Mathematical Programming 35 (1986) 365-367
North-Holland

SHORT COMMUNICATION

# A NOTE ON DEGENERACY IN LINEAR PROGRAMMING

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Received 20 September 1985
Revised manuscript received 10 February 1986

We show that the problem of exiting a degenerate vertex is as hard as the general linear programming problem. More precisely, every linear programming problem can easily be reduced to one where the second best vertex (which is highly degenerate) is already given. So, to solve the latter, it is sufficient to exit that vertex in a direction that improves the objective function value.

*Key words*: Degeneracy, strongly polynomial time, randomized simplex.

## 1. Introduction

An interesting question is raised in [1] about the role of degeneracy in the worst-case complexity of the randomized simplex algorithm. It is well known that every linear programming problem can be perturbed into a non-degenerate problem [3, 4]. However, it is interesting to know how hard the problem is of "exiting" a degenerate *vertex* in a direction that improves the objective function value. To formulate the question more precisely, consider the following definition.

**Definition 1.1.** The following problem will be called the *degeneracy problem.* A linear programming problem is given in the form

$$\begin{aligned} &\text{minimize} && c^{\mathrm{T}}x \\ &\text{subject to} && Ax \geqslant b \end{aligned}$$

(where $m > n$ and $A \in R^{m\times n}$, $b \in R^m$ and $\{c, x\} \subseteq R^n$) together with a degenerate solution (possibly a vertex) $u \in R^n$ such that $Au = b$. Decide whether $u$ is an optimal solution; if $u$ is not optimal, then provide a feasible direction of improvement, that is, a vector $w$ such that $c^{\mathrm{T}}w < 0$ and $Aw \geqslant 0$.

The degeneracy problem can obviously be solved as a linear programming problem. Thus, it is in the class *P.* Moreover, there are standard techniques for dealing with degeneracy [3, 4] and finite pivoting rules were developed by Bland [2]. We are interested here in the question whether the degeneracy problem is easier than the general linear programming problem. The question is of interest in the context of the uniform cost model. It is not known whether the linear programming problem can be solved in strongly polynomial time, that is, in a polynomial number $p(m, n)$ of arithmetic operations. The result proved below implies that if the degeneracy problem is solvable in strongly polynomial time, then so is the general linear programming problem.

## 2. The result

Suppose a linear programming problem $P$ with rational coefficients is given in the form

$$\begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x \\ \text{subject to} & Ax \geq b \end{array}$$

and assume $A$ is of full rank. Without loss of generality let us assume that $P$ has an optimal solution, and furthermore, assume that the optimal value of the objective function is equal to zero. A justification for this assumption can be found in [5]. Moreover, assume that $P$ has a unique optimal basic solution, that is, a unique (nonsingular) submatrix $B \in R^{n\times n}$ of $A$ such that $B^{-1}b_B$ (where $b_B$ is the restriction of $b$ to the coordinates corresponding to $B$) is an optimal solution. This assumption can be justified by small perturbations.

Since the coefficients are rational, we can easily derive a lower bound $\delta > 0$ on the value of the objective function at any nonoptimal basic feasible solution. Such a $\delta$ depends on $m$, $n$ and the largest integer occurring as a numerator or denominator in any coefficient. A suitable $\delta$ can easily be found in a linear number of arithmetic operations, and its size (that is, the length of its binary encoding) is bounded by a polynomial in the size of the problem. Now, let $v \in R^n$ be any vector such that $c^{\mathrm{T}}v = \delta$. Since by our assumptions $c \neq 0$, we may simply choose $v$ so that $v_j = \delta / c_j$ for some $j$ such that $c_j \neq 0$, and $v_i = 0$ for $i \neq j$. Now, consider the following linear programming problem $P^*$ in $n+1$ variables:

$$\begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x \\ \text{subject to} & Ax + (b - Av)x_{n+1} \geq b, \\ & 0 \leq x_{n+1} \leq 1. \end{array}$$

Obviously, $(v, 1)$ is a degenerate solution.

**Proposition 2.1.** *The feasible region of $P^*$ has no vertices with $0 < x_{n+1} < 1$.*

**Proof.** Suppose $(x, x_{n+1})$ is any feasible solution of $P^*$ with $0 < x_{n+1} < 1$ and consider the straight line determined by $(x, x_{n+1})$ and $(v, 1)$. For any $t \geq 0$, $A[tx + (1-t)v] + (b - Av)[tx_{n+1} + (1-t)1] \geq b$. Thus, for any $t \geq 0$ such that $tx_{n+1} + 1 - t \geq 0$, the point $t(x, x_{n+1}) + (1-t)(v, 1)$ is feasible in $P^*$. This range of values of $t$ is equal to the interval $[0, 1/(1-x_{n+1})]$ and hence includes $t=1$ in its interior. It follows that $(x, x_{n+1})$ is not a vertex.

It can similarly be shown that the feasible region has no vertex of the form $(x, 1)$ where $x \neq v$.

**Proposition 2.2.** *If $(x, x_{n+1})$ is feasible in $P^*$ and $0 < x_{n+1} < 1$ then $c^T x > 0$.*

**Proof.** Suppose, to the contrary, that $(x, x_{n+1})$ is feasible and $c^T x \leq 0$. Let $t = 1/(1-x_{n+1})$ and consider the point $(y, y_{n+1}) = t(x, x_{n+1}) + (1-t)(v, 1)$. It follows that $y_{n+1} = 0$, $Ay \geq b$ and $c^T y < 0$, which contradicts our assumption that the optimal value of $P$ is zero.

**Proposition 2.3.** *There is no feasible solution $(x, 1)$ of $P^*$ with $c^T x < c^T v$.*

**Proof.** The existence of such a point implies the existence of feasible points $(y, 1)$ with *any* negative value for $c^T y$. This contradicts Proposition 2.2.

**Corollary 2.4.** *The optimal objective function value of $P^*$ is also* 0.

Consider the degeneracy problem at $(v, 1)$. We have to find a vector $(w, w_{n+1})$ such that for sufficiently small $\varepsilon > 0$, $(v, 1) + \varepsilon(w, w_{n+1})$ is feasible and $c^T w < 0$. It follows from the previous propositions that such a vector $(w, w_{n+1})$ would lead us to a point $y$ which is feasible in $P$ and $0 \leq c^T y < \delta$. Once at $y$, finding an optimal solution to $P$ is straightforward. This establishes the equivalence of the two problems from the worst-case viewpoint in the sense that one is solvable in strongly polynomial time or randomized strongly polynomial time if and only if the other is also. It is still an open problem to study the role of degeneracy in the 'average case' performance of algorithms for linear programming.

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