

A NOTE ON CUMULATIVE SUMS

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Let $\{Z_i\}$ be a denumerable sequence of identical independent real-valued random variables. Two constants $a > 0 > b$ are chosen and the random variable n defined as the smallest integer for which one of the inequalities $\sum_1^n Z_i \geq a$ or $\sum_1^n Z_i \leq b$ holds. For any events E_1 and E_2 , $P\{E_1\}$ will denote the probability of the event E_1 and $P\{E_1 | E_2\}$ the conditional probability of the event E_1 given that E_2 has occurred.

It will be shown that there exists $t_0 > 0$ such that the moment generating function, Ee^{nt} exists for any complex number t whose real part is less than or equal to t_0 , and as an immediate consequence that n has finite moments of all orders.

If d is any constant satisfying $b < d < a$, then, for fixed m ,

$$(1) \quad P\left\{b < \sum_1^m Z_i + d < a\right\} \leq P\left\{\left|\sum_1^m Z_i\right| < c\right\}$$

where $c = |a| + |b|$. We exclude the case $P\{Z_i = 0\} = 1$. Then there exists $\epsilon > 0$ such that either

$$\delta_1 = P\{Z_i \geq \epsilon\} > 0 \quad \text{or} \quad \delta_2 = P\{Z_i \leq -\epsilon\} > 0.$$

Taking, for example, the former alternative with $m_1 = \left\lceil \frac{c}{\epsilon} \right\rceil + 1$,

$$(2) \quad P\left\{\left|\sum_1^{m_1} Z_i\right| \geq c\right\} \geq P\left\{Z_i \geq \epsilon \text{ for } i = 1, \dots, m_1\right\} = \delta_1^{m_1} > 0$$

where $[w]$ denotes the largest integer less than or equal to w . For any positive integer k ,

$$\begin{aligned} \frac{P\{n > km_1\}}{P\{n > (k-1)m_1\}} &= P\{n > km_1 | n > (k-1)m_1\} \\ &\leq P\left\{b < \sum_1^{km_1} Z_i < a \mid b < \sum_1^s Z_i < a \text{ for } s = 1, \dots, (k-1)m_1\right\} \end{aligned}$$

since $n > km$ implies $b < \sum_1^{km_1} Z_i < a$.

But $\sum_1^{km_1} Z_i = \sum_1^{(k-1)m_1} Z_i + \sum_{(k-1)m_1+1}^{km_1} Z_i$ and the second sum on the right hand side is independent of all terms in the first sum.

Thus the distribution of $\sum_1^{km_1} Z_i$ given $\sum_1^s Z_i$ for $s = 1, \dots, (k - 1)m_1$ depends only on $\sum_1^{(k-1)m_1} Z_i$ so that

$$(3) \quad \frac{P\{n > km_1\}}{P\{n > (k-1)m_1\}} \leq P\left\{b < \sum_{(k-1)m_1+1}^{km_1} Z_i + \sum_1^{(k-1)m_1} Z_i < a \mid b < \sum_1^{(k-1)m_1} Z_i < a\right\} \\ \leq P\left\{\sum_{(k-1)m_1+1}^{km_1} Z_i < c\right\} \leq 1 - \delta_1^{m_1} \text{ by (1) and (2).}$$

Consequently, by induction on k ,

$$(4) \quad P\{n > m\} \leq P\left\{n > \left[\frac{m}{m_1}\right]m_1\right\} \leq (1 - \delta_1^{m_1})^{\lceil \frac{m}{m_1} \rceil}.$$

Let t_0 be any positive number less than $-\frac{1}{m_1} \log(1 - \delta_1^{m_1})$.

Then

$$(5) \quad Ee^{nt_0} = \sum_{m=1}^{\infty} e^{m t_0} P\{n = m\} \\ \leq \sum_{k=1}^{\infty} e^{k m_1 t_0} P\{(k-1)m_1 < n \leq k m_1\} \\ \leq \sum_{k=1}^{\infty} e^{k m_1 t_0} P\{n > (k-1)m_1\} \\ \leq \sum_{k=1}^{\infty} e^{k m_1 t_0} (1 - \delta_1^{m_1})^{k-1} \\ = \frac{1}{1 - \delta_1^{m_1}} \sum_{k=1}^{\infty} \{e^{m_1 t_0} (1 - \delta_1^{m_1})\}^k.$$

But this is a geometric series with decreasing terms, and is consequently convergent. Thus for any t whose real part $R(t) \leq t_0$, the moment generating function Ee^{nt} exists. Since, for all positive l , $m^l < e^{m t_0}$ for sufficiently large m , n has finite moments of all orders.