A NOTE ON EIGENVALUES OF BICOMPLEX MATRIX

JOGENDRA KUMAR

ABSTRACT. In this paper, we have studied eigenvalues and eigenvectors of the bicomplex matrix and investigated their properties and established some results. We have also established some results on the eigenvalues of some special bicomplex matrices.

2010 AMS SUBJECT CLASSIFICATION. 30G35, 32A30, 32A10, 15B57, 15B10, 15A18.

KEYWORDS AND PHRASES. Bicomplex matrix,Bicomplex Hermitian Matrix, Bicomplex Unitary Matrix, Eigenvalues and Eigenvectors of Bicomplex Matrix.

1. INTRODUCTION

Throughout this paper, the set of Bicomplex numbers is denoted by $\mathbb{C}_2$ and the sets of complex and real numbers are denoted by $\mathbb{C}_1$ and $\mathbb{C}_0$,respectively. For detail of the theory (cf. [1, 2, 3, 4]).

Definition 1.1 (Bicomplex numbers). The set of bicomplex numbers is defined as:

$$\mathbb{C}_2 = \{x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 : x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$$

where $i_1 \neq i_2, i_1^2 = i_2^2 = -1$ and, $i_1i_2 = i_2i_1$.

We shall use the notations $\mathbb{C}(i_1)$ and $\mathbb{C}(i_2)$ for the following sets:
$\mathbb{C}(i_1) = \{x + i_1y : x, y \in \mathbb{C}_0\}$
$\mathbb{C}(i_2) = \{x + i_2y : x, y \in \mathbb{C}_0\}$

1.1. **Bicomplex Matrix.** In this section, we discussed the bicomplex matrices along with their properties and some results. [5, 6, 7, 8].

Here, we denote $\mathbb{C}_2^{m\times n} = \{A = [\xi_{ij}]_{m\times n} : \xi_{ij} \in \mathbb{C}_2\}$ as the set of $m \times n$ matrices with bicomplex entries. Let $A = [\xi_{ij}]_{m\times n} \in \mathbb{C}_2^{m\times n} \Rightarrow \xi_{ij} \in \mathbb{C}_2$

1.2. **Idempotent Representation of Bicomplex Matrix.** We can represent the Bicomplex Matrix in two different idempotent forms in terms of $\mathbb{C}^{m\times n}(i_1)$ and $\mathbb{C}^{m\times n}(i_2)$, explained as follows :

(i) The $\mathbb{C}^{m\times n}(i_1)$-idempotent representation of Bicomplex Matrix $A = [\xi_{ij}]_{m\times n} \in \mathbb{C}_2^{m\times n}$ is given by

$$A = [\xi_{ij}]_{m\times n} = [{}^1\xi_{ij}]_{m\times n}e_1 + [{}^2\xi_{ij}]_{m\times n}e_2 = {}^1Ae_1 + {}^2Ae_2,$$

where ${}^1A = [{}^1\xi_{ij}]_{m\times n}, {}^2A = [{}^2\xi_{ij}]_{m\times n} \in \mathbb{C}^{m\times n}(i_1)$

(ii) The $\mathbb{C}^{m\times n}(i_2)$-idempotent representation of Bicomplex Matrix $A = [\xi_{ij}]_{m\times n} \in \mathbb{C}_2^{m\times n}$ is given by

$$A = [\xi_{ij}]_{m\times n} = [\xi_{1,ij}]_{m\times n} e_1 + [\xi_{2,ij}]_{m\times n} e_2 = A_1 e_1 + A_2 e_2,$$

where $A_1 = [\xi_{1,ij}]_{m\times n}, A_2 = [\xi_{2,ij}]_{m\times n} \in \mathbb{C}^{m\times n}(i_2)$

1.3. **Determinant of Bicomplex Matrices.** As, only square matrices can have determinant, so let $A = [\xi_{ij}]_{n\times n} \in \mathbb{C}_2^{n\times n}$. Then determinant of A is denoted as $\det(A)$.

1.4. **Non-Singular and Singular Bicomplex Matrix.**

A matrix $A = [\xi_{ij}]_{n\times n} \in \mathbb{C}_2^{n\times n}$ is said to be Non-Singular (Invertible) if $\det(A) \notin \mathbb{O}_2$. It is said to be Singular (Non-invertible) matrix if $\det(A) \in \mathbb{O}_2$.

1.5. **Conjugation of Bicomplex Matrix.** Here, we have given three different types of conjugations of the bicomplex matrices. Let $A = [\xi]_{m\times n} \in \mathbb{C}_2^{n\times n}$ be a bicomplex matrix. Then, its conjugations are defined as follows:

(a) i_1**-Conjugation**

$A^* = ({}^1Ae_1 + {}^2Ae_2)^* = \overline{{}^2A}e_1 + \overline{{}^1A}e_2$ when ${}^1A, {}^2A \in \mathbb{C}^{m\times n}(i_1)$,

$A^* = (A_1e_1 + A_2e_2)^* = A_2e_1 + A_1e_2$ when $A_1, A_2 \in \mathbb{C}^{m\times n}(i_2)$.

(b) i_2**-Conjugation**

$A^{\#} = ({}^1Ae_1 + {}^2Ae_2)^{\#} = {}^2Ae_1 + {}^1Ae_2$ when ${}^1A, {}^2A \in \mathbb{C}^{m\times n}(i_1)$,

$A^{\#} = (A_1e_1 + A_2e_2)^{\#} = \overline{A}_2e_1 + \overline{A}_1e_2$ when $A_1, A_2 \in \mathbb{C}^{m\times n}(i_2)$.

(c) i_1i_2**-Conjugation**

$A' = ({}^1Ae_1 + {}^2Ae_2)' = \overline{{}^1A}e_1 + \overline{{}^2A}e_2$ when ${}^1A, {}^2A \in \mathbb{C}^{m\times n}(i_1)$,

$A' = (A_1e_1 + A_2e_2)' = \overline{A}_1e_1 + \overline{A}_2e_2$ when $A_1, A_2 \in \mathbb{C}^{m\times n}(i_2)$.

Definition 1.2 (**Hyperbolic Matrix**)**.** The set of Hyperbolic Matrix is defined as : $\mathbb{H}^{m\times n} = \{A = [\xi_{ij}]_{m\times n} : \xi_{ij} \in \mathbb{H}\}$. If $A = {}^1Ae_1 + {}^2Ae_2 \in \mathbb{H}^{m\times n}$, then ${}^1A, {}^2A \in \mathbb{C}_0^{m\times n}$

2. Eigenvalues and Eigenvectors of a Bicomplex Matrix

2.1. **Eigenvalues of a Bicomplex Matrix.** Let $A = [\xi_{ij}]_{n\times n} \in \mathbb{C}_2^{n\times n}$ and $\lambda \in \mathbb{C}_2$, then the matrix $A - \lambda I$ is called the characteristic matrix of A and $P(\lambda) = \det(A - \lambda I)$ is called characteristic polynomial of A.

The equation $\det(A - \lambda I) = 0$ is called the characteristic equation of A and the roots of this equation are called characteristic roots or latent roots or characteristic values or eigenvalues of the matrix A. The set of all eigenvalues of the matrix A is denoted by $A(\lambda)$.

Note 2.1. $A(\lambda) = {}^1A({}^1\lambda)e_1 + {}^2A({}^2\lambda)e_2$ is the collection of all eigenvalues of A iff ${}^1A({}^1\lambda)$ and ${}^2A({}^2\lambda)$ are spectrum of ${}^1A$ and ${}^2A$ respectively.

2.2. **Eigenvectors of a Bicomplex Matrix.** Let $A = [\xi_{ij}]_{n\times n} \in \mathbb{C}_2^{n\times n}$ and $\lambda \in \mathbb{C}_2$ is a eigenvalue of A, then there exist $X = {}^1Xe_1 + {}^2Xe_2 \in \mathbb{C}_2^{n\times 1}$, ${}^1X \neq O$ and ${}^2X \neq O$ such that $AX = \lambda X$, then X is called characteristic vector or eigenvector of A corresponding to the eigenvalue λ.

2.3. **Properties of Eigenvalues of Some Special Matrices.**

Proposition 2.1. *If $\lambda = {}^1\lambda e_1 + {}^2\lambda e_2$ is the eigenvalue of $A = {}^1Ae_1 + {}^2Ae_2 \in \mathbb{H}^{n\times n}$, then $\lambda' = \overline{{}^1\lambda}e_1 + \overline{{}^2\lambda}e_2$ is also eigenvalue of A.*

Proof. Let $\lambda = {}^1\lambda e_1 + {}^2\lambda e_2$ is the eigenvalue of $A = {}^1Ae_1 + {}^2Ae_2 \in \mathbb{H}^{n\times n}$

$\Rightarrow P(\lambda) = 0$

$\Rightarrow \det(A - \lambda I) = 0$

$\Rightarrow \det({}^1A - {}^1\lambda I) = 0$ and $\det({}^2A - {}^2\lambda I) = 0$

$\Rightarrow {}^1P({}^1\lambda) = 0$ and ${}^2P({}^2\lambda) = 0$

$\Rightarrow \overline{{}^1P({}^1\lambda)} = 0$ and $\overline{{}^2P({}^2\lambda)} = 0$

As, ${}^1A, {}^2A \in \mathbb{C}_0^{n\times n}$

$\Rightarrow \overline{{}^1P({}^1\lambda)} = {}^1P(\overline{{}^1\lambda})$ and $\overline{{}^2P({}^2\lambda)} = {}^2P(\overline{{}^2\lambda})$

$\Rightarrow {}^1P(\overline{{}^1\lambda}) = 0$ and ${}^2P(\overline{{}^2\lambda}) = 0$

$\Rightarrow {}^1P(\overline{{}^1\lambda})e_1 + {}^2P(\overline{{}^2\lambda})e_2 = 0$

$\Rightarrow P(\overline{{}^1\lambda}e_1 + \overline{{}^2\lambda}e_2) = 0$

$\Rightarrow P(\lambda') = 0$

Hence $\lambda' = \overline{{}^1\lambda}e_1 + \overline{{}^2\lambda}e_2$ is also eigenvalue of A.

□

Proposition 2.2. *Let $X = {}^1Xe_1 + {}^2Xe_2$ be the eigenvector of the matrix $A = {}^1Ae_1 + {}^2Ae_2 \in \mathbb{H}^{n\times n}$. Then $X' = \overline{{}^1X}e_1 + \overline{{}^2X}e_2$ is the eigenvector of A for the eigenvalue $\lambda' = \overline{{}^1\lambda}e_1 + \overline{{}^2\lambda}e_2$.*

Proof. Let $X = {}^1Xe_1 + {}^2Xe_2$ be the eigenvector of the matrix $A = {}^1Ae_1 + {}^2Ae_2 \in \mathbb{H}^{n\times n}$ associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$

$\Rightarrow (AX)' = (\lambda X)'$

$\Rightarrow A'X' = \lambda' X'$

$\Rightarrow AX' = \lambda' X'$ $(\because A = {}^1Ae_1 + {}^2Ae_2 \in \mathbb{H}^{n\times n} \Rightarrow A' = A)$

$\Rightarrow X' = \overline{{}^1X}e_1 + \overline{{}^2X}e_2$ is the eigenvector of A for the eigenvalue $\lambda' = \overline{{}^1\lambda}e_1 + \overline{{}^2\lambda}e_2$.

□

Proposition 2.3. *Let $A \in \mathbb{H}^{n\times n}$ be a Symmetric matrix and let λ be an eigenvalue of A, then $\lambda' = \lambda$ or equivalently $\lambda \in \mathbb{H}$.*

Proof. Let $A = {}^1Ae_1 + {}^2Ae_2 \in \mathbb{H}^{n\times n}$ be a symmetric matrix i.e. $A^t = A$

Let $X = {}^1Xe_1 + {}^2Xe_2$ be the eigenvector of the matrix $A = {}^1Ae_1 + {}^2Ae_2 \in \mathbb{H}^{n\times n}$ associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$
$\Rightarrow (AX)' = (\lambda X)'$
$\Rightarrow A'X' = \lambda' X'$
$\Rightarrow AX' = \lambda' X'$ $(\because A = {}^1Ae_1 + {}^2Ae_2 \in \mathbb{H}^{n\times n} \Rightarrow A' = A)$
Again now,
$(AX)^t = (\lambda X)^t$
$\Rightarrow X^tA^t = \lambda X^t$
$\Rightarrow X^tA = \lambda X^t$ $(\because A^t = A)$
$\Rightarrow (X^tA)X' = (\lambda X^t)X'$
$\Rightarrow X^t(AX') = \lambda(X^tX')$
$\Rightarrow X^t(\lambda' X') = \lambda(X^tX')$
$\Rightarrow \lambda'(X^tX') = \lambda(X^tX')$
$\Rightarrow (\lambda' - \lambda)X^tX' = 0$
$\Rightarrow (\overline{{}^1\lambda} - {}^1\lambda)({}^1X)^t\,\overline{{}^1X}e_1 + (\overline{{}^2\lambda} - {}^2\lambda)({}^2X)^t\,\overline{{}^2X}e_2 = 0$
$\Rightarrow (\overline{{}^1\lambda} - {}^1\lambda)({}^1X)^t\,\overline{{}^1X} = 0$ and $(\overline{{}^2\lambda} - {}^2\lambda)({}^2X)^t\,\overline{{}^2X} = 0$
$\Rightarrow \overline{{}^1\lambda} - {}^1\lambda = 0$ and $\overline{{}^2\lambda} - {}^2\lambda = 0$ $(\because ({}^1X)^t\,\overline{{}^1X} \neq 0$ and $({}^2X)^t\,\overline{{}^2X} \neq 0)$
$\Rightarrow \overline{{}^1\lambda} = {}^1\lambda$ and $\overline{{}^2\lambda} = {}^2\lambda$
$\Rightarrow \lambda = {}^1\lambda e_1 + {}^2\lambda e_2 = \overline{{}^1\lambda}e_1 + \overline{{}^2\lambda}e_2 = \lambda'$
$\Rightarrow \lambda \in \mathbb{H}$

□

Proposition 2.4. *Let $A \in \mathbb{C}_2^{n\times n}$ be a Bicomplex idempotent matrix and let λ be an eigenvalue of A, then $\lambda \in \{0, 1, e_1, e_2\}$.*

Proof. Let A be any Bicomplex idempotent matrix i.e. $A^2 = A$

Let $X = {}^1Xe_1 + {}^2Xe_2$ be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$
$\Rightarrow A(AX) = A(\lambda X)$
$\Rightarrow (AA)X = \lambda(AX)$
$\Rightarrow A^2X = \lambda(AX)$
$\Rightarrow AX = \lambda(AX)$ $(\because A^2 = A)$
$\Rightarrow \lambda X = \lambda(\lambda X)$ $(\because AX = \lambda X)$
$\Rightarrow \lambda X = \lambda^2 X$
$\Rightarrow (\lambda - \lambda^2)X = O$
$\Rightarrow ({}^1\lambda - {}^1\lambda^2)\,{}^1Xe_1 + ({}^2\lambda - {}^2\lambda^2)\,{}^2Xe_2 = O$
$\Rightarrow ({}^1\lambda - {}^1\lambda^2)\,{}^1X = O$ and $({}^2\lambda - {}^2\lambda^2)\,{}^2X = O$
$\Rightarrow {}^1\lambda - {}^1\lambda^2 = 0$ and ${}^2\lambda - {}^2\lambda^2 = 0$ $(\because {}^1X \neq O$ and ${}^2X \neq O$)
$\Rightarrow {}^1\lambda = 0, 1$ and ${}^2\lambda = 0, 1$
$\Rightarrow \lambda \in \{0, 1, e_1, e_2\}$

□

Proposition 2.5. *Let $A \in \mathbb{C}_2^{n\times n}$ be a Bicomplex skew-idempotent matrix and let λ be an eigenvalue of A, then $\lambda \in \{0, -1, -e_1, -e_2\}$.*

Proof. Let A be any Bicomplex Skew-idempotent matrix i.e. $A^2 = -A$

Let $X = {}^1Xe_1 + {}^2Xe_2$ be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$

$\Rightarrow A(AX) = A(\lambda X)$

$\Rightarrow (AA)X = \lambda(AX)$

$\Rightarrow A^2X = \lambda(AX)$

$\Rightarrow -AX = \lambda(AX)$ $(\because A^2 = -A)$

$\Rightarrow -\lambda X = \lambda(\lambda X)$ $(\because AX = \lambda X)$

$\Rightarrow -\lambda X = \lambda^2 X$

$\Rightarrow (-\lambda - \lambda^2)X = O$

$\Rightarrow (\lambda + \lambda^2)X = O$

$\Rightarrow ({}^1\lambda + {}^1\lambda^2)\,{}^1Xe_1 + ({}^2\lambda + {}^2\lambda^2)\,{}^2Xe_2 = O$

$\Rightarrow ({}^1\lambda + {}^1\lambda^2)\,{}^1X = O$ and $({}^2\lambda + {}^2\lambda^2)\,{}^2X = O$

$\Rightarrow {}^1\lambda + {}^1\lambda^2 = 0$ and ${}^2\lambda + {}^2\lambda^2 = 0$ $(\because {}^1X \neq O$ and ${}^2X \neq O$)

$\Rightarrow {}^1\lambda = 0, -1$ and ${}^2\lambda = 0, -1$

$\Rightarrow \lambda \in \{0, -1, -e_1, -e_2\}$

□

Proposition 2.6. *Let $A \in \mathbb{C}_2^{n\times n}$ such that $A^2 = \eta A, \eta \in \mathbb{C}_2$ and let λ be an eigenvalue of A, then $\lambda \in \{0, \eta, {}^1\eta e_1, {}^2\eta e_2\}$.*

Proof. Let $A \in \mathbb{C}_2^{n\times n}$ such that $A^2 = \eta A, \eta \in \mathbb{C}_2$

Let $X = {}^1Xe_1 + {}^2Xe_2$ be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$

$\Rightarrow A(AX) = A(\lambda X)$

$\Rightarrow (AA)X = \lambda(AX)$

$\Rightarrow A^2X = \lambda(AX)$

$\Rightarrow (\eta A)X = \lambda(AX)$ $(\because A^2 = \eta A)$

$\Rightarrow \eta(AX) = \lambda(AX)$

$\Rightarrow \eta(\lambda X) = \lambda(\lambda X)$ $(\because AX = \lambda X)$

$\Rightarrow \eta\lambda X = \lambda^2 X$

$\Rightarrow (\eta\lambda - \lambda^2)X = O$

$\Rightarrow ({}^1\eta\,{}^1\lambda - {}^1\lambda^2)\,{}^1Xe_1 + ({}^2\eta\,{}^2\lambda - {}^2\lambda^2)\,{}^2Xe_2 = O$

$\Rightarrow ({}^1\eta\,{}^1\lambda - {}^1\lambda^2)\,{}^1X = O$ and $({}^2\eta\,{}^2\lambda - {}^2\lambda^2)\,{}^2X = O$

$\Rightarrow {}^1\eta\,{}^1\lambda - {}^1\lambda^2 = 0$ and ${}^2\eta\,{}^2\lambda - {}^2\lambda^2 = 0$ $(\because {}^1X \neq O$ and ${}^2X \neq O$ $)$
$\Rightarrow {}^1\lambda = 0, {}^1\eta$ and ${}^2\lambda = 0, {}^2\eta$
$\Rightarrow \lambda \in \{0, \eta, {}^1\eta e_1, {}^2\eta e_2\}$ □

Note 2.2.
(i) For $\eta = 1$, we get **proposition 2.4**
(ii) For $\eta = -1$, we get **proposition 2.5**

Proposition 2.7. *Let $A \in \mathbb{C}_2^{n\times n}$ be a Bicomplex involutory matrix and let λ be an eigenvalue of A, then $\lambda \in \{1, -1, i_1 i_2, -i_1 i_2\}$.*

Proof. Let A be any Bicomplex involutory matrix i.e. $A^2 = I$

Let $X = {}^1Xe_1 + {}^2Xe_2$ be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$
$\Rightarrow A(AX) = A(\lambda X)$
$\Rightarrow (AA)X = \lambda(AX)$
$\Rightarrow A^2X = \lambda(AX)$
$\Rightarrow IX = \lambda(AX)$ $(\because A^2 = I)$
$\Rightarrow X = \lambda(\lambda X)$ $(\because AX = \lambda X)$
$\Rightarrow X = \lambda^2 X$
$\Rightarrow (1 - \lambda^2)X = O$
$\Rightarrow (1 - {}^1\lambda^2)\,{}^1Xe_1 + (1 - {}^2\lambda^2)\,{}^2Xe_2 = O$
$\Rightarrow (1 - {}^1\lambda^2)\,{}^1X = O$ and $(1 - {}^2\lambda^2)\,{}^2X = O$
$\Rightarrow 1 - {}^1\lambda^2 = 0$ and $1 - {}^2\lambda^2 = 0$ $(\because {}^1X \neq O$ and ${}^2X \neq O$ $)$
$\Rightarrow {}^1\lambda = 1, -1$ and ${}^2\lambda = 1, -1$
$\Rightarrow \lambda \in \{1, -1, i_1 i_2, -i_1 i_2\}$

□

Proposition 2.8. *Let $A \in \mathbb{C}_2^{n\times n}$ be a Bicomplex Skew-involutory matrix and let λ be an eigenvalue of A, then $\lambda \in \{i_1, -i_1, i_2, -i_2\}$.*

Proof. Let $A \in \mathbb{C}_2^{n\times n}$ be a Bicomplex Skew-involutory matrix i.e. $A^2 = -I$

Let X be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$
$\Rightarrow A(AX) = A(\lambda X)$
$\Rightarrow (AA)X = \lambda(AX)$
$\Rightarrow A^2X = \lambda(AX)$
$\Rightarrow (-I)X = \lambda(AX)$ $(\because A^2 = -I)$
$\Rightarrow -(IX) = \lambda(AX)$
$\Rightarrow -X = \lambda(AX)$
$\Rightarrow -X = \lambda(\lambda X)$ $(\because AX = \lambda X)$

$\Rightarrow -X = \lambda^2 X$
$\Rightarrow (-1 - \lambda^2)X = O$
$\Rightarrow (1 + \lambda^2)X = O$
$\Rightarrow (1 + {}^1\lambda^2)\,{}^1Xe_1 + (1 + {}^2\lambda^2)\,{}^2Xe_2 = O$
$\Rightarrow (1 + {}^1\lambda^2)\,{}^1X = O$ and $(1 + {}^2\lambda^2)\,{}^2X = O$
$\Rightarrow 1 + {}^1\lambda^2 = 0$ and $1 + {}^2\lambda^2 = 0$ ($\because {}^1X \neq O$ and ${}^2X \neq O$)
$\Rightarrow {}^1\lambda = i_1, -i_1$ and ${}^2\lambda = i_1, -i_1$
$\Rightarrow \lambda \in \{i_1, -i_1, i_2, -i_2\}$

□

Proposition 2.9. *Let $A \in \mathbb{C}_2^{n\times n}$ such that $A^2 = \eta I, \eta \in \mathbb{C}_2$ and let λ be an eigenvalue of A, then $\lambda \in \{\sqrt{{}^1\eta}e_1 + \sqrt{{}^2\eta}e_2, -(\sqrt{{}^1\eta}e_1 + \sqrt{{}^2\eta}e_2), \sqrt{{}^1\eta}e_1 - \sqrt{{}^2\eta}e_2, -\sqrt{{}^1\eta}e_1 + \sqrt{{}^2\eta}e_2\}$.*

Proof. Let $A \in \mathbb{C}_2^{n\times n}$ be any Bicomplex matrix such that $A^2 = \eta I$
Let X be the eigenvector of the matrix A associated with the eigenvalue λ
$\Rightarrow AX = \lambda X$
$\Rightarrow A(AX) = A(\lambda X)$
$\Rightarrow (AA)X = \lambda(AX)$
$\Rightarrow A^2X = \lambda(AX)$
$\Rightarrow (\eta I)X = \lambda(AX)$ ($\because A^2 = \eta I$)
$\Rightarrow \eta(IX) = \lambda(AX)$
$\Rightarrow \eta X = \lambda(AX)$
$\Rightarrow \eta X = \lambda(\lambda X)$ ($\because AX = \lambda X$)
$\Rightarrow \eta X = \lambda^2 X$
$\Rightarrow (\eta - \lambda^2)X = O$
$\Rightarrow ({}^1\eta - {}^1\lambda^2)\,{}^1Xe_1 + ({}^2\eta - {}^2\lambda^2)\,{}^2Xe_2 = O$
$\Rightarrow ({}^1\eta - {}^1\lambda^2)\,{}^1X = O$ and $({}^2\eta - {}^2\lambda^2)\,{}^2X = O$
$\Rightarrow {}^1\eta - {}^1\lambda^2 = 0$ and ${}^2\eta - {}^2\lambda^2 = 0$ ($\because {}^1X \neq O$ and ${}^2X \neq O$)
$\Rightarrow {}^1\lambda = \sqrt{{}^1\eta}, -\sqrt{{}^1\eta}$ and ${}^2\lambda = \sqrt{{}^2\eta}, -\sqrt{{}^2\eta}$
$\Rightarrow \lambda \in \{\sqrt{{}^1\eta}e_1 + \sqrt{{}^2\eta}e_2, -(\sqrt{{}^1\eta}e_1 + \sqrt{{}^2\eta}e_2), \sqrt{{}^1\eta}e_1 - \sqrt{{}^2\eta}e_2, -\sqrt{{}^1\eta}e_1 + \sqrt{{}^2\eta}e_2\}$

□

Note 2.3.
(i) For $\eta = 1$, we get **proposition 2.7**
(ii) For $\eta = -1$, we get **proposition 2.8**

Proposition 2.10.
(i) *Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1-Hermitian Matrix and λ is an eigenvalue of A with corresponding eigenvector $X = {}^1Xe_1 + {}^2Xe_2$, such that $(\overline{{}^2X})^t\,{}^1X \neq 0$. Then $\lambda = \lambda^*$ or equivalently $\lambda \in \mathbb{C}(i_2)$.*

(ii) *Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1 - Skew - Hermitian Matrix and λ is a eigenvalue of A with corresponding eigenvector $X = {}^1Xe_1 + {}^2Xe_2$, such that $(\overline{{}^2X})^t\,{}^1X \neq 0$. Then $\lambda = -\lambda^*$ or equivalently $\lambda \in i_1\mathbb{C}(i_2)$.*

Proof.

(i) Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1 - Hermitian Matrix i.e. $A^{*t} = A$

Let $X = {}^1Xe_1 + {}^2Xe_2$ be the eigenvector of the matrix A associated with the eigenvalue λ such that $(\overline{{}^2X})^t\,{}^1X \neq 0$

$\Rightarrow AX = \lambda X$

$\Rightarrow (AX)^{*t} = (\lambda X)^{*t}$

$\Rightarrow X^{*t}A^{*t} = \lambda^* X^{*t}$

$\Rightarrow X^{*t}A = \lambda^* X^{*t}$ (As A is i_1 - Hermitian matrix $A^{*t} = A$)

$\Rightarrow (X^{*t}A)X = (\lambda^* X^{*t})X$

$\Rightarrow X^{*t}(AX) = \lambda^*(X^{*t}X)$

$\Rightarrow X^{*t}(\lambda X) = \lambda^*(X^{*t}X)$ ($\because AX = \lambda X$)

$\Rightarrow \lambda(X^{*t}X) = \lambda^*(X^{*t}X)$

$\Rightarrow (\lambda - \lambda^*)X^{*t}X = 0$

$\Rightarrow ({}^1\lambda - \overline{{}^2\lambda})(\overline{{}^2X})^t\,{}^1Xe_1 + ({}^2\lambda - \overline{{}^1\lambda})(\overline{{}^1X})^t\,{}^2Xe_2 = 0$

$\Rightarrow ({}^1\lambda - \overline{{}^2\lambda})(\overline{{}^2X})^t\,{}^1X = 0$ and $({}^2\lambda - \overline{{}^1\lambda})(\overline{{}^1X})^t\,{}^2X = 0$

$\Rightarrow {}^1\lambda - \overline{{}^2\lambda} = 0$ and ${}^2\lambda - \overline{{}^1\lambda} = 0$ ($\because (\overline{{}^2X})^t\,{}^1X \neq 0 \Rightarrow (\overline{{}^1X})^t\,{}^2X \neq 0$)

$\Rightarrow {}^1\lambda = \overline{{}^2\lambda}$ and ${}^2\lambda = \overline{{}^1\lambda}$

$\Rightarrow \lambda = {}^1\lambda e_1 + {}^2\lambda e_2 = \overline{{}^2\lambda}e_1 + \overline{{}^1\lambda}e_2 = \lambda^*$

$\Rightarrow \lambda \in \mathbb{C}(i_2)$

(ii) Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1 - Skew - Hermitian Matrix i.e. $A^{*t} = -A$

Let $X = {}^1Xe_1 + {}^2Xe_2$ be the eigenvector of the matrix A associated with the eigenvalue λ such that $(\overline{{}^2X})^t\,{}^1X \neq 0$

$\Rightarrow AX = \lambda X$

$\Rightarrow (\mathrm{AX})^{*t} = (\lambda X)^{*t}$

$\Rightarrow X^{*t}A^{*t} = \lambda^* X^{*t}$

$\Rightarrow X^{*t}(-A) = \lambda^* X^{*t}$ (As A is i_1 - Skew - Hermitian matrix $A^{*t} = -A$)

$\Rightarrow -(X^{*t}A)X = (\lambda^* X^{*t})X$

$\Rightarrow -X^{*t}(AX) = \lambda^*(X^{*t}X)$

$\Rightarrow -X^{*t}(\lambda X) = \lambda^*(X^{*t}X)$ ($\because AX = \lambda X$)

$\Rightarrow -\lambda(X^{*t}X) = \lambda^*(X^{*t}X)$

$\Rightarrow (\lambda + \lambda^*)X^{*t}X = 0$

$\Rightarrow ({}^1\lambda + \overline{{}^2\lambda})(\overline{{}^2X})^t\,{}^1Xe_1 + ({}^2\lambda + \overline{{}^1\lambda})(\overline{{}^1X})^t\,{}^2Xe_2 = 0$

$\Rightarrow ({}^1\lambda + \overline{{}^2\lambda})(\overline{{}^2X})^t\,{}^1X = 0$ and $({}^2\lambda + \overline{{}^1\lambda})(\overline{{}^1X})^t\,{}^2X = 0$

$\Rightarrow {}^1\lambda + \overline{{}^2\lambda} = 0$ and ${}^2\lambda + \overline{{}^1\lambda} = 0$ ($\because (\overline{{}^2X})^t\,{}^1X \neq 0 \Rightarrow (\overline{{}^1X})^t\,{}^2X \neq 0$)

$\Rightarrow {}^1\lambda = -\overline{{}^2\lambda}$ and ${}^2\lambda = -\overline{{}^1\lambda}$

$\Rightarrow \lambda = {}^1\lambda e_1 + {}^2\lambda e_2 = -(\overline{{}^2\lambda} e_1 + \overline{{}^1\lambda} e_2) = -\lambda^*$
$\Rightarrow \lambda \in i_1\mathbb{C}(i_2)$

□

Proposition 2.11.
(i) *Let $A \in \mathbb{C}_2^{n\times n}$ be a i_2 - Hermitian Matrix and λ is a eigenvalue of A with the corresponding eigenvector $X = {}^1Xe_1 + {}^2Xe_2$, such that $({}^2X)^t\, {}^1X \neq 0$. Then $\lambda = \lambda^{\#}$ or equivalently $\lambda \in \mathbb{C}(i_1)$.*
(ii) *Let $A \in \mathbb{C}_2^{n\times n}$ be a i_2 - Skew-Hermitian Matrix and λ is a eigenvalue of A with corresponding eigenvector $X = {}^1Xe_1 + {}^2Xe_2$, such that $({}^2X)^t\, {}^1X \neq 0$. Then $\lambda = -\lambda^{\#}$ or equivalently $\lambda \in i_2\mathbb{C}(i_1)$.*

Proof.
(i) Let $A \in \mathbb{C}_2^{n\times n}$ be a i_2 - Hermitian Matrix i.e. $A^{\#t} = A$

Let $X = {}^1Xe_1 + {}^2Xe_2$ be the eigenvector of the matrix A associated with the eigenvalue λ such that $({}^2X)^t\, {}^1X \neq 0$

$\Rightarrow AX = \lambda X$
$\Rightarrow (AX)^{\#t} = (\lambda X)^{\#t}$
$\Rightarrow X^{\#t}A^{\#t} = \lambda^{\#}X^{\#t}$
$\Rightarrow X^{\#t}A = \lambda^{\#}X^{\#t}$ (As A is i_2 - Hermitian matrix $A^{\#t} = A$)
$\Rightarrow (X^{\#t}A)X = (\lambda^{\#}X^{\#t})X$
$\Rightarrow X^{\#t}(AX) = \lambda^{\#}(X^{\#t}X)$
$\Rightarrow X^{\#t}(\lambda X) = \lambda^{\#}(X^{\#t}X)$ ($\because AX = \lambda X$)
$\Rightarrow \lambda(X^{\#t}X) = \lambda^{\#}(X^{\#t}X)$
$\Rightarrow (\lambda - \lambda^{\#})X^{\#t}X = 0$
$\Rightarrow ({}^1\lambda - {}^2\lambda)({}^2X)^t\, {}^1Xe_1 + ({}^2\lambda - {}^1\lambda)({}^1X)^t\, {}^2Xe_2 = 0$
$\Rightarrow ({}^1\lambda - {}^2\lambda)({}^2X)^{t1}X = 0$ and $({}^2\lambda - {}^1\lambda)({}^1X)^{t2}X = 0$
$\Rightarrow {}^1\lambda - {}^2\lambda = 0$ and ${}^2\lambda - {}^1\lambda = 0$ ($\because ({}^2X)^t\, {}^1X = ({}^1X)^t\, {}^2X \neq 0$)
$\Rightarrow {}^1\lambda = {}^2\lambda$
$\Rightarrow \lambda = {}^1\lambda e_1 + {}^2\lambda e_2 = {}^2\lambda e_1 + {}^1\lambda e_2 = \lambda^{\#}$
$\Rightarrow \lambda \in \mathbb{C}(i_1)$

(ii) Let $A \in \mathbb{C}_2^{n\times n}$ be a i_2 - Skew- Hermitian Matrix i.e. $A^{\#t} = -A$

Let $X = {}^1Xe_1 + {}^2Xe_2$ be the eigenvector of the matrix A associated with the eigenvalue λ such that $({}^2X)^t\, {}^1X \neq 0$

$\Rightarrow AX = \lambda X$
$\Rightarrow (AX)^{\#t} = (\lambda X)^{\#t}$
$\Rightarrow X^{\#t}A^{\#t} = \lambda^{\#}X^{\#t}$
$\Rightarrow X^{\#t}(-A) = \lambda^{\#}X^{\#t}$ (As A is i_2 - Skew - Hermitian Matrix i.e. $A^{\#t} = -A$)
$\Rightarrow -X^{\#t}A = \lambda^{\#}X^{\#t}$
$\Rightarrow -(X^{\#t}A)X = (\lambda^{\#}X^{\#t})X$

$\Rightarrow -X^{\#t}(AX) = \lambda^{\#}(X^{\#t}X)$
$\Rightarrow -X^{\#t}(\lambda X) = \lambda^{\#}(X^{\#t}X)$ $(\because AX = \lambda X)$
$\Rightarrow -\lambda(X^{\#t}X) = \lambda^{\#}(X^{\#t}X)$
$\Rightarrow (-\lambda - \lambda^{\#})X^{\#t}X = 0$
$\Rightarrow (\lambda + \lambda^{\#})X^{\#t}X = 0$
$\Rightarrow ({}^1\lambda + {}^2\lambda)({}^2X)^t\,{}^1Xe_1 + ({}^2\lambda + {}^1\lambda)({}^1X)^t\,{}^2Xe_2 = 0$
$\Rightarrow ({}^1\lambda + {}^2\lambda)({}^2X)^t\,{}^1X = 0$ and $({}^2\lambda + {}^1\lambda)({}^1X)^t\,{}^2X = 0$
$\Rightarrow {}^1\lambda + {}^2\lambda = 0$ and ${}^2\lambda + {}^1\lambda = 0$ $(\because ({}^2X)^t\,{}^1X = ({}^1X)^t\,{}^2X \neq 0)$
$\Rightarrow {}^1\lambda = -{}^2\lambda$
$\Rightarrow \lambda = {}^1\lambda e_1 + {}^2\lambda e_2 = -{}^2\lambda e_1 - {}^1\lambda e_2 = -\lambda^{\#}$
$\Rightarrow \lambda \in i_2\mathbb{C}(i_1)$

□

Proposition 2.12.

(i) *i_1i_2 - Hermitian Matrix have Hyperbolic Eigenvalues.*
Suppose $A \in \mathbb{C}_2^{n\times n}$ be a i_1i_2 - Hermitian Matrix and λ is a eigenvalue of A. Then $\lambda = \lambda'$ or equivalently $\lambda \in \mathbb{H}$.

(ii) *Suppose $A \in \mathbb{C}_2^{n\times n}$ be a i_1i_2 - Skew - Hermitian Matrix and λ is a eigenvalue of A. Then $\lambda = -\lambda'$ or equivalently $\lambda \in i_1\mathbb{H}$ or $i_2\mathbb{H}$.*

Proof.

(i) Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1i_2 - Hermitian Matrix i.e. $A'^t = A$
Let $X = {}^1Xe_1 + {}^2Xe_2$ be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$
$\Rightarrow (\mathrm{AX})'^t = (\lambda X)'^t$
$\Rightarrow X'^tA'^t = \lambda'\ X'^t$
$\Rightarrow X'^tA = \lambda'\ X'^t$ (As A is i_1i_2 -Hermitian matrix i.e. $A'^t = A$)
$\Rightarrow (X'^tA)X = (\lambda' X'^t)X$
$\Rightarrow X'^t(AX) = \lambda'(X'^tX)$
$\Rightarrow X'^t(\lambda X) = \lambda'(X'^tX)$ $(\because AX = \lambda X)$
$\Rightarrow \lambda(X'^tX) = \lambda'(X'^tX)$
$\Rightarrow (\lambda - \lambda')X'^tX = 0$
$\Rightarrow ({}^1\lambda - \overline{{}^1\lambda})(\overline{{}^1X})^t\,{}^1Xe_1 + ({}^2\lambda - \overline{{}^2\lambda})(\overline{{}^2X})^t\,{}^2Xe_2 = 0$
$\Rightarrow ({}^1\lambda - \overline{{}^1\lambda})(\overline{{}^1X})^t\,{}^1X = 0$ and $({}^2\lambda - \overline{{}^2\lambda})(\overline{{}^2X})^t\,{}^2X = 0$
$\Rightarrow {}^1\lambda - \overline{{}^1\lambda} = 0$ and ${}^2\lambda - \overline{{}^2\lambda} = 0$ $(\because (\overline{{}^1X})^t\,{}^1X \neq 0$ and $(\overline{{}^2X})^t\,{}^2X \neq 0)$
$\Rightarrow {}^1\lambda = \overline{{}^1\lambda}$ and ${}^2\lambda = \overline{{}^2\lambda}$
$\Rightarrow \lambda = {}^1\lambda e_1 + {}^2\lambda e_2 = \overline{{}^1\lambda}e_1 + \overline{{}^2\lambda}e_2 = \lambda'$
$\Rightarrow \lambda \in \mathbb{H}$

(ii) Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1i_2 - Skew - Hermitian Matrix i.e. $A'^t = -A$

Let $X = {}^1Xe_1 + {}^2Xe_2$ be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$
$\Rightarrow (AX)^{'t} = (\lambda X)^{'t}$
$\Rightarrow X^{'t}A^{'t} = \lambda^{'}\ X^{'t}$
$\Rightarrow X^{'t}(-A) = \lambda^{'}X^{'t}$ (As A is i_1i_2 - Skew - Hermitian matrix i.e. $A^{'t} = -A$)
$\Rightarrow -(X^{'t}A)X = (\lambda^{'}X^{'t})X$
$\Rightarrow -X^{'t}(AX) = \lambda^{'}(X^{'t}X)$
$\Rightarrow -X^{'t}(\lambda X) = \lambda^{'}(X^{'t}X)$ $(\because AX = \lambda X)$
$\Rightarrow -\lambda(X^{'t}X) = \lambda^{'}(X^{'t}X)$
$\Rightarrow (\lambda + \lambda^{'})X^{'t}X = 0$
$\Rightarrow ({}^1\lambda + \overline{{}^1\lambda})(\overline{{}^1X})^t\,{}^1Xe_1 + ({}^2\lambda + \overline{{}^2\lambda})(\overline{{}^2X})^t\,{}^2Xe_2 = 0$
$\Rightarrow ({}^1\lambda + \overline{{}^1\lambda})(\overline{{}^1X})^t\,{}^1X = 0$ and $({}^2\lambda + \overline{{}^2\lambda})(\overline{{}^2X})^t\,{}^2X = 0$
$\Rightarrow {}^1\lambda + \overline{{}^1\lambda} = 0$ and ${}^2\lambda + \overline{{}^2\lambda} = 0$ $(\because (\overline{{}^1X})^t\,{}^1X \neq 0$ and $(\overline{{}^2X})^t\,{}^2X \neq 0)$
$\Rightarrow {}^1\lambda = -\overline{{}^1\lambda}$ and ${}^2\lambda = -\overline{{}^2\lambda}$
$\Rightarrow \lambda = {}^1\lambda e_1 + {}^2\lambda e_2 = -(\overline{{}^1\lambda}e_1 + \overline{{}^2\lambda}e_2) = -\lambda^{'}$
$\Rightarrow \lambda \in i_1\mathbb{H}$ or $i_2\mathbb{H}$

□

Proposition 2.13. *Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1 - Unitary Matrix and λ is a eigenvalue of A with corresponding eigenvector $X = {}^1Xe_1 + {}^2Xe_2$, such that $(\overline{{}^2X})^t\,{}^1X \neq 0$. Then $\lambda = (\lambda^{-1})^*$.*

Proof. Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1 - Unitary Matrix i.e. $A^{*t}A = AA^{*t} = I$

Let $X = {}^1Xe_1 + {}^2Xe_2$ be the eigenvector of the matrix A associated with the eigenvalue λ such that $(\overline{{}^2X})^t\,{}^1X \neq 0$

$\Rightarrow AX = \lambda X$
$\Rightarrow (AX)^{*t} = (\lambda X)^{*t}$
$\Rightarrow X^{*t}A^{*t} = \lambda^* X^{*t}$
$\Rightarrow (X^{*t}A^{*t})AX = (\lambda^* X^{*t})\lambda X$ $(\because AX = \lambda X)$
$\Rightarrow X^{*t}(A^{*t}A)X = \lambda^*\lambda(X^{*t}X)$
$\Rightarrow X^{*t}(I)X = \lambda^*\lambda(X^{*t}X)$ $(\because A^{*t}A = AA^{*t} = I)$
$\Rightarrow X^{*t}X = \lambda^*\lambda(X^{*t}X)$
$\Rightarrow (1 - \lambda^*\lambda)X^{*t}X = 0$
$\Rightarrow (1 - \overline{{}^2\lambda}\,{}^1\lambda)(\overline{{}^2X})^t\,{}^1Xe_1 + (1 - \overline{{}^1\lambda}\,{}^2\lambda)(\overline{{}^1X})^t\,{}^2Xe_2 = 0$
$\Rightarrow (1 - \overline{{}^2\lambda}\,{}^1\lambda)(\overline{{}^2X})^t\,{}^1X = 0$ and $(1 - \overline{{}^1\lambda}\,{}^2\lambda)(\overline{{}^1X})^t\,{}^2X = 0$
$\Rightarrow 1 - \overline{{}^2\lambda}\,{}^1\lambda = 0$ and $1 - \overline{{}^1\lambda}\,{}^2\lambda = 0$ $(\because (\overline{{}^2X})^t\,{}^1X \neq 0 \Rightarrow (\overline{{}^1X})^t\,{}^2X \neq 0)$
$\Rightarrow \overline{{}^2\lambda}\,{}^1\lambda = 1$ and $\overline{{}^1\lambda}\,{}^2\lambda = 1$
$\Rightarrow {}^1\lambda = \frac{1}{\overline{{}^2\lambda}}$ and ${}^2\lambda = \frac{1}{\overline{{}^1\lambda}}$
$\Rightarrow \lambda = {}^1\lambda e_1 + {}^2\lambda e_2 = \frac{1}{\overline{{}^2\lambda}}e_1 + \frac{1}{\overline{{}^1\lambda}}e_2 = \frac{1}{\lambda^*} = (\lambda^{-1})^*$

□

Note 2.4. λ is a non-singular and λ^* is a multiplicative inverse of λ.

Proposition 2.14. *Let $A \in \mathbb{C}_2^{n\times n}$ be a i_2 - Unitary Matrix and λ is a eigenvalue of A with corresponding eigenvector $X = {}^1Xe_1 + {}^2Xe_2$, such that $({}^2X)^t\,{}^1X \neq 0$. Then $\lambda = (\lambda^{-1})^{\#}$.*

Proof. Let $A \in \mathbb{C}_2^{n\times n}$ be a i_2 -Unitary Matrix i.e. $A^{\#t}A = AA^{\#t} = I$

Let $X = {}^1Xe_1 + {}^2Xe_2$ be the eigenvector of the matrix A associated with the eigenvalue λ such that $({}^2X)^t\,{}^1X \neq 0$

$\Rightarrow AX = \lambda X$
$\Rightarrow (AX)^{\#t} = (\lambda X)^{\#t}$
$\Rightarrow X^{\#t}A^{\#t} = \lambda^{\#}X^{\#t}$
$\Rightarrow (X^{\#t}A^{\#t})AX = (\lambda^{\#}X^{\#t})\lambda X$ $(\because AX = \lambda X)$
$\Rightarrow X^{\#t}(A^{\#t}A)X = \lambda^{\#}\lambda(X^{\#t}X)$
$\Rightarrow X^{\#t}(I)X = \lambda^{\#}\lambda(X^{\#t}X)$ $(\because A^{\#t}A = AA^{\#t} = I)$
$\Rightarrow X^{\#t}X = \lambda^{\#}\lambda(X^{\#t}X$
$\Rightarrow (1 - \lambda^{\#}\lambda)X^{\#t}X = 0$
$\Rightarrow (1 - {}^2\lambda{}^1\lambda)({}^2X)^t\,{}^1Xe_1 + (1 - {}^1\lambda{}^2\lambda)({}^1X)^t\,{}^2Xe_2 = 0$
$\Rightarrow (1 - {}^2\lambda{}^1\lambda)({}^2X)^t\,{}^1X = 0$ and $(1 - {}^1\lambda{}^2\lambda)({}^1X)^t\,{}^2X = 0$
$\Rightarrow 1 - {}^2\lambda{}^1\lambda = 0$ and $1 - {}^1\lambda{}^2\lambda = 0$ $(\because ({}^2X)^t\,{}^1X = ({}^1X)^t\,{}^2X \neq 0)$
$\Rightarrow {}^2\lambda{}^1\lambda = 1$
$\Rightarrow {}^1\lambda = \frac{1}{{}^2\lambda}$ and ${}^2\lambda = \frac{1}{{}^1\lambda}$
$\Rightarrow \lambda = {}^1\lambda e_1 + {}^2\lambda e_2 = \frac{1}{{}^2\lambda}e_1 + \frac{1}{{}^1\lambda}e_2 = \frac{1}{\lambda^{\#}} = (\lambda^{-1})^{\#}$

□

Note 2.5. λ is non-singular and $\lambda^{\#}$ is a multiplicative inverse of λ.

Proposition 2.15. *Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1i_2 - Unitary Matrix and λ is a eigenvalue of A. Then $\lambda = (\lambda^{-1})'$.*

Proof. Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1i_2 - Unitary Matrix i.e. $A'^tA = AA'^t = I$

Let $X = {}^1Xe_1 + {}^2Xe_2$ be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$
$\Rightarrow (AX)'^t = (\lambda X)'^t$
$\Rightarrow X'^tA'^t = \lambda'X'^t$
$\Rightarrow (X'^tA'^t)AX = (\lambda'X'^t)\lambda X$ $(\because AX = \lambda X)$
$\Rightarrow X'^t(A'^tA)X = \lambda'\lambda(X'^tX)$
$\Rightarrow X'^t(I)X = \lambda'\lambda(X'^tX)$ $(\because A'^tA = AA'^t = I)$
$\Rightarrow X'^tX = \lambda'\lambda(X'^tX)$
$\Rightarrow (1 - \lambda'\lambda)X'^tX = 0$
$\Rightarrow (1 - \overline{{}^1\lambda}{}^1\lambda)(\overline{{}^1X})^t\,{}^1Xe_1 + (1 - \overline{{}^2\lambda}{}^2\lambda)(\overline{{}^2X})^t\,{}^2Xe_2 = 0$
$\Rightarrow (1 - \overline{{}^1\lambda}{}^1\lambda)(\overline{{}^1X})^t\,{}^1X = 0$ and $(1 - \overline{{}^2\lambda}{}^2\lambda)(\overline{{}^2X})^t\,{}^2X = 0$

$\Rightarrow 1 - \overline{^1\lambda}{}^1\lambda = 0$ and $1 - \overline{^2\lambda}{}^2\lambda = 0$ $(\because (\overline{^1X})^t\, {}^1X \neq 0$ and $(\overline{^2X})^t\, {}^2X \neq 0)$
$\Rightarrow \overline{^1\lambda}{}^1\lambda = 1$ and $\overline{^2\lambda}{}^2\lambda = 1$
$\Rightarrow {}^1\lambda = \frac{1}{\overline{^1\lambda}}$ and ${}^2\lambda = \frac{1}{\overline{^2\lambda}}$
$\Rightarrow \lambda = {}^1\lambda e_1 + {}^2\lambda e_2 = \frac{1}{\overline{^1\lambda}} e_1 + \frac{1}{\overline{^2\lambda}} e_2 = \frac{1}{\lambda'} = (\lambda^{-1})'$

□

Note 2.6. λ is non-singular and λ' is a multiplicative inverse of λ.

Proposition 2.16. *Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1i_2 - Hermitian Matrix and X and Y are two eigenvectors of A associated with the eigenvalues λ and μ such that $\lambda - \mu \notin O_2$. Then $X'^tY = 0$.*

Proof. As A is i_1i_2 - Hermitian Matrix, $A'^t = A$ and $\lambda' = \lambda$, $\mu' = \mu$

Let X and Y are the eigenvectors of the matrix A associated with the eigenvalues λ and μ respectively
$\Rightarrow AX = \lambda X$ and $AY = \mu Y$
Now,
$AX = \lambda X$
$\Rightarrow (AX)'^t = (\lambda X)'^t$
$\Rightarrow X'^t A'^t = \lambda' X'^t$
$\Rightarrow X'^t A = \lambda X'^t$ $(\because A'^t = A$ and $\lambda' = \lambda)$
$\Rightarrow (X'^t A)Y = (\lambda X'^t)Y$
$\Rightarrow X'^t(AY) = \lambda(X'^tY)$
$\Rightarrow X'^t(\mu Y) = \lambda(X'^tY)$ $(\because AY = \mu Y)$
$\Rightarrow \mu(X'^tY) = \lambda(X'^tY)$
$\Rightarrow (\mu - \lambda)X'^tY = 0$
$\Rightarrow X'^tY = 0$ $(\because \mu - \lambda \notin O_2)$

□

Acknowledgments

I am heartily thankful to Prof. M.A. Gauri, Dr. R.S. Giri and Dr. Chitranjan Singh, Govt. Degree College, Raza Nagar, Swar, Rampur (U.P), Dr. Sukhdev Singh, Lovely Professional University, Punjab and all staff of Govt. Degree College, Raza Nagar,Swar,Rampur (U.P) for their encouragement and support during the preparation of this paper.

References

[1] Price, G. B.(1991) "An introduction to multicomplex space and Functions" Marcel Dekker.
[2] Luna-Elizarrarás, M.E., Shapiro, M., Struppa, D. C., Vajiac, A.(2015) "Bicomplex Holomorphic Functions:The Algebra, Geometry and Analysis of Bicomplex Numbers" Springer International Publishing.

[3] Kumar, J. (2018) ”On Some Properties of Bicomplex Numbers •Conjugates •Inverse •Modulii” Journal of Emerging Technologies and Innovative Research (JETIR),5(9), 475-499.

[4] Srivastava, Rajiv K. (2008) ”Certain Topological Aspects of Bicomplex Space” Bull. Pure and Appl. Math, 222-234.

[5] Alpay, D., Luna-Elizarrarás, M. E., Shapiro, M. , Struppa, D.C. (2014) “Basics of Functional Analysis with Bicomplex Scalars and Bicomplex Schur Analysis” Springer International Publishing, 19-30.

[6] Kumar,J. (2016) “Conjugation of Bicomplex Matrix” J. of Science and Tech. Res. (JSTR), 1(1), 24-28.

[7] Kumar,J. (2022)“On Some Properties of Determinants of Bicomplex Matrices”. Cambridge Open Engage. doi:10.33774/coe-2022-bghhb-v2.

[8] Beezer, Robert A. (2012)”A First Course in Linear Algebra” Congruent Press, Gig Harbor, Washington, USA.

Department of Mathematics, Govt. Degree College, Raza Nagar, Swar, Rampur (U.P), India.

Email address: jogendra.ibs@gmail.com