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# A note on convective heat transfer of an MHD Jeffrey fluid over a stretching sheet

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This article focuses on the exact solution regarding convective heat transfer of a magnetohydrodynamic (MHD) Jeffrey fluid over a stretching sheet. The effects of joule and viscous dissipation, internal heat source/sink and thermal radiation on the heat transfer characteristics are taken in account in the presence of a transverse magnetic field for two types of boundary heating process namely prescribed power law surface temperature (PST) and prescribed heat flux (PHF). Similarity transformations are used to reduce the governing non-linear momentum and thermal boundary layer equations into a set of ordinary differential equations. The exact solutions of the reduced ordinary differential equations are developed in the form of confluent hypergeometric function. The influence of the pertinent parameters on the temperature profile is examined. In addition the results for the wall temperature gradient are also discussed in detail.  [http://dx.doi.org/10.1063/1.4935571]

## INTRODUCTION

It is well known that investigation regarding boundary layer flows over continuous stretching surfaces finds numerous important applications occurring in many engineering process. Drawing of plastic films and wires, manufacture of foods, crystal growing, liquid films in condensation process etc. are a few examples of practical importance. Further it has several practical applications in the field of metallurgy and chemical engineering such as extrusion process, heat materials traveling between a feed roll and a wind up roll. Sakiadis[1,2] performed the pioneering works by studying the flow induced by a moving surface. The closed form exponential solution for the flow by a linear stretching is examined by Crane.[3] Later on Vajravelu and Roper[4] analyzed the flow and heat transfer over a stretching sheet with prescribed surface temperature. Many researchers investigated the stretching sheet problems by considering different effects (some studies can be cited[5–9]).

A considerable interest has been revealed by different researchers in studying the flow and heat transfer of an electrically conducting and heat generating fluid due to its important applications in the process of purification of molten metals from non-metallic inclusions. In polymer technology and specific metallurgical operations the rate of cooling of continuous stretched strips is controlled by applying the principles of MHD techniques. This application is well marked in case of drawing, annealing, and thinning of copper wires. Chen[10] examined the combined effects of viscous dissipation joule heating for the momentum and thermal transport of MHD Newtonian fluid. Liu[11] studied the flow and heat transfer in an electrically conducting second grade fluid with transverse magnetic field with power law surface temperature. Parida et al.[12] have examined the MHD heat and mass

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transfer in a rotating system with periodic suction. Anjli devi and Ganja[13] have studied the MHD flow over a stretching porous surface with viscous dissipation.

At high operating temperatures in engineering process radiation may play a dynamic role under many non-isothermal situations. In polymer processing industry if the entire system involving the polymer extrusion process is placed in a thermally controlled environment then thermal radiation might be great important in controlling heat transfer process. Hayat et al.[14] studied the radiative flow of Jeffrey fluid in a porous medium with power law heat flux and heat source. Vyas and Rai[15] examined Radiative flow with variable thermal conductivity over a non-isothermal stretching sheet in a porous medium. Singh[16] studied the magneto convection flow of viscoelastic fluid over stretching sheet with heat source and radiation effects. Many other researchers have studied the flow and heat transfer of fluids by taking the radiation heat source/sink effects.[17–21]

Motivated by above studies we intend to investigate the flow and heat transfer of an MHD Jeffrey fluid over a stretching sheet subject to power law temperature in the heat source/ sink with radiation. Also we are considering two general cases of non-isothermal boundary conditions, namely (1) the sheet with prescribed power law surface temperature (PST case) (2) and the sheet with prescribed power law heat flux (PHF case). Using appropriate similarity transformations the highly non-linear partial differential equations are reduced to a set of non-linear ordinary differential equations. We derived closed form analytical solution for non-dimensional velocity and temperature distribution in the form of confluent hypergeometric function (Kummer's function).[22] The effects of non-dimensional parameters, magnetic number, Eckert number, radiation parameter, heat source/sink parameter and Deborah number on the temperature profile are explored graphically and discussed in details.

## MATHEMATICAL FORMULATION

The constitutive equations for a Jeffrey fluid are given by[23]

$$\boldsymbol{\tau} = -p\mathbf{I} + \mathbf{S}, \tag{1}$$

with $\mathbf{S}$ as the extra stress tensor defined by

$$\mathbf{S} = \frac{\mu}{1+\lambda}\left[\mathbf{R}_1 + \lambda_1\left(\frac{\partial \mathbf{R}_1}{\partial t} + \mathbf{V}.\boldsymbol{\nabla}\right)\mathbf{R}_1\right], \tag{2}$$

where $\boldsymbol{\tau}$ is the Cauchy stress tensor, $p$ the pressure, $\mu$ the dynamic viscosity, $\lambda$ and $\lambda_1$ are the material parameters of the Jeffrey fluid and $\mathbf{R}_1$ the Rivlin-Ericksen tensor defined by

$$\mathbf{R}_1 = (\nabla V) + (\nabla V)^t. \tag{3}$$

We consider the steady two dimensional boundary layer flow of an electrically conducting Jeffrey fluid past a stretching sheet in the presence of heat source and chemical reaction. The flow is generated due to linear stretching of the sheet caused by simultaneous application of equal and opposite forces along the $x$ -axis while keeping the origin fixed (Fig. 1). A uniform magnetic field of strength $B_o$ is imposed to the flow perpendicular to the sheet.

The boundary layer equations which govern the flow and heat transfer of a Jeffrey fluid over a stretching surface are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\upsilon}{1+\lambda}\left[\frac{\partial^2 u}{\partial y^2} + \lambda_1\left(u\frac{\partial^3 u}{\partial x \partial y^2} + v\frac{\partial^3 u}{\partial y^3} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y}\right)\right] - \frac{\sigma B_o^2}{\rho}u, \tag{5}$$

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{1+\lambda}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \lambda_1\frac{\partial u}{\partial y}\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\right]$$
$$+q\left(T - T_\infty\right) + \sigma B_o^2 u^2 - \frac{\partial q_r}{\partial y}, \tag{6}$$

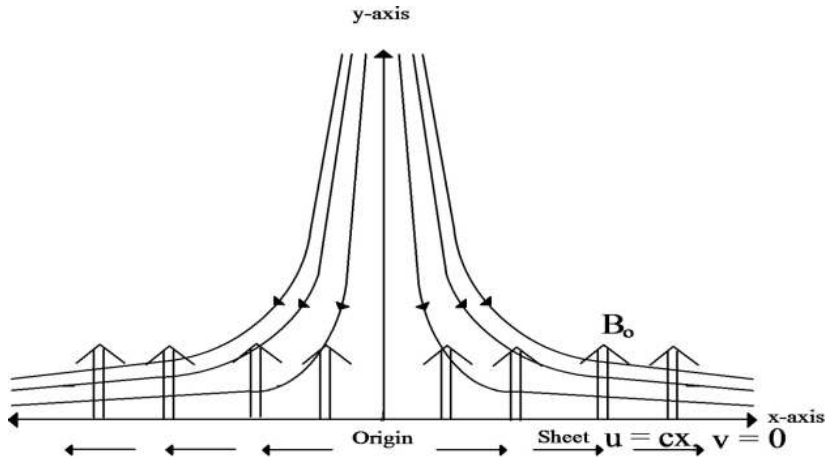


FIG. 1. Schematic diagram of flow towards a stretching sheet.

where $u$ and $v$ are the velocity components in the $x$ and $y$ direction respectively, $T$ the fluid temperature, $\upsilon$ the kinematic viscosity, $\rho$ the fluid density, $c_p$ the specific heat $\lambda$ indicates the ratio of relaxation and retardation times and $\lambda_1$ the relaxation time, $q_r$ the radiative heat flux and $q$ is the rate of volumetric heat generation/absorption. The last three terms in Eq. (6) are the work done due to deformation, internal heat generation/absorption and the Joule heating. By using Rosseland approximation for thermal radiation[24] the radiative heat flux can be written as

$$q_r = -\frac{4\sigma_1}{3\alpha_R}\frac{\partial T^4}{\partial y}, \tag{7}$$

where $\sigma_1$ and $\alpha_r$ are the Stephan-Boltzmann constant and the mean absorption coefficient. We assume that the difference in the temperature within the flow is such that $T^4$ can be expressed as a linear combination of the temperature. This can be accomplished by expanding $T^4$ in the Taylor series about $T_\infty$ and neglecting the higher order terms as

$$T^4 \cong 4T_\infty^3 T - 3T_\infty^4. \tag{8}$$

Using Eq. (7) in Eq. (6), the energy equation becomes

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p(1+\lambda)}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \lambda_1\frac{\partial u}{\partial y}\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\right]$$
$$+\frac{q}{\rho c_p}(T - T_\infty) + \frac{\sigma B_o^2}{\rho c_p}u^2 + \frac{16\sigma_1 T_\infty^3}{3\rho c_p \alpha_R}\frac{\partial^2 T}{\partial y^2}. \tag{9}$$

In this paper we intend to investigate the thermal transport phenomenon for two non-isothermal conditions, namely prescribed surface temperature (PST) and prescribed surface heat flux (PHF). Correspondingly, we consider the following boundary conditions for the flow and temperature

$$u = u_w = cx, \qquad v = 0, \qquad \text{at} \quad y = 0, \tag{10}$$

$$u = 0, \qquad \frac{\partial u}{\partial y} \to 0 \quad \text{as } y \to \infty \tag{11}$$

$$\left.\begin{aligned} T = T_w(x) = T_\infty + A_1\left(\frac{x}{l}\right)^2 \quad & \text{PST case} \\ q_w(x) = -k\frac{\partial T}{\partial y} = A_2\left(\frac{x}{l}\right)^2 \quad & \text{PHF case} \end{aligned}\right\} \text{ at } y = 0 \tag{12}$$

$$T \to T_\infty \qquad \text{as} \quad y \to \infty, \tag{13}$$

where $A_1$ and $A_2$ are the constants depending upon the properties of the fluid, $l$ the characteristic length, $T_w$ and $T_\infty$ are the temperatures at the wall and far away from the wall respectively.

## SOLUTION OF THE FLOW FIELD

To facilitate the analysis we introduce the following similarity transformation and dimensionless variable

$$\eta = \sqrt{\frac{c}{\upsilon}}y, \ u = cxf'(\eta), \quad v = -\sqrt{c\upsilon}f(\eta). \tag{14}$$

The governing equation of motion (4) reduces to

$$f''' + (1+\lambda)\left[ff'' - f'^2\right] + \beta\left(f''^2 - ff^{iv}\right) - Mf' = 0, \tag{15}$$

where prime denotes differentiation with respect to $\eta$. Furthermore $\beta = \lambda_1 c$ is the Deborah number and $M = \frac{\sigma B_o^2}{\rho c}$ the magnetic parameter. The corresponding boundary conditions are then

$$f'(\eta) = 1, \ f(\eta) = 0 \text{ at } \eta = 0, \tag{16}$$

$$f'(\eta) \to 0 \text{ as } \eta \to \infty. \tag{17}$$

Lawrence and Rao[25] have given a general method and obtained all non-unique solution of the reduced Eq. (14), among all these solutions of the form, we consider the realistic solution of the form

$$f(\eta) = \frac{1 - \exp(-\alpha\eta)}{\alpha}, \tag{18}$$

where

$$\alpha = \sqrt{\frac{1+\lambda+M}{1+\beta}}. \tag{19}$$

## SOLUTION OF THE TEMPERATURE FIELD

### Case 1: Prescribed surface temperature (PST)

In PST case we define the following dimensionless temperature variable as

$$T(\eta) = T_\infty + \Delta T\,\theta(\eta), \tag{20}$$

with $\Delta T = T_w - T_\infty$. The dimensionless energy equation can be written as

$$(1+N)\,\theta'' + \Pr f\theta' + \Pr(Q - 2f')\,\theta = -\Pr E_c\left[Mf'^2 + \frac{1}{1+\lambda}\left(f''^2 + \beta f''\left(f'f'' - ff^{iv}\right)\right)\right], \tag{21}$$

subject to the boundary conditions

$$\theta(0) = 1, \quad \theta(\infty) = 0, \tag{22}$$

here the dimensionless parameters Pr, $N$, $Q$ and $E_c$ are the Prandtl number, radiation parameter, internal heat generation/absorption parameter and Eckert number, respectively and are defined as

$$\Pr = \frac{\rho c_p \nu}{k}, \ N = \frac{16\sigma_1 T_\infty^3}{3\alpha_R k}, \ Q = \frac{q}{c\rho c_p}, \ E_c = \frac{c^2 l^2}{A_1 c_p}. \tag{23}$$

By introducing the new variable

$$\xi = -\Pr^* e^{-\alpha\eta}, \tag{24}$$

where $\text{Pr}^* = \frac{\text{Pr}}{\alpha^2(1+N)}$ is the modified Prandlt number. Making use of Eqs. (18) and (24) in Eqs. (21) and (22) the transform problem is

$$\xi\frac{d^2\theta}{d\xi^2} + (1 - \text{Pr}^* - \xi)\frac{d\theta}{d\xi} + \left(2 + \frac{\text{Pr}^*Q}{\xi}\right)\theta = -\frac{E_c\alpha^2}{\text{Pr}^*}\left(\frac{M}{\alpha^2} + \frac{1}{1+\lambda}(1+\beta)\right), \tag{25}$$

$$\theta(\xi = -\text{Pr}^*) = 1, \qquad \theta(\xi = 0) = 0. \tag{26}$$

Eqs. (25) and (26) constitute a non- homogeneous boundary value problem. Denoting the solution of homogenous part of Eq. (25) by $\theta_c$ and non-homogenous part by $\theta_p$, so we can write

$$\theta(\xi) = \theta_c(\xi) + \theta_p(\xi). \tag{27}$$

A closed form particular solution of Eq. (24) can be written as

$$\theta_p(\xi) = -\frac{E_c\,\text{Pr}\left(\frac{M}{\alpha^2} + \frac{1}{1+\lambda}(1+\beta)\right)}{(1+N)\,(4 - 2\text{Pr}^* + \text{Pr}^*Q)}\left(\frac{\xi}{\text{Pr}^*}\right)^2, \tag{28}$$

Making use of the boundary condition (26) the homogenous part of the temperature can be written in the form of confluent hypergeometric function (Kummer's function) as

$$\theta_c(\xi) = C_1 M(\chi - 2,\ Z + 1,\ \xi). \tag{29}$$

Using the boundary conditions (26) the solution of the Eq. (25) is determined to be

$$\theta(\xi) = \left(1 + \frac{E_c\,\text{Pr}\left(\frac{M}{\alpha^2} + \frac{1}{1+\lambda}(1+\beta)\right)}{(1+N)\,(4 - 2\text{Pr}^* + \text{Pr}^*Q)}\right)\left(\frac{\xi}{-\text{Pr}^*}\right)^{\chi}\frac{M(\chi - 2,\ Z + 1,\ \xi)}{M(\chi - 2,\ Z + 1,\ -\text{Pr}^*)} - \frac{E_c\,\text{Pr}\left(\frac{M}{\alpha^2} + \frac{1}{1+\lambda}(1+\beta)\right)}{(1+N)\,(4 - 2\text{Pr}^* + \text{Pr}^*Q)}\left(\frac{\xi}{\text{Pr}^*}\right)^2, \tag{30}$$

where

$$\chi = \frac{\text{Pr}^* + \sqrt{(\text{Pr}^*)^2 - 4\text{Pr}^*Q}}{2}, \qquad Z = \sqrt{(\text{Pr}^*)^2 - 4\text{Pr}^*Q}, \tag{31}$$

and $M(a_\circ,\ b_\circ,\ \xi)$ is the Kummer's function (Abramowitz and Stegun[22]) and it is defined as

$$M(a_\circ,\ b_\circ,\ \xi) = 1 + \sum_{n=1}^{\infty}\frac{(a_\circ)_n\xi^n}{(b_\circ)_n n!}, \tag{32}$$
$$(a_\circ)_n = a_\circ(a_\circ+1)(a_\circ+2)\cdots(a_\circ+n-1),$$
$$(b_\circ)_n = b_\circ(b_\circ+1)(b_\circ+2)\cdots(b_\circ+n-1).$$

The temperature profile in term of $\eta$ can be written by invoking Eq. (24) in Eq. (30),

$$\theta(\eta) = \left(1 + \frac{E_c\,\text{Pr}\left(\frac{M}{\alpha^2} + \frac{1}{1+\lambda}(1+\beta)\right)}{(1+N)\,(4 - 2\text{Pr}^* + \text{Pr}^*Q)}\right)e^{-\chi\alpha\eta}\frac{M(\chi - 2,\ Z + 1,\ -\text{Pr}^*e^{-\alpha\eta})}{M(\chi - 2,\ Z + 1, -\text{Pr}^*)} - \frac{E\,\text{Pr}\left(\frac{M}{\alpha^2} + \frac{1}{1+\lambda}(1+\beta)\right)}{(1+N)\,(4 - 2\text{Pr}^* + \text{Pr}^*Q)}e^{-2\alpha\eta}. \tag{33}$$

Note that this solution also satisfies the boundary conditions given in Eq. (22)

The local Nusselt number is defined as

$$Nu_x = \frac{hx}{k} = \frac{q_w}{T_w - T_\infty}\frac{x}{k}. \tag{34}$$

In terms of dimensionless parameters it reduces to

$$Nu_x\text{Re}_x^{\frac{-1}{2}} = -\theta'(0). \tag{35}$$

The dimensionless wall temperature gradient can be written as

$$\theta'(0) = \alpha\left(1 + \frac{E_c\,\mathrm{Pr}\left(\frac{M}{\alpha^2} + \frac{1}{1+\lambda}(1+\beta)\right)}{(1+N)\,(4 - 2\mathrm{Pr}^* + \mathrm{Pr}^* Q)}\right)\left[\mathrm{Pr}^*\left(\frac{\chi-2}{Z+1}\right)\frac{M(\chi-1,\ Z+2,\ -\mathrm{Pr}^*)}{M(\chi-2,\ Z+1, -\mathrm{Pr}^*)} - \chi\right]$$
$$+2\alpha\frac{E_c\,\mathrm{Pr}\left(1 + \frac{M}{\alpha^2}\right)}{(1+N)\,(4 - 2\mathrm{Pr}^* + \mathrm{Pr}^* Q)}. \tag{36}$$

### Case 2: Prescribed heat flux (PHF)

In PHF case we define the following dimensionless temperature variable as

$$\Phi = \frac{T - T_\infty}{q_w x/k}\mathrm{Re}_x^{\frac{1}{2}}. \tag{37}$$

The transformed energy equation along with the boundary conditions can be written as

$$(1+N)\,\Phi'' + \mathrm{Pr}\, f\Phi' + \mathrm{Pr}\,(Q - 2f')\,\Phi = -\mathrm{Pr}\,E_c\left[M f'^2 + \frac{1}{1+\lambda}\left(f''^2 + \beta f''(f'f'' - ff''')\right)\right], \tag{38}$$

$$\Phi'(0) = -1, \quad \Phi(\infty) = 0, \tag{39}$$

where the Eckert number in PHF case is defined as $E_c = kc^2 l^2\sqrt{c/\upsilon}/A_2 c_p$ while the other parameters are same as defined in PST case. Making use of Eqs. (18) and (24) in Eqs. (38) and (39) we obtained the following

$$\xi\frac{d^2\Phi}{d\xi^2} + (1 - \mathrm{Pr}^* - \xi)\,\frac{d\Phi}{d\xi} + \left(2 + \frac{\mathrm{Pr}^* Q}{\xi}\right)\Phi = -\frac{E\alpha^2}{\mathrm{Pr}^*}\left(\frac{M}{\alpha^2} + \frac{1}{1+\lambda}(1+\beta)\right), \tag{40}$$

$$\Phi\,(\xi = -\mathrm{Pr}^*) = -\frac{1}{\alpha\mathrm{Pr}^*}, \qquad \Phi(\xi = 0) = 0. \tag{41}$$

The analytical solution of Eq. (40) subject to the boundary condition in Eq. (41) by following the same procedure as describe in PST case, is given by

$$\Phi(\xi) = \frac{\left(2\frac{E_c\,\mathrm{Pr}\left(\frac{M}{\alpha^2} + \frac{1}{1+\lambda}(1+\beta)\right)}{(1+N)\,(4-2\mathrm{Pr}^*+\mathrm{Pr}^* Q)} + 1/\alpha\right)\left(\frac{\xi}{-\mathrm{Pr}^*}\right)^{\chi} M(\chi-2,\ Z+1,\ \xi)}{\chi M(\chi-2,\ Z+1,\ -\mathrm{Pr}^*) - \mathrm{Pr}^*\left(\frac{\chi-2}{Z+1}\right)M(\chi-1,\ Z+2,\ -\mathrm{Pr}^*)} -$$
$$\frac{E_c\,\mathrm{Pr}\left(\frac{M}{\alpha^2} + \frac{1}{1+\lambda}(1+\beta)\right)}{(1+N)\,(4 - 2\mathrm{Pr}^* + \mathrm{Pr}^* Q)}\left(\frac{\xi}{\mathrm{Pr}^*}\right)^2. \tag{42}$$

The temperature profile in term of $\eta$ can be written by substituting Eq. (24) in Eq. (42)

$$\Phi(\eta) = \frac{\left(2\frac{E_c\,\mathrm{Pr}\left(\frac{M}{\alpha^2} + \frac{1}{1+\lambda}(1+\beta)\right)}{(1+N)\,(4-2\mathrm{Pr}^*+\mathrm{Pr}^* Q)} + 1/\alpha\right)e^{-\chi\alpha\eta} M(\chi-2,\ Z+1,\ -\mathrm{Pr}^* e^{-\alpha\eta})}{\chi M(\chi-2,\ Z+1,\ -\mathrm{Pr}^*) - \mathrm{Pr}^*\left(\frac{\chi-2}{Z+1}\right)M(\chi-1,\ Z+2,\ -\mathrm{Pr}^*)} -$$
$$\frac{E_c\mathrm{Pr}^*\left(\frac{M}{\alpha^2} + \frac{1}{1+\lambda}(1+\beta)\right)}{(1+N)\,(4 - 2\mathrm{Pr}^* + \mathrm{Pr}^* Q)}e^{-2\alpha\eta}. \tag{43}$$

The expression for the dimensionless wall temperature is obtained as

$$\Phi(0) = \frac{\left(2\frac{E_c \Pr\left(\frac{M}{\alpha^2}+\frac{1}{1+\lambda}(1+\beta)\right)}{(1+N)(4-2\Pr^*+\Pr^* Q)} + 1/\alpha\right) M(\chi-2,\ Z+1,\ -\Pr^*)}{\chi M(\chi-2,\ Z+1,\ -\Pr^*) - \Pr^*\left(\frac{\chi-2}{Z+1}\right) M(\chi-1,\ Z+2,\ -\Pr^*)} - \frac{E_c \Pr\left(\frac{M}{\alpha^2}+\frac{1}{1+\lambda}(1+\beta)\right)}{(1+N)(4-2\Pr^*+\Pr^* Q)}. \tag{44}$$

For PHF case the local Nusselt number can be expressed as

$$Nu_x \mathrm{Re}_x^{\frac{-1}{2}} = 1/\Phi(0). \tag{45}$$

## RESULTS AND DISCUSSION

In this article an MHD boundary layer problem for momentum and heat transfer in Jeffrey fluid flow over a non-isothermal stretching sheet in the presence of dissipative energy, thermal radiation and internal heat source/ sink is investigated. The highly non-linear governing partial differential equations are reduced into a set of non-linear ordinary differential equations by applying suitable similarity transformations and their analytic solutions are obtained in the form of confluent hypergeometric function.

This section highlights the influence of pertinent parameters on the temperature profiles. Figures 2(a) and 2(b) describe the effect of ratio of relaxation and retardation times parameter $\lambda$ on temperature profiles $\theta(\eta)$ and $\Phi(\eta)$ in both PST and PHF cases, respectively for the selected values of Pr, $M$, $Q$, $N$, $E_c$ and $\beta$. It is quite clear that an increase in the value of $\lambda$ results in an increase in the temperature profiles $\theta(\eta)$ and $\Phi(\eta)$ for both PST and PHF cases. Thus broadened the thermal boundary layer thickness such that the wall temperature gradient decreases in PST case and the surface temperature increases in PHF case. In order to see the effects of Eckert number $E_c$ figures 3(a) and 3(b) are displayed for both PST and PHF cases on the temperature distribution $\theta(\eta)$ and $\Phi(\eta)$. It is evident from these figures that the thermal boundary is enhanced by increasing the values of Eckert number $E_c$ so the dissipative energy becomes more important with an enlargement of temperature profile. Also the energy dissipation represents an increase in temperature gradient in PST case and the wall temperature in PHF case. Figures 4(a) and 4(b) are plotted to see the effects of radiation parameter $N$ on the temperature profiles $\theta(\eta)$ and $\Phi(\eta)$ for both PST and PHF cases. It is obvious that by increasing the radiation parameter $N$ the temperature profiles increases in both cases. Thus the thermal boundary layer thickness enhanced in the presence of thermal radiation such that the wall temperature decreases in PST case and the surface temperature increases in PHF case. This result indicates that thermal radiation is to reduce the heat transfer rate. The influence of heat generation/absorption parameter $Q$ on the temperature profiles $\theta(\eta)$ and $\Phi(\eta)$ is displayed in figures 5(a) and 5(b) for both PST and PHF cases respectively. It is worth noting that for the case of $Q > 0$ the energy may generates in the boundary layer, in consequences, the temperature profiles

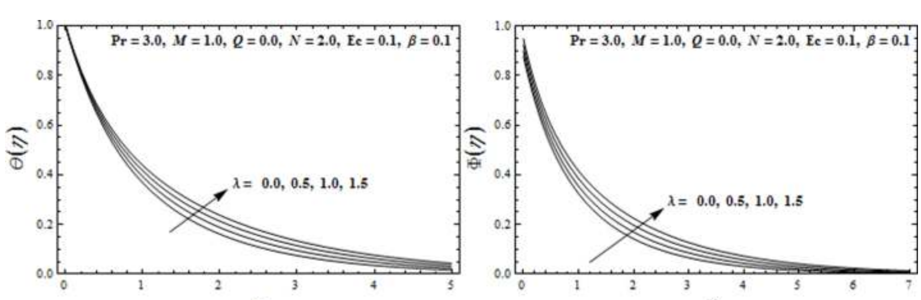


FIG. 2. Influence of $\lambda$ on the temperature profiles: (a) $\theta(\eta)$ for PST case and (b) $\Phi(\eta)$ for PHF case.

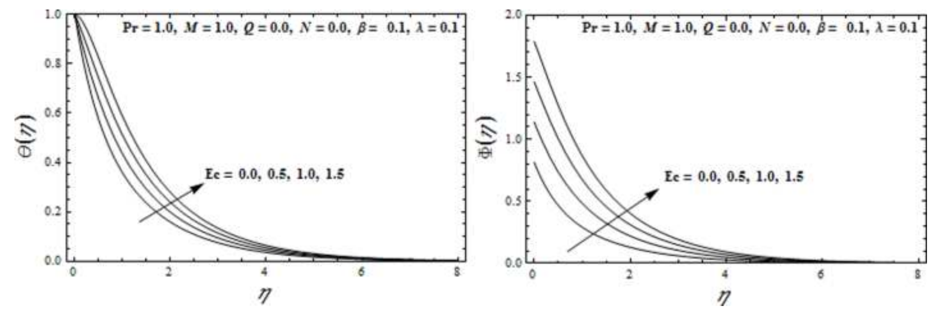


FIG. 3. Influence of the Eckert number $E_c$ on the temperatures profiles: (a) $\theta(\eta)$ for PST case and (b) $\Phi(\eta)$ for PHF case.

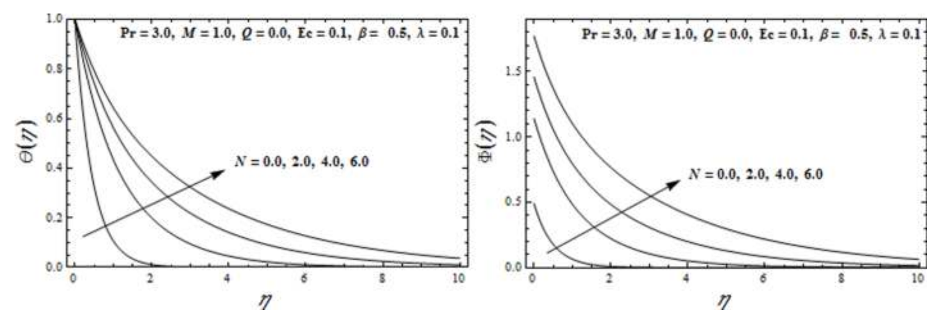


FIG. 4. Influence of $N$ on the temperature profiles: (a) $\theta(\eta)$ for PST case and (b) $\Phi(\eta)$ for PHF case.

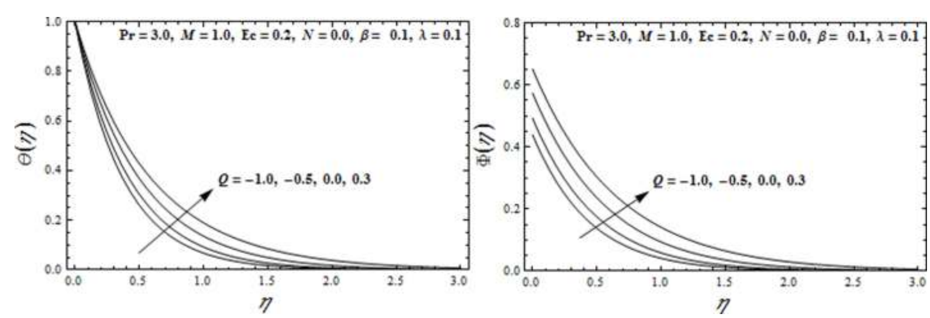


FIG. 5. Influence of $Q$ on the temperature profiles: (a) $\theta(\eta)$ for PST case and (b) $\Phi(\eta)$ for PHF case.

$\theta(\eta)$ and $\Phi(\eta)$ of the fluid temperature will increase. In the other hand, for $Q < 0$ (heat sink) energy absorbs in the boundary layer which causes the temperature profile $\theta(\eta)$ and $\Phi(\eta)$ to decreases in both PST and PHF cases. The effects of the magnetic number $M$ on temperature profile for both PST and PHF cases are plotted in figures 6(a) and 6(b). An increase in the magnetic number $M$ enhances the boundary layer thickness for both cases. Which occurs due to the fact that a resistive Lorentz force is produced by applying the transverse magnetic field to the electrically conducting fluid. This force has a tendency to slow down the motion of the fluid in the boundary layer and increases the temperature distribution. Figures 7(a) and 7(b) demonstrate the effects of Deborah number $\beta$ on the temperature distribution $\theta(\eta)$ and $\Phi(\eta)$ respectively. For both PST and PHF cases, keeping different values of other parameters fixed. As we increase the value of Deborah number $\beta$ the temperature distribution $\theta(\eta)$ and $\Phi(\eta)$ decreases in both PST and PHF cases. Since the Deborah number $\beta$ depend on the retardation time $\lambda_1$ , physically a large retardation time of any material makes it less viscous, which may results an increase in its motion, which consequently reduces the thermal boundary layer thickness and temperature distribution.

Table I displays the effect of various parameters on dimensionless temperature at the wall. From this table we can observe that an increasing values of the magnetic parameter $M$, heat generation/absorption parameter $Q$, Eckert number $E_c$, relaxation times parameter $\lambda$, and radiation number

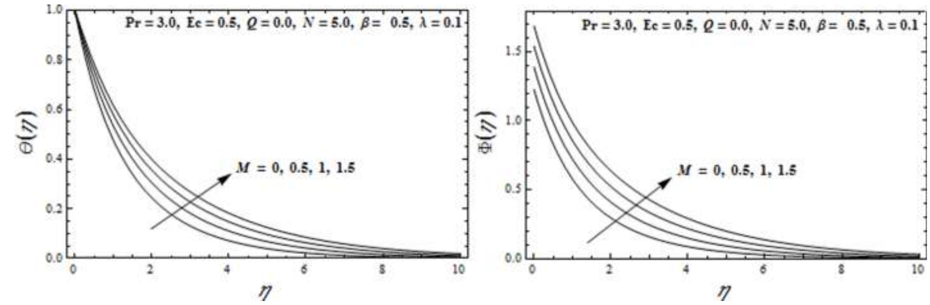


FIG. 6. Influence of $M$ on temperature profiles: (a) $\theta(\eta)$ for PST case and (b) $\Phi(\eta)$ for PHF case.

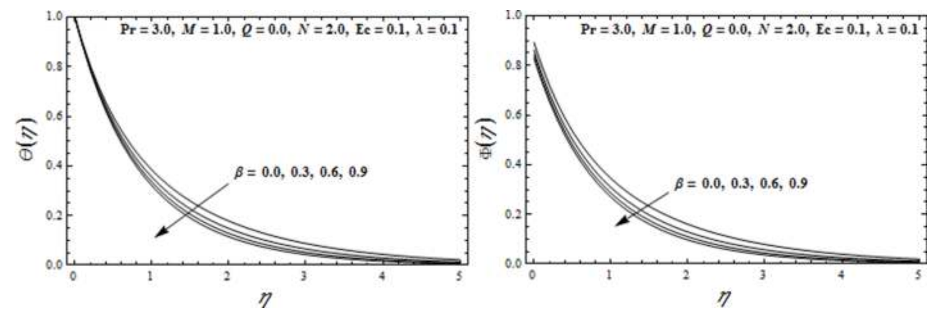


FIG. 7. Influence of $\beta$ on temperature profiles: (a) $\theta(\eta)$ for PST case and (b) $\Phi(\eta)$ for PHF case.

TABLE I. Wall temperature gradient $\theta'(0)$ for PST case and wall temperature $\Phi(0)$ for PHF case.

| $M$ | $Q$ | $E$ | $\beta$ | $\lambda$ | $N$ | Pr | $-\theta'(0)$ PST case | $\Phi(0)$ PHF case |
|---|---|---|---|---|---|---|---|---|
| 0.3 | 0.1 | 0.1 | 0.1 | 0.2 | 0.2 | 1 | 1.04481 | 0.95875 |
| 0.5 | | | | | | | 1.00949 | 0.99104 |
| 0.7 | | | | | | | 0.97464 | 1.02458 |
| 0.3 | -0.1 | | | | | | 1.15458 | 0.87057 |
| | 0.0 | | | | | | 1.10430 | 0.90889 |
| | 0.1 | | | | | | 1.04481 | 0.95875 |
| 0.3 | 0.1 | 0.0 | | | | | 1.08648 | 0.92040 |
| | | 0.2 | | | | | 1.00314 | 0.99711 |
| | | 0.4 | | | | | 0.91978 | 1.07383 |
| 0.3 | 0.1 | 0.1 | 0.0 | | | | 1.02542 | 0.97616 |
| | | | 0.3 | | | | 1.07510 | 0.93288 |
| | | | 0.6 | | | | 1.10750 | 0.90686 |
| 0.3 | 0.1 | 0.1 | 0.1 | 0.0 | | | 1.06961 | 0.93754 |
| | | | | 0.4 | | | 1.01989 | 0.98121 |
| | | | | 0.8 | | | 0.96770 | 1.03219 |
| 0.3 | 0.1 | 0.1 | 0.1 | 0.2 | 0.0 | | 1.18053 | 0.85308 |
| | | | | | 0.3 | | 0.98829 | 1.01140 |
| | | | | | 0.6 | | 0.84836 | 1.17204 |
| 0.3 | 0.1 | 0.1 | 0.1 | 0.2 | 0.2 | 0.7 | 0.80269 | 1.23660 |
| | | | | | | 1.0 | 1.04481 | 0.95875 |
| | | | | | | 1.2 | 1.18053 | 0.853084 |

$N$ is to increase the wall temperature gradient $\theta'(0)$ in PST case and the wall temperature $\Phi(0)$ in PHF case. Also we observe that the effect of Deborah number $\beta$ is to increase the wall temperature gradient $\theta'(0)$ in PST case and the wall temperature $\Phi(0)$ in PHF case.

The influence of Nusselt number in terms of $-\theta'(0)$ in PST case and $1/\Phi(0)$ in PHF case with different values observed through Table I. A remarkable decrease is noticed in the value of Nusselt number with an increase in the value of magnetic parameter $M$ for both PST and PHF cases. The effect of heat source/ sink parameter $Q$ is noticed. The heat transfer rate increases as $Q < 0$ and decreases as $Q > 0$. An increasing value of Eckert number $E_c$ results a decrease in heat transfer rate for both cases. This is due to the fact that large value of Eckert number $E_c$ indicates more energy dissipating in the boundary layer, consequently the fluid temperature increases and the heat transfer rate decreases. A decreasing effect in the value of Nusselt number is observed by increasing the value of radiation parameter $N$. Furthermore, by increasing the value of Prandtl number Pr the Nusselt number increases. As the fluid with larg-e Prandtl number Pr has higher heat capacity and hence increases the heat transfer rate. Moreover, a decrease in the heat transfer rate is seen from the Table I with an increase in the values of ratio of relaxation and retardation times parameter $\lambda$, while a quite opposite trend is observed in case of Deborah number $\beta$.

## CONCLUDING REMARKS

An analytic investigation has been carried out on an MHD Jeffrey fluid past over a stretching sheet with viscous dissipation as well as energy dissipation due to joule heating, internal heat generation/absorption and thermal radiation. By invoking suitable transformation highly nonlinear partial differential equations were reduced into non-linear ordinary differential equation. The closed form solution of the reduced heat equation is obtained in the form of Kummer's function for two types of thermal boundary condition that is prescribed surface temperature (PST) and prescribed heat flux (PHF). The effects of various physical parameters such as magnetic field, thermal radiation, heat source/sink, Eckert number, relaxation time parameter on thermal behavior are examined and discussed graphically in details.


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