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A NOTE ON END PROPERTIES OF MARCINKIEWICZ INTEGRAL

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ABSTRACT. In this note we give the mapping properties of the *Marcinkiewicz integral* μ_Ω *at some end spaces. More precisely,* we first prove that μ_Ω is a bounded operator from $H^{1,\infty}(\mathbb{R}^n)$ to $L^{1,\infty}$ $(\mathbb{R}^n)$. As a corollary of the results above, we obtain again the weak *type (1,1)* boundedness of μ_Ω, *but the condition assumed on* Ω is weaker than Stein's condition. Finally, we show that μ_Ω is bounded from BMO$(\mathbb{R}^n)$ to BMO$(\mathbb{R}^n)$. The results in this note are the extensions of the results obtained by Lee and Rim recently.

1. Introduction

Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the Lebesgue measure $d\sigma$. Let Ω be a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying $\Omega \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.1}$$

where $x' = x/|x|$ for any $x \neq 0$. Then the Marcinkiewicz integral operator μ_Ω of higher dimension is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

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It is well known that the operator μ_Ω was first defined by Stein in [9]. He proved that if Ω satisfies a Lip_α $(0 < \alpha \leq 1)$ condition on S^{n-1}, then μ_Ω is of type (p,p) for $1 < p \leq 2$ and of weak type (1,1). Subsequently Benedek, Calderón and Panzone showed in [1] that if $\Omega \in \mathcal{C}^1(S^{n-1})$ then μ_Ω is of type (p,p) for $1 < p < \infty$. In 2000, Ding, Fan and Pan[4] proved the following result:

THEOREM A. *Suppose that* $\Omega \in H^1(S^{n-1})$ *and satisfies* (1.1). *Then for* $1 < p < \infty$, *there exists a constant* $C > 0$, *independent of* f, *such that* $\|\mu_\Omega(f)\|_{L^p} \leq C\|f\|_{L^p}$.

Here $H^1(S^{n-1})$ denotes the Hardy space on the unit sphere (see [2] and [3] for the definition and properties of $H^1(S^{n-1})$). Note that the following relationship between $H^1(S^{n-1})$ and the other function spaces on S^{n-1}:

$$\begin{aligned} \mathrm{Lip}_\alpha(S^{n-1}) \subset L^q(S^{n-1}) \subset L\log^+ L(S^{n-1}) \\ \subset H^1(S^{n-1}) \subset L^1(S^{n-1}), \end{aligned} \tag{1.2}$$

where all inclusions are proper for any $0 < \alpha \leq 1$ and $q > 1$. Hence, Theorem A improved the results in [9] (for $1 < p \leq 2$)) and [1] mentioned above.

Recently, Lee and Rim[8] discussed the (H^1, L^1), $(L^2 \cap L^\infty, BMO)$ and (L^p, L^p) $(1 < p < \infty)$ boundedness of μ_Ω when there exist constants $C > 0$ and $\rho > 1$ such that

$$|\Omega(x') - \Omega(y')| \leq \frac{C}{(\log \frac{1}{|x'-y'|})^\rho}, \tag{1.3}$$

holds uniformly in $x', y' \in S^{n-1}$.

It is easy to see that the condition (1.3) is weaker than the Lip_α $(0 < \alpha \leq 1)$ condition on S^{n-1}. In addition, it is also obvious if Ω satisfies (1.3) for some $\rho > 1$, then by (1.2)

$$\Omega \in L^\infty(S^{n-1}) \subset L^q(S^{n-1})\,(1 < q < \infty) \subset H^1(S^{n-1}). \tag{1.4}$$

In this note we will consider the mapping properties of the Marcinkiewicz integral μ_Ω on some end function spaces when Ω satisfies (1.3) for $\rho > 2$. More precise, we will first prove that μ_Ω is bounded from $H^{1,\infty}(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. Here $H^{1,\infty}(\mathbb{R}^n)$ denotes the weak Hardy space, which was introduced first by Fefferman and Soria in [6]. To state our results, let us recall the definition of $H^{1,\infty}(\mathbb{R}^n)$.

DEFINITION 1. Suppose that $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\int \phi \neq 0$. Denote $f_+^*(x) = \sup_{t>0} |(\phi_t * f)(x)|$, where $\phi_t(x) = t^{-n}\phi(x/t)$. A function f is said to belong to the weak Hardy space $H^{1,\infty}(\mathbb{R}^n)$ if $f_+^* \in L^{1,\infty}(\mathbb{R}^n)$, i.e., there exists a constant $C > 0$ such that for any $\beta > 0$

$$\sup_{\beta>0} \beta|\{x \in \mathbb{R}^n : f_+^*(x) > \beta\}| \leq C.$$

The smallest constant C satisfying the above inequality is called the $H^{1,\infty}(\mathbb{R}^n)$ norm of f, which is denoted by $\|f\|_{H^{1,\infty}}$.

We have the following conclusions.

THEOREM 1. *Let Ω satisfy (1.1) and (1.3) for some $\rho > 2$. Then μ_Ω is bounded operator from $H^{1,\infty}(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. That is, there exists a constant $C > 0$ such that for any $f \in H^{1,\infty}(\mathbb{R}^n)$ and $\beta > 0$,*

$$|\{x : \mu_\Omega f(x) > \beta\}| \leq C\|f\|_{H^{1,\infty}}/\beta. \tag{1.5}$$

REMARK 1. Let M be the Hardy-Littlewood maximal operator. It is known that if $\phi \in C_0^\infty(\mathbb{R}^n)$ and $f \in L^1(\mathbb{R}^n)$, then $f_+^*(x) \leq CM(f)(x)$ (see [10, pp. 62, Theorem 2]). Hence, by the weak $(1,1)$ boundedness of M, it is easy to see that the space $L^1(\mathbb{R}^n)$ is continuously embedded as a subspace of the space $H^{1,\infty}(\mathbb{R}^n)$, and $\|f\|_{H^{1,\infty}} \leq C\|f\|_{L^1}$ for any $f \in L^1(\mathbb{R}^n)$. Thus we get immediately the following corollary of Theorem 1.

COROLLARY. *If Ω satisfy (1.1) and (1.3) for some $\rho > 2$, then μ_Ω is of weak type $(1,1)$. That is, there exists a constant $C > 0$ such that for any $f \in L^1(\mathbb{R}^n)$ and $\beta > 0$, $|\{x : \mu_\Omega f(x) > \beta\}| \leq C\|f\|_{L^1}/\beta$.*

REMARK 2. In [9], Stein proved that μ_Ω is of weak (1,1) if $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ for $0 < \alpha \leq 1$. In [5], Fan and Sato showed that if $\Omega \in L\log^+ L(S^{n-1})$, then μ_Ω is also of weak (1,1). Hence, the condition (1.3) of Corollary is weaker than Stein's condition but stronger than Fan-Sato's condition.

The second result in this note shows that μ_Ω is also bounded from $\mathrm{BMO}(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$.

DEFINITION 2. A locally integrable function $f(x)$ is said to belong to $\mathrm{BMO}(\mathbb{R}^n)$ if

$$\|f\|_{BMO} := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty,$$

where the supremum is taken over all cubes in R^n with sides parallel to the coordinate axes, and f_Q denotes the average of f over Q, i.e. $f_Q = \frac{1}{|Q|}\int_Q f(x)dx$.

A well-known important fact on $\mathrm{BMO}(\mathbb{R}^n)$ is that for any $1 \le p < \infty$,

$$\|f\|_{BMO} \sim \sup_Q \left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p dx\right)^{1/p} < \infty.$$

THEOREM 2. *Let Ω satisfy* (1.1) *and* (1.3) *for some $\rho > 2$. Suppose that $f(x) \in BMO(\mathbb{R}^n)$ and there is a measurable set $E \subset \mathbb{R}^n$ with $|E| > 0$ such that $\mu_\Omega(f)(x) < \infty$ for any $x \in E$. Then $\mu_\Omega(f)(x) < \infty$ a.e. on $\mathbb{R}^n$ and*

$$\|\mu_\Omega(f)\|_{BMO} \le C\|f\|_{BMO}, \tag{1.6}$$

where the constant C is independent of f.

REMARK 3. In [7], Han gave an example to show that there exist an Ω and $f_0 \in L^\infty(\mathbb{R}^n)$ such that $\mu_\Omega(f_0)(x) = \infty$ for any $x \in \mathbb{R}^n$. Therefore, μ_Ω can not map an $L^\infty(\mathbb{R}^n)$ function into a $\mathrm{BMO}(\mathbb{R}^n)$ function in general.

2. Proof of Theorem 1

We need the following Fefferman-Soria's decomposition theorem of function in $H^{1,\infty}(\mathbb{R}^n)$ (See [6]).

THEOREM B. *Given a function $f \in H^{1,\infty}(\mathbb{R}^n)$, there exists a sequence of bounded functions $\{f_k\}_{k=-\infty}^{\infty}$ with the following properties:*

(a) *$f - \sum_{|k|\le N} f_k$ tends to zero in the sense of distributions;*

(b) *Each f_k may be further decomposed as $f_k = \sum_i h_i^k$ in L^1 and $\{h_i^k\}$ satisfies*

  (i) *$\mathrm{supp}(h_i^k) \subset B_i^k := B(x_i^k, r_i^k)$, where $B(x,r)$ denotes the ball in $\mathbb{R}^n$ with the center at x and radius r. Moreover, $\sum_i |B_i^k| \le C_1 2^{-k}$ and $\sum_i \chi_{B_i^k}(x) \le C$, where $C_1 \sim \|f\|_{H^{1,\infty}}$;*

  (ii) *$\|h_i^k\|_\infty \le C2^k$, where C is independent of i and k;*

  (iii) *$\int h_i^k(x)dx = 0$, for every i and k.*

Now let us turn to the proof of Theorem 1. We need to show that there exists a constant $C > 0$ such that (1.5) holds for any $f \in H^{1,\infty}(\mathbb{R}^n)$

and $\beta > 0$. To do this, for any given $\beta > 0$, we take k_0 satisfying $2^{k_0} \le \beta < 2^{k_0+1}$, then by Theorem B we may write

$$f = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1}^{\infty} f_k := F_1 + F_2 \quad \text{and} \quad f_k = \sum_i h_i^k,$$

where h_i^k satisfies (i), (ii) and (iii). Now denote $A_k = \operatorname{supp} f_k$, then $A_k = \cup_i B_i^k$ and $|A_k| \le \sum_i |B_i^k| \le C2^{-k}\|f\|_{H^{1,\infty}}$. Note that $\|f_k\|_\infty \le C2^k$, we have

$$\begin{aligned}\|F_1\|_2 &\le \sum_{k=-\infty}^{k_0} \|f_k\|_2 \le C \sum_{k=-\infty}^{k_0} 2^k |A_k|^{1/2} \\ &\le C \sum_{k=-\infty}^{k_0} 2^{k/2} \|f\|_{H^{1,\infty}}^{1/2} \le C\|f\|_{H^{1,\infty}}^{1/2} \beta^{1/2}.\end{aligned}$$

Using (1.4) and Theorem A, it is easy to see that

$$\begin{aligned}|\{x : \mu_\Omega(F_1)(x) > \beta\}| &\le \|\mu_\Omega(F_1)\|_2^2/\beta^2 \\ &\le C\|F_1\|_2^2/\beta^2 \\ &\le C\|f\|_{H^{1,\infty}}/\beta.\end{aligned} \tag{2.1}$$

On the other hand, let

$$\bar{B_i^k} = B(x_i^k, 2(3/2)^{(k-k_0)/n} r_i^k) \quad \text{and} \quad B_{k_0} = \bigcup_{k=k_0+1}^{\infty} \bigcup_i \bar{B_i^k}.$$

We have

$$\begin{aligned}|B_{k_0}| &\le \sum_{k=k_0+1} \sum_i |\bar{B_i^k}| \\ &= \sum_{k=k_0+1} \sum_i 2^n (3/2)^{k-k_0} |B_i^k| \\ &\le C \sum_{k=k_0+1}^{\infty} (3/2)^{k-k_0} 2^{-k} \|f\|_{H^{1,\infty}} \\ &\le C\|f\|_{H^{1,\infty}}/\beta.\end{aligned} \tag{2.2}$$

Thus, to prove (1.5) it suffices to show

$$|\{x \in (B_{k_0})^c : \mu_\Omega(F_2)(x) > \beta\}| \le C\|f\|_{H^{1,\infty}}/\beta. \tag{2.3}$$

Using the Minkowski inequality, we get

$$\begin{aligned}
&\int_{(B_{k_0})^c} \mu_\Omega(F_2)(x)dx \\
&= \int_{(B_{k_0})^c} \left(\int_0^\infty \left|\sum_{k=k_0+1}^\infty \sum_i \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} h_i^k(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2} dx \\
&\le \int_{(B_{k_0})^c} \sum_{k=k_0+1}^\infty \sum_i \left(\int_0^\infty \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} h_i^k(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2} dx \\
&\le C \sum_{k=k_0+1}^\infty \sum_i (I_1 + I_2),
\end{aligned}$$

where

$$I_1 = \int_{(B_{k_0})^c} \left(\int_0^{|x-x_i^k|+2r_i^k} \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} h_i^k(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2} dx$$

and

$$I_2 = \int_{(B_{k_0})^c} \left(\int_{|x-x_i^k|+2r_i^k}^\infty \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} h_i^k(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2} dx.$$

It is easy to check that when $y \in B_i^k$, for any $x \in (B_{k_0})^c$, $|x - x_i^k| \sim |x - y| \sim |x - x_i^k| + 2r_i^k$, and

$$\left|\frac{1}{(|x-x_i^k|+2r_i^k)^2} - \frac{1}{|x-y|^2}\right| \le C\frac{r_i^k}{|x-y|^3}. \tag{2.4}$$

Hence by $\Omega \in L^\infty(S^{n-1})$ and (2.4), we have

$$\begin{aligned}
I_1 &\le \int_{(B_{k_0})^c} \int_{B_i^k} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |h_i^k(y)| \left(\int_{|x-y|\le t\le |x-x_i^k|+2r_i^k} \frac{1}{t^3} dt\right)^{1/2} dydx \\
&\le C2^k \|\Omega\|_{L^\infty(S^{n-1})} \\
&\quad \times \int_{(B_{k_0})^c} \int_{B_i^k} \frac{1}{|x-y|^{n-1}} \left|\frac{1}{(|x-x_i^k|+2r_i^k)^2} - \frac{1}{|x-y|^2}\right|^{1/2} dydx \\
&\le C2^k (r_i^k)^{1/2} \int_{B_i^k} \int_{2(3/2)^{(k-k_0)/n} r_i^k}^\infty \frac{1}{s^{1+1/2}} dsdy \\
&\le C2^k (r_i^k)^{1/2} |B_i^k| \left(\frac{2}{3}\right)^{(k-k_0)/(2n)} (r_i^k)^{-1/2}.
\end{aligned}$$

Thus

$$\sum_{k=k_0+1}^{\infty}\sum_i I_1 \le C \sum_{k=k_0+1}^{\infty}\left(\frac{2}{3}\right)^{(k-k_0)/(2n)}\|f\|_{H^{1,\infty}} \le C\|f\|_{H^{1,\infty}}. \tag{2.5}$$

Now let us consider I_2. As above we know that if $x \in (B_{k_0})^c$, $y \in B_i^k$, then $|x-x_i^k| \sim |x-y| \sim |x-x_i^k| + 2r_i^k$, and

$$\left|\frac{x-y}{|x-y|} - \frac{x-x_i^k}{|x-x_i^k|}\right| \le C\frac{r_i^k}{|x-x_i^k|}. \tag{2.6}$$

Thus by (1.3) and (2.6), we get

$$\begin{aligned}\left|\Omega(x-y)-\Omega(x-x_i^k)\right| &= \left|\Omega\left(\frac{x-y}{|x-y|}\right) - \Omega\left(\frac{x-x_i^k}{|x-x_i^k|}\right)\right| \\ &\le \frac{C}{\left(\log\frac{|x-x_i^k|}{r_i^k}\right)^{\rho}}.\end{aligned} \tag{2.7}$$

Applying (2.7) we have

(2.8)

$$\begin{aligned}&\left|\frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_i^k)}{|x-x_i^k|^{n-1}}\right| \\ &\le \frac{|\Omega(x-y)-\Omega(x-x_i^k)|}{|x-x_i^k|^{n-1}} + |\Omega(x-y)|\left|\frac{1}{|x-y|^{n-1}} - \frac{1}{|x-x_i^k|^{n-1}}\right| \\ &\le \frac{C}{|x-x_i^k|^{n-1}\left(\log\frac{|x-x_i^k|}{r_i^k}\right)^{\rho}} + \frac{r_i^k}{|x-x_i^k|^n} \\ &\le \frac{C}{|x-x_i^k|^{n-1}\left(\log\frac{|x-x_i^k|}{r_i^k}\right)^{\rho}}.\end{aligned}$$

Now let us return to the estimate of I_2. Note that when $y \in B_i^k$, for any $x \in (B_{k_0})^c$ and $t > |x-x_i^k| + 2r_i^k$, we have $B_i^k \subset \{y \in \mathbb{R}^n : |x-y| \le t\}$. By the cancellation property of $h_i^k(y)$ and (2.8), we get

$$\begin{aligned}I_2 = \int_{(B_{k_0})^c}\Bigg(\int_{|x-x_i^k|+2r_i^k}^{\infty}\Bigg|\int_{|x-y|\le t}&\left(\frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_i^k)}{|x-x_i^k|^{n-1}}\right) \\ &\times h_i^k(y)dy\Bigg|^2\frac{dt}{t^3}\Bigg)^{1/2}dx\end{aligned}$$

$$\le C2^k \int_{(B_{k_0})^c} \int_{B_i^k} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_i^k)}{|x-x_i^k|^{n-1}} \right| \times \left(\int_{|x-y|\le t, |x-x_i^k|<t} \frac{1}{t^3} dt \right)^{1/2} dydx$$

$$\le C2^k \int_{B_i^k} \int_{|x-x_i^k|>(3/2)^{(k-k_0)/n} r_i^k} \frac{1}{|x-x_i^k|^{n-1} \left(\log \frac{|x-x_i^k|}{r_i^k} \right)^\rho} \cdot \frac{1}{|x-x_i^k|} dxdy$$

$$\le C2^k |B_i^k| \int_{(3/2)^{(k-k_0)/n} r_i^k}^\infty \frac{ds}{s\left(\log \frac{s}{r_i^k} \right)^\rho} ds.$$

Note that $\rho > 2$

$$\int_{(3/2)^{(k-k_0)/n} r_i^k}^\infty \frac{ds}{s\left(\log \frac{s}{r_i^k} \right)^\rho} ds$$

$$= \sum_{j=1}^\infty \int_{(3/2)^{j(k-k_0)/n}}^{(3/2)^{(j+1)(k-k_0)/n}} \frac{ds}{s(\log s)^\rho} ds$$

$$\le \sum_{j=1}^\infty \frac{1}{\left(\log (3/2)^{j(k-k_0)/n} \right)^\rho} \int_{(3/2)^{j(k-k_0)/n}}^{(3/2)^{(j+1)(k-k_0)/n}} \frac{ds}{s} \tag{2.9}$$

$$= \sum_{j=1}^\infty \frac{\log (3/2)^{(k-k_0)/n}}{\left(\log (3/2)^{j(k-k_0)/n} \right)^\rho}$$

$$\le C(k-k_0)^{1-\rho}.$$

We therefore get by (2.9)

$$I_2 \le C2^k |B_i^k| (k-k_0)^{1-\rho}. \tag{2.10}$$

Thus

$$\sum_{k=k_0+1}^\infty \sum_i I_2 \le C \sum_{k=k_0+1}^\infty (k-k_0)^{1-\rho} \|f\|_{H^{1,\infty}} \le C\|f\|_{H^{1,\infty}}. \tag{2.11}$$

By (2.5) and (2.11), we have

$$\int_{(B_{k_0})^c} \mu_\Omega(F_2)(x) dx \le C\|f\|_{H^{1,\infty}}. \tag{2.12}$$

From (2.12) we get (2.3). Hence we finish the proof of Theorem 1.

3. Proof of Theorem 2

Let us begin by recalling a known result.

LEMMA 1. *Suppose that $1 \le p < \infty$, $\eta > 0$ and $f \in BMO(\mathbb{R}^n)$. Then there exists a $C = C(n,p,\eta) > 0$ such that for any cube Q with its center at x_0 and side length d,*

$$\int_{R^n} \frac{d^\eta |f(x) - f_Q|^p}{d^{n+\eta} + |x - x_0|^{n+\eta}} dx \le C\|f\|_{BMO}^p.$$

Now let us turn to the proof of Theorem 2. Take any density point $\bar{x}$ of E and any cube Q with center at $\bar{x}$, and denote by d the side length of Q. First we show that $\mu_\Omega(f)(x) < \infty$ a.e. on Q. To do this, we denote $Q^* = 16Q$ to be the sixteen times extension of Q with its center at $\bar{x}$. Decompose $f(x)$ as

$$\begin{aligned} f(x) &= f_{Q^*} + (f(x) - f_{Q^*})\chi_{Q^*} + (f(x) - f_{Q^*})(1 - \chi_{Q^*}) \\ &:= f_1(x) + f_2(x) + f_3(x). \end{aligned} \tag{3.1}$$

By (1.1) we get $\mu_\Omega(f_1)(x) \equiv 0$. From (1.4) and Theorem A, we obtain

$$\begin{aligned} \int_Q |\mu_\Omega(f_2)(x)|^2 dx &\le \int_{\mathbb{R}^n} |\mu_\Omega(f_2)(x)|^2 dx \\ &\le C \int_{\mathbb{R}^n} |f_2(x)|^2 dx \\ &\le C|Q|\, \|f\|_{BMO}^2. \end{aligned}$$

Hence

$$\begin{aligned} \int_Q |\mu_\Omega(f_2)(x)| dx &\le |Q|^{1/2} \Big(\int_Q |\mu_\Omega(f_2)(x)|^2 dx \Big)^{1/2} \\ &\le C|Q|\, \|f\|_{BMO}. \end{aligned} \tag{3.2}$$

This shows that $\mu_\Omega(f_2)(x) < \infty$ a.e. on Q. Since $|E| > 0$, we have $|Q \cap E| > 0$. Hence there exists an $x_0 \in Q \cap E$ such that $\mu_\Omega(f)(x_0) < \infty$ and $\mu_\Omega(f_2)(x_0) < \infty$ hold at the same time, and we may chose x_0 to close to $\bar{x}$ enough, say, $|x_0 - \bar{x}| < d/4$. Thus

$$\mu_\Omega(f_3)(x_0) \le \mu_\Omega(f)(x_0) + \mu_\Omega(f_2)(x_0) < \infty. \tag{3.3}$$

On the other hand, for any $x \in Q$ we have

(3.4)
$$\begin{aligned}
&|\mu_\Omega(f_3)(x) - \mu_\Omega(f_3)(x_0)| \\
&\le \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x-y) f_3(y)}{|x-y|^{n-1}} dy \right.\right. \\
&\quad \left.\left. - \int_{|x_0-y|\le t} \frac{\Omega(x_0-y) f_3(y)}{|x_0-y|^{n-1}} dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\le \left(\int_0^\infty \left(\int_{\substack{|x-y|\le t \\ |x_0-y|>t}} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} f_3(y) \right| dy \right)^2 \frac{dt}{t^3} \right)^{1/2} \\
&\quad + \left(\int_0^\infty \left(\int_{\substack{|x-y|>t \\ |x_0-y|\le t}} \left| \frac{\Omega(x_0-y)}{|x_0-y|^{n-1}} f_3(y) \right| dy \right)^2 \frac{dt}{t^3} \right)^{1/2} \\
&\quad + \left(\int_0^\infty \left(\int_{\substack{|x-y|\le t \\ |x_0-y|\le t}} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n-1}} \right| |f_3(y)| dy \right)^2 \frac{dt}{t^3} \right)^{1/2} \\
&:= I_1(x) + I_2(x) + I_3(x).
\end{aligned}$$

Below we shall give the estimates of I_1, I_2 and I_3, respectively. For I_1, we have

$$\begin{aligned}
&I_1(x) \\
&\le \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f_3(y)| \left(\int_{\substack{|x-y|\le t \\ |x_0-y|>t}} \frac{dt}{t^3} \right)^{1/2} dy \\
&\le C\|\Omega\|_{L^\infty(S^{n-1})} \int_{(Q^*)^c} \frac{1}{|x-y|^{n-1}} |f_3(y)| \left| \frac{1}{|x-y|^2} - \frac{1}{|x_0-y|^2} \right|^{1/2} dy.
\end{aligned}$$

Note that $y \in (Q^*)^c$ and $x, x_0 \in Q$, we have $|x-y| \sim |x_0-y| \sim |\bar{x}-y|$ and
$$\left| \frac{1}{|x-y|^2} - \frac{1}{|x_0-y|^2} \right| \le \frac{Cd}{|x-y|^3}.$$
From this and applying Hölder's inequality, we get

$$\begin{aligned}
I_1(x) &\le C\|\Omega\|_{L^\infty(S^{n-1})} \int_{(Q^*)^c} \frac{d^{1/2}|f_3(y)|}{|x-y|^{n+1/2}} dy \\
&\le C\|\Omega\|_{L^\infty(S^{n-1})} \left(\int_{(Q^*)^c} \frac{d^{1/2}}{|x-y|^{n+1/2}} dy \right)^{1/2} \qquad (3.5) \\
&\quad \times \left(\int_{(Q^*)^c} \frac{d^{1/2}|f_3(y)|^2}{|x-y|^{n+1/2}} dy \right)^{1/2}.
\end{aligned}$$

It is easy to check that

$$d^{n+1/2}+|x-y|^{n+1/2}\le 2|x-y|^{n+1/2}$$

for $x\in Q$ and $y\in (Q^*)^c$. Hence by (3.5) and Lemma 1

$$\begin{aligned}I_1(x)&\le C\Big(\int_{(Q^*)^c}\frac{d^{1/2}|f_3(y)|^2}{d^{n+1/2}+|x-y|^{n+1/2}}dy\Big)^{1/2}\\&\le C\|f\|_{BMO}.\end{aligned}\tag{3.6}$$

Similarly, we have $I_2(x)\le C\|f\|_{BMO}$ in the same way of estimating $I_1(x)$. Now let us consider $I_3(x)$. Using Hölder's inequality again, we get

$$\begin{aligned}I_3(x)&\le C\int_{(Q^*)^c}\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n-1}}\right||f_3(y)|\\&\quad\times\Big(\int_{|x-y|\le t,|x_0-y|\le t}\frac{dt}{t^3}\Big)^{1/2}dy\\&\le C\int_{(Q^*)^c}\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n-1}}\right|\frac{|f_3(y)|}{|x_0-y|}dy.\end{aligned}$$

Similar to the estimates of (2.6)–(2.8), we get

$$I_3(x)\le C\int_{(Q^*)^c}\frac{|f_3(y)|dy}{|y-x_0|^n\big(\log\frac{|y-x_0|}{d}\big)^\rho}.$$

Note that $y\in(Q^*)^c$ and $x_0,\bar{x}\in Q$, and hence

$$|y-x_0|\ge|y-\bar{x}|-|\bar{x}-x_0|\ge 8d-d/4>4d.$$

Thus

$$\begin{aligned}I_3(x)&\le C\sum_{j=2}^\infty\int_{2^jd\le|y-x_0|<2^{j+1}d}\frac{|f_3(y)|dy}{|y-x_0|^n\big(\log\frac{|y-x_0|}{d}\big)^\rho}.\\&\le C\sum_{j=2}^\infty\Big(\int_{2^jd\le|y-x_0|<2^{j+1}d}\frac{dy}{|y-x_0|^n\big(\log\frac{|y-x_0|}{d}\big)^{2\rho}}\Big)^{1/2}\\&\quad\times\Big(\int_{2^jd\le|y-x_0|<2^{j+1}d}\frac{|f_3(y)|^2dy}{|y-x_0|^n}\Big)^{1/2}\end{aligned}$$

Observe

$$
\begin{aligned}
&\left(\int_{2^j d\le|y-x_0|<2^{j+1}d}\frac{dy}{|y-x_0|^n\left(\log\frac{|y-x_0|}{d}\right)^{2\rho}}\right)^{1/2}\\
&\le C\left(\int_{2^j d}^{2^{j+1}d}\frac{ds}{s\left(\log(s/d)\right)^{2\rho}}\right)^{1/2}\\
&\le \frac{C}{\left(\log 2^j\right)^{\rho}}\left(\int_{2^j}^{2^{j+1}}\frac{ds}{s}\right)^{1/2}\\
&\le \frac{C}{j^{\rho}}.
\end{aligned}\tag{3.7}
$$

On the other hand, denote by Q_j the 2^j times extension of Q, then we have

(3.8)

$$
\begin{aligned}
&\left(\int_{2^j d\le|y-x_0|<2^{j+1}d}\frac{|f_3(y)|^2dy}{|y-x_0|^n}\right)^{1/2}\\
&\le\left(\frac{1}{(2^jd)^n}\int_{2^j d\le|y-x_0|<2^{j+1}d}|f_3(y)|^2dy\right)^{1/2}\\
&\le\left(\frac{1}{(2^jd)^n}\int_{2^j d\le|y-x_0|<2^{j+1}d}(|f(y)-f_{Q_{j+1}}|^2+|f_{Q_{j+1}}-f_{Q^*}|^2)dy\right)^{1/2}\\
&\le C\left(\frac{1}{|Q_{j+1}|}\int_{Q_{j+1}}|f(y)-f_{Q_{j+1}}|^2dy+(j+1)^2 2^{2n}\|f\|_{BMO}^2\right)^{1/2}.
\end{aligned}
$$

Thus by (3.7) and (3.8) and note that $\rho>2$

$$
I_3(x)\le C\sum_{j=2}^{\infty}\frac{1}{j^{\rho}}\left[1+(j+1)2^n\right]\|f\|_{BMO}\le C\|f\|_{BMO}.\tag{3.9}
$$

Summing up (3.4), (3.6) and (3.9), we get

$$
|\mu_\Omega(f_3)(x)-\mu_\Omega(f_3)(x_0)|\le C\|f\|_{BMO}.\tag{3.10}
$$

Thus we prove that

$$
\begin{aligned}
\mu_\Omega(f)(x)&\le\mu_\Omega(f_1)(x)+\mu_\Omega(f_2)(x)\\
&\qquad+|\mu_\Omega(f_3)(x)-\mu_\Omega(f_3)(x_0)|+\mu_\Omega(f_3)(x_0)\\
&<\infty\quad\text{a.e. on } Q.
\end{aligned}
$$

Because Q is any cube with center at $\bar{x} \in E$, we get actually $\mu_\Omega(f)(x) < \infty$ a.e. on $\mathbb{R}^n$.

Finally, let us show that (1.6) holds. In fact, from the process of the above proof we know that for any cube $Q \subset \mathbb{R}^n$, there exists an $x_0 \in Q$ such that $\mu_\Omega(f_3)(x_0) < \infty$. Let $Q^* = 16Q$ and write $f = f_1+f_2+f_3$ as in (3.1). From the process of the above proof, we know that $\mu_\Omega(f_1)(x) = 0$ on Q, and $\int_Q |\mu_\Omega(f_2)(x)|dx \leq C|Q|\,\|f\|_{BMO}$. And by (3.2) and (3.10) we get

$$\begin{aligned} &\int_Q |\mu_\Omega(f)(x) - \mu_\Omega(f_3)(x_0)|dx \\ &\leq \int_Q |\mu_\Omega(f_2)(x)|dx + \int_Q |\mu_\Omega(f_3)(x) - \mu_\Omega(f_3)(x_0)|dx \\ &\leq C|Q|\,\|f\|_{BMO}. \end{aligned} \tag{3.11}$$

Using (3.10) and (3.11) again, we have

$$\begin{aligned} &\int_Q |\mu_\Omega(f)(x) - (\mu_\Omega(f))_Q|dx \\ &\leq \int_Q |\mu_\Omega(f)(x) - \mu_\Omega(f_3)(x_0)|dx + \int_Q |\mu_\Omega(f_3)(x_0) - (\mu_\Omega(f))_Q|dx \\ &\leq C|Q|\|f\|_{BMO} + \int_Q |\mu_\Omega(f)(x) - \mu_\Omega(f_3)(x_0)|dx \\ &\leq C|Q|\|f\|_{BMO}. \end{aligned}$$

Thus we obtain (1.6) and complete the proof of Theorem 2.

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References

[1] A. Benedek, A. Calderón, and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Natl. Acad. Sci. USA **48** (1962), 356–365.

[2] L. Colzani, *Hardy space on sphere*, Ph.D.Thesis, Washington University, St. Louis, Mo, 1982.

[3] L. Colzani, M. Taibleson, and G. Weiss, *Maximal estimates for cesáro and Riesz means on spheres*, Indiana Univ. Math. J. **33** (1984), 873–889.

[4] Y. Ding, D. Fan, and Y. Pan, *L^p-boundedness of Marcinkiewicz integrals with Hardy space function kernel*, Acta. Math. Sinica (English series) **16** (2000), 593–600.

[5] D. Fan and S. Sato, *Weak type (1,1) estimates for Marcinkiewicz integrals with rough kernels*, Tôhoku Math. J. **53** (2001), 265–284.

[6] R. Fefferman and F. Soria, *The Weak space H^1*, Studia Math. **85** (1987), 1–16.

[7] Y. Han, *Some properties of S-function and Marcinkiewicz integrals*, Acta Sci. Natur. Univ. Pekinensis **5** (1987), 21–34.

[8] J. Lee and K. S. Rim, *Estimates of Marcinkiewicz integrals with bounded homogeneous kernels of degree zero*, Integral Equations Operator Theory **48** (2004), 213–223.

[9] E. Stein, *On the function of Littlewood-Paley, Lusin and Marcinkiewicz*, Trans. Amer. Math. Soc. **88** (1958), 430–466.

[10] ———, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N.J., 1970.

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