pi_1 = \int_0^{v_0} vf(v)dv \quad . \tag{25}$$

12

The seller collects $q(v)$ for all tracts of worth $v > v_0$. Thus:

$$\pi_s = \int_{v_0}^{\infty} q^*(v)f(v)dv = \int_{v_0}^{\infty} \frac{\int_{v_0}^{v} rf(r)dr}{F(v)} f(v)dv$$

$$= -\int_{v_0}^{\infty} rf(r) \log F(r)dr \qquad (26)$$

The informed profits from the difference between the true value and the bid he submits for the tracts he wins, $v > v_0$:

$$\pi_2 = \int_{v_0}^{\infty} \left[v - q^*(v)\right] f(v)dv = \int_{v_0}^{\infty} rf(r)\left[1 + \log F(r)\right]dr \quad . \qquad (27)$$

Let us recapitulate the implications of our analysis:

1) The informed will bid the truncated conditional mean of all tracts exceeding a cutoff value, v_0. This tells us the degree to which he is shading his bid.

2) The informed will submit a losing bid for the worst 37% ($\approx 1/e$) of the tracts. The uninformed will win these tracts. It is true, in particular, that the uninformed will not drop out of the auction so long as there are sufficiently many good tracts (mathematically, so long as the mass at zero, if one exists, does not exceed $1/e$). Within this structure, the performance of a manager preparing bids can be evaluated simply by looking at the ratio of winning bids.

3) Returning to our diagram we can schematically represent the surplus division as shown in Figure 3.

These empirical implications clearly depend on the special assumptions we have made. In particular, that there are only two bidders and that the informed is
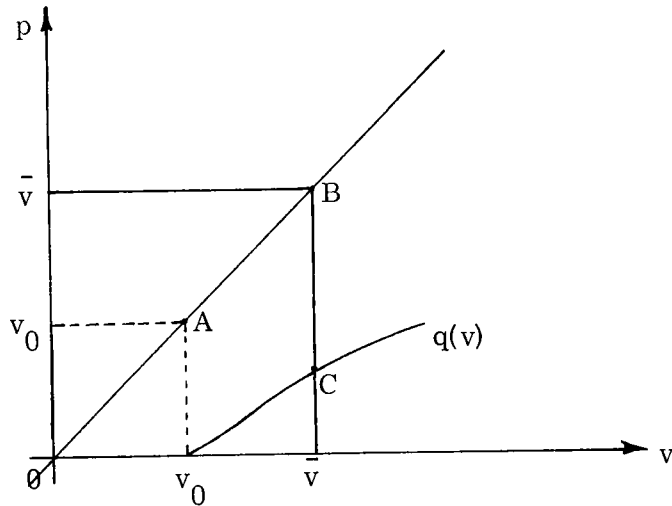
13

Figure 3

3a) The uninformed gains the triangle $0Av_0$, Eq. (25).

3b) The seller collects the triangle $v_0C\bar{v}$, Eq. (26).

3c) The informed wins $ABCv_0$, Eq. (27).

a Stackelberg leader. The universal prediction about the proportion of tracts going to the uninformed is, nonetheless, surprisingly strong.

Armed with these results let us compute two additional examples.

1) Exponential, $F(v) = f(v) = e^{v-1}$, $0 \leqslant v \leqslant 1$.

In this case there are many bad tracts, $v_0 = 0$, and the uninformed is just bid out (i.e., the point mass at zero is exactly $1/e$).

$$q^*(v) = \frac{\int_0^v rf(r)}{F(v)} = v - 1 + e^{-v}$$

$$\pi_1 = 0, \qquad \pi_s = \frac{3}{e} - 1, \qquad \pi_2 = 1 - \frac{2}{e}.$$

Normalizing, we obtain the following rent sharing ratios: $(\bar{\pi}_1, \bar{\pi}_s, \bar{\pi}_2) = (0, 72, 28)$.

14

2) Uniform, $F(v) = v$, $0 \leqslant v \leqslant 1$.

Here, $v_0 = 1/e$.

$$q^*(v) = \frac{\int_{v_0}^{v} rf(r)dr}{v} = \frac{v^2 - v_0^2}{2v}$$

$$\pi_1 = v_0^2/2 = 0.067 \; , \qquad \pi_s = (1 - v_0^2 - 2v_0)/4 \; , \qquad \pi_2 = 0.401 \; .$$

Or, $(\bar{\pi}_1, \bar{\pi}_s, \bar{\pi}_2) = (13, 7, 80)$.

3) For the sake of comparison we reconsider our previous example with $F(v) = \sin(v-1)$ on $\left[1, \frac{\pi}{2} + 1\right]$. Some computation shows that $(\pi_1, \pi_s, \pi_2) = (0.45, 0.68, 0.44)$ while $(\bar{\pi}_1, \bar{\pi}_s, \bar{\pi}_2) = (28, 44, 28)$. It is interesting to note that both the informed and the uninformed gain when the informed assumes the role of a Stackelberg leader. It is the seller who pays for these gains. Moreover, the uninformed gains relative to the informed. The intuitive reason that this sequencing of moves is to the advantage of player 1 is that allowing player 2 to move first reveals some useful information to the uninformed.

References

[1] Gelfand, I.M. and S.V. Fomin, Calculus of Variation, Prentice Hall, Englewood Cliffs, NJ (1963).

[2] Milgrom, P. and R. Weber, "The Value of Information in a Sealed-Bid Auction," Journal of Mathematical Economics, 10 (1982), 105-114.

[3] Ryder, H.E. and G.M. Heal, "Optimal Growth with Intertemporally Dependent Preferences," Review of Economic Studies, 40 (1973), 1-31.

[4] Weverbergh, M., "Competitive Bidding with Asymmetric Information Reanalyzed," Management Science, 25 (1979), 290-294.

[5] Wilson, R.B., "Competitive Bidding with Asymmetric Information," Management Science, 13 (1967), 816-820.

[6] Woods, D.H., Decision Making under Uncertainty in Hierarchical Organizations, Dissertation, Harvard Business School, Boston (1965).

[7] Zahl, S., "An Allocation Problem with Applications to Operations Research and Statistics," Operations Research, 11 (1963), 426-441.

Appendix

1. Consequences of log-concavity:

Assume that $\log G(p)$ is concave. Then

$$\frac{d}{dp}\left\{\log G(p)\right\} = \frac{g(p)}{G(p)}$$

is decreasing, which implies that the right hand side of (1) is increasing. It also implies, by differentiation, that

$$g'G - g^2 < 0 \quad . \tag{1'}$$

Looking now at the second derivative of the profit function and substituting from (1) we see that

$$\pi''(q) = g'(q)(v-q) - 2g(q) = \frac{G(q)}{g(q)} g'(q) - 2g(q) \quad .$$

(1') shows that $\pi''(q) < 0$ so that the second order conditions are satisfied too.

2. The differential equation associated with Wilson's easy example is (see (9)):

$$\frac{G}{g} + p + \frac{G}{g}\left(\frac{G}{g}\right)' = 0 \quad .$$

If we define $T(p) = \log G(p)$ and note that $1/T' = G/g$ we can rewrite this as

$$\frac{1}{T'} + p - \frac{T''}{(T')^3} = 0 \quad .$$

Using the inverse, $S (= T^{-1})$, the latter is equivalent to

$$S'' + S' + S = 0$$

But the above is a second-order linear equation whose solution is given by

$$S(y) = e^{-y/2}\left[A \sin \frac{\sqrt{3}}{2} y + B \cos \frac{\sqrt{3}}{2} y\right]$$

The desired G is just $G(p) = e^{S^{-1}(p)}$.

3. The variational derivative

Using the definition of $g^\triangle$, we obtain

$$J(g^\triangle) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{h^\triangle(p)} (v-p)f(v)\,dv \right] g^\triangle(p)\,dp$$

$$= \int_{-\infty}^{c} \left[\int_{-\infty}^{h(p)} (v-p)f(v)\,dv \right] g(p)\,dp + \int_{c}^{c+\alpha} \left[\int_{-\infty}^{h^\triangle(p)} (v-p)f(v)\,dv \right] g(p)+\beta\,dp$$

$$+ \int_{c+\alpha}^{\infty} \left[\int_{-\infty}^{h(p)} (v-p)f(v)\,dv + \int_{h(p)}^{h(p)+\frac{\alpha\beta}{g(p)}} (v-p)f(v)\,dv \right] g(p)\,dp$$

$$= \int_{-\infty}^{c} \int_{-\infty}^{h(p)} + \int_{c}^{c+\alpha} \left[\int_{-\infty}^{h(p)} (v-p)f(v)\,dv - \int_{h^\triangle(p)}^{h(p)} (v-p)f(v)\,dv \right] \left[g(p)+\beta \right] dp$$

$$+ \int_{c+\alpha}^{\infty} \int_{-\infty}^{h(p)} + \int_{c+\alpha}^{\infty} \int_{h(p)}^{h(p)+\frac{\alpha\beta}{g(p)}}$$

$$= \int_{-\infty}^{c} \int_{-\infty}^{h(p)} + \int_{c}^{c+\alpha} \int_{-\infty}^{h(p)} + \int_{c+\alpha}^{\infty} \int_{-\infty}^{h(p)}$$

$$- \int_{c}^{c+\alpha} \left[\int_{h^\triangle(p)}^{h(p)} (v-p)f(v)\,dv \right] \left[g(p)+\beta \right] dp + \beta \int_{c}^{c+\alpha} \int_{-\infty}^{h(p)} (v-p)f(v)\,dv\,dp$$

$$+ \int_{c+\alpha}^{\infty} \left[\int_{h(p)}^{h(p)+\frac{\alpha\beta}{g(p)}} (v-p)f(v)\,dv \right] g(p)\,dp$$

where $\quad \displaystyle\int\int \equiv \int \left[\int (v-p)f(v)\,dv \right] g(p)\,dp$

18

$$\approx \ J(g) + \alpha\beta \int_{-\infty}^{h(c)} (v-c)f(v)dv + \int_{c+\alpha}^{\infty} (h(p) - p) f(h(p)) \frac{\alpha\beta}{g(p)} g(p)dp$$

$$- \int_{c}^{c+\alpha} (h(p) - p) f(h(p)) \frac{\beta\left[G - (p-c)g\right]}{g(g+\beta)} (g+\beta)dp$$

where we have used the first order approximation

$$\int_{y_0}^{y_0+\epsilon} a(y)dy \approx a(y_0)\epsilon \quad .$$

Using the same approximation once more we obtain

$$\approx \ J(g) + \alpha\beta \int_{-\infty}^{h(c)} (v-c)f(v)dv$$

$$+ \alpha\beta \int_{c+\alpha}^{\infty} (h(p) - p)f(h(p))dp - \alpha\beta(h(c) - c)f(h(c)) \frac{G(c)}{g(c)}$$

so that:

$$\lim_{\alpha,\beta \to 0} \frac{J(g^{\triangle}) - J(g)}{\alpha\beta} = \int_{-\infty}^{h(c)} (v-c)f(v)dv + \int_{c}^{\infty} (h(p) - p)f(h(p)) \ dp$$

$$- (h(c) - c)^2 f(h)c)) \quad ,$$

from which equation (14) follows. Differentiation of the above, using Leibnitz' rule gives us equation (9) in turn.