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A NOTE ON ENESTRÖM-KAKEYA THEOREM

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Abstract. In this paper, the restriction on the coefficients of a polynomial with complex coefficients is weakened in order to obtain an extension of Eneström-Kakeya's Theorem. Our method of proofs is of independent interest. Moreover, remark at the end simplifies several known results in this area of research.

1. Introduction

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. One of the fundamental problem of finding out the region which contains all or a prescribed number of zeros of a polynomial was first studied by Gauss [9]. He proved:

Theorem 1.1. *If $P(z) = z^n + \sum_{j=1}^{n-1} a_j z^j$, where a_j are all real, then $P(z)$ has all its zeros in $|z| \le R$, where*

(i) $R = \max(1, 2^{\frac{1}{2}} s)$, *s being the sum of positive a_j*
(ii) $R = \max(n 2^{\frac{1}{2}} |a_j|)^j$.

In 1829, Cauchy [4] gave more exact bounds for the moduli of zeros of a polynomial than those given by Gauss [9]. He proved the following result.

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Theorem 1.2. *All the zeros of the polynomial* $P(z) = \sum_{j=0}^{n} a_j z^j$ *of degree* n *lie in the circle* $|z| \leq R$*, where* R *is the root of the equation*

$$|a_0| + |a_1|z + |a_2|z^2 + ... + |a_{n-1}|z^{n-1} + |a_n|z^n = 0.$$

Several generalisations and improvements of this result are available in the literature (see [1-6, 11-12]). The following elegant results on the location of zeros of a polynomial with restricted coefficients is known as the Eneström-Kakeya theorem [13-14].

Theorem 1.3. (Eneström-Kakeya) *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *whose coefficients* a_j *satisfy*

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 > 0.$$

Then all the zeros of $P(z)$ *lie in the closed unit disk* $|z| \leq 1$.

Joyal, Labella and Rahman[11] extended Theorem 1.3 to polynomials whose coefficients are monotonic but need not be non-negative as follows:

Theorem 1.4. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that*

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0.$$

Then all the zeros of $P(z)$ *lie in*

$$|z| \leq \frac{a_n + |a_0| - a_0}{|a_n|}.$$

Aziz and Zargar [2] relaxed the conditions of Theorem 1.3 and proved the following generalisation of Theorem 1.4.

Theorem 1.5. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that for some* $k \geq 1$,

$$ka_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0.$$

Then all the zeros of $P(z)$ *lie in*

$$|z + k - 1| \leq \frac{ka_n + |a_0| - a_0}{|a_n|}.$$

Aziz and Zargar[3] obtained some extensions of Theorem 1.3 by relaxing the hypothesis as follows:

Theorem 1.6. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n*. If for some positive numbers* k *and* ρ *with* $k \geq 1$ *and* $0 < \rho \leq 1$,

$$ka_n \geq a_{n-1} \geq ... \geq a_1 \geq \rho a_0 \geq 0,$$

then all the zeros of $P(z)$ *lie in the closed unit disk*

$$|z + k - 1| \leq k + 2\frac{a_0}{a_n}(1 - \rho).$$

Theorem 1.7. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some positive number ρ, $0 < \rho \leq 1$, and some non-negative integer λ, $0 \leq \lambda \leq n-1$,*

$$a_n \leq a_{n-1} \leq ... \leq a_{\lambda+1} \leq a_\lambda \geq a_{\lambda-1} \geq ... \geq \rho a_0,$$

then all the zeros of $P(z)$ lie in

$$|z + \frac{a_{n-1}}{a_n} - 1| \leq \frac{2a_\lambda - a_{n-1} + (2-\rho)|a_0| - \rho a_0}{a_n}.$$

In this paper, we further weaken the hypothesis of Theorems 1.6 and 1.7 to prove following result for polynomials with complex coefficients. Our result is an extension of Theorem 1.3 (Eneström-Kakeya) among others.

2. Main Results

Theorem 2.1. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some real $t > 0$, $\mu \geq 0$, $0 \leq \lambda \leq n-1$ and $0 < \rho \leq 1$,*

$$t^n a_n \leq t^{n-1} a_{n-1} \leq ... \leq t^{\lambda+1} a_{\lambda+1} \leq t^\lambda a_\lambda + \mu t^{\lambda-1} \geq t^{\lambda-1} a_{\lambda-1} \geq ... \geq \rho a_0.$$

Then all the zeros of $P(z)$ lie in

$$|z - \frac{\mu}{a_n}| \leq \frac{1}{|a_n|}\left\{-a_n t + \mu + \frac{a_0}{t^{n-1}} - \rho\frac{(a_0 + |a_0|)}{t^{n-1}} + \frac{2a_\lambda}{t^{n-\lambda-1}} + \frac{|a_0|}{t^{n-1}}\right\}.$$

Proof. Consider the polynomial

$$\begin{aligned}
F(z) &= (t-z)P(z) \\
&= a_0 t + \sum_{j=1}^{n}(a_j t - a_{j-1})z^j - a_n z^{n+1} \\
&= -a_n z^{n+1} + \sum_{j=1}^{n}(a_j t - a_{j-1})z^j + a_0 t \\
&= -a_n z^{n+1} + (a_n t - a_{n-1})z^n + \sum_{j=1}^{n-1}(a_j t - a_{j-1})z^j + a_0 t \\
&= -a_n z^{n+1} + (\mu - a_n t)z^n + a_n t z^n + (a_n t - \mu - a_{n-1})z^n \\
&\quad + (a_1 t - a_0)z + \sum_{j=2}^{n-1}(a_j t - a_{j-1})z^j + a_0 t.
\end{aligned}$$

This gives

$$
\begin{aligned}
|F(z)| &= |-a_n z^{n+1} + (\mu - a_n t)z^n + a_n t z^n + (a_n t - \mu - a_{n-1})z^n \\
&\quad + (a_1 t - a_0)z + \sum_{j=2}^{n-1}(a_j t - a_{j-1})z^j + a_0 t|. \\
&= |-a_n z^{n+1} + (\mu - a_n t)z^n + a_n t z^n + (a_n t - \mu - a_{n-1})z^n \\
&\quad + (a_1 t - a_0)z + \sum_{j=2}^{\lambda}(a_j t - a_{j-1})z^j \\
&\quad + \sum_{j=1+\lambda}^{n-1}(a_j t - a_{j-1})z^j + a_0 t| \\
&\geq |z|^n |a_n z - \mu| \\
&\quad - |z|^n \Bigg[|a_n t - \mu - \alpha_{n-1}| + \frac{|a_1 t - a_0|}{|z|^{n-j}} \\
&\quad + \frac{|a_0|t}{|z|^n} + \sum_{j=2}^{\lambda} \frac{|a_j t - a_{j-1}|}{|z|^{n-j}} + \sum_{j=1+\lambda}^{n-1} \frac{|a_j t - a_{j-1}|}{|z|^{n-j}} \Bigg] \\
&\geq |z|^n |a_n z - \mu| \\
&\quad - |z|^n \Bigg[|a_n t - \mu - a_{n-1}| + \frac{|a_1 t - \rho a_0|}{|t|^{n-1}} + \frac{|a_0 - \rho a_0|}{|t|^{n-1}} \\
&\quad + \frac{|a_0|t}{|z|^n} + \sum_{j=2}^{\lambda} \frac{|a_j t - a_{j-1}|}{|z|^{n-j}} + \sum_{j=1+\lambda}^{n-1} \frac{|a_j t - a_{j-1}|}{|z|^{n-j}} \Bigg].
\end{aligned}
$$

Now, let $|z| \geq t$, so that $\frac{1}{|z|^{n-j}} \leq \frac{1}{|t|^{n-j}}$ for $0 \leq j \leq n$. Then, we have

$$
\begin{aligned}
|F(z)| &\geq |z|^n \Bigg[|a_n z - \mu| \\
&\quad - \Bigg\{ |a_n t - \mu - a_{n-1}| + \frac{|a_1 t - \rho a_0|}{|t|^{n-1}} + \frac{|a_0 - \rho a_0|}{|t|^{n-1}} \\
&\quad + \frac{|a_0|t}{|t|^n} + \sum_{j=2}^{\lambda} \frac{|a_j t - a_{j-1}|}{|t|^{n-j}} + \sum_{j=1+\lambda}^{n-1} \frac{|a_j t - a_{j-1}|}{|t|^{n-j}} \Bigg\} \Bigg] \\
&= |z|^n \Bigg[|a_n z - \mu| - \Bigg\{ -a_n t + \mu + a_{n-1} \\
&\quad + \frac{a_1}{|t|^{n-2}} - \rho \frac{|a_0|}{|t|^{n-1}} + \frac{a_0}{|t|^{n-1}} - \rho \frac{|a_0|}{|t|^{n-1}}
\end{aligned}
$$

$$+\sum_{j=2}^{\lambda}\frac{a_jt-a_{j-1}}{|t|^{n-j}}+\sum_{j=1+\lambda}^{n-1}\frac{a_{j-1}t-a_j}{|t|^{n-j}}+\frac{|a_0|t}{|t|^n}\Bigg\}\Bigg]$$

$$=|z|^n\Bigg[|a_nz-\mu|-\Bigg\{-a_nt+\mu+a_{n-1}+\frac{a_1}{t^{n-2}}-\rho\frac{|a_0|}{t^{n-1}}$$

$$+\frac{a_0}{t^{n-1}}-\rho\frac{|a_0|}{t^{n-1}}+\frac{a_\lambda t^{1+\lambda}}{t^n}+\sum_{j=2}^{\lambda-1}\frac{a_jt^{1+j}}{t^n}-\frac{a_0}{t^{n-1}}-\sum_{j=2}^{\lambda-1}\frac{a_jt^{1+j}}{t^n}$$

$$+\frac{a_\lambda t^{1+\lambda}}{t^n}+\sum_{j=1+\lambda}^{n-2}\frac{a_jt^{1+j}}{t^n}-a_{n-1}-\sum_{j=1+\lambda}^{n-2}\frac{a_jt^{1+j}}{t^n}+\frac{|a_0|t}{|t|^n}\Bigg\}\Bigg]$$

$$=|z|^n\Bigg[|a_nz-\mu|-\Bigg\{-a_nt+\mu+\frac{a_1}{t^{n-2}}-\rho\frac{(a_0+|a_0|)}{t^{n-1}}+\frac{2a_\lambda}{t^{n-\lambda-1}}$$

$$+\frac{|a_0|}{t^{n-1}}\Bigg\}\Bigg]$$

$$\geq|z|^n\Bigg[|a_nz-\mu|-\Bigg\{-a_nt+\mu+\frac{a_0}{t^{n-1}}-\rho\frac{(a_0+|a_0|)}{t^{n-1}}$$

$$+\frac{2a_\lambda}{t^{n-\lambda-1}}+\frac{|a_0|}{t^{n-1}}\Bigg\}\Bigg].$$

If

$$|a_nz-\mu|>\left\{-a_nt+\mu+\frac{a_0}{t^{n-1}}-\rho\frac{(a_0+|a_0|)}{t^{n-1}}+\frac{2a_\lambda}{t^{n-\lambda-1}}+\frac{|a_0|}{t^{n-1}}\right\}.$$

i.e.,

$$|z-\frac{\mu}{a_n}|>\frac{1}{|a_n|}\left\{-a_nt+\mu+\frac{a_0}{t^{n-1}}-\rho\frac{(a_0+|a_0|)}{t^{n-1}}+\frac{2a_\lambda}{t^{n-\lambda-1}}+\frac{|a_0|}{t^{n-1}}\right\},$$

then all the zeros of $F(z)$ whose modulus is greater than or equal to t lie in

$$|z-\frac{\mu}{a_n}|\leq\frac{1}{|a_n|}\left\{-a_nt+\mu+\frac{a_0}{t^{n-1}}-\rho\frac{(a_0+|a_0|)}{t^{n-1}}+\frac{2a_\lambda}{t^{n-\lambda-1}}+\frac{|a_0|}{t^{n-1}}\right\}.$$

But those zeros of $F(z)$ whose modulus is less than t already satisfy the above inequality and all the zeros of $P(z)$ are also the zeros of $F(z)$. Hence it follows that all the zeros of $F(z)$ and hence of $P(z)$ lie in

$$|z-\frac{\mu}{a_n}|\leq\frac{1}{|a_n|}\left\{-a_nt+\mu+\frac{a_0}{t^{n-1}}-\rho\frac{(a_0+|a_0|)}{t^{n-1}}+\frac{2a_\lambda}{t^{n-\lambda-1}}+\frac{|a_0|}{t^{n-1}}\right\}.$$

This completes the proof. □

Remark 2.2. (1) For $t = 1$ and $\lambda = n$, $\mu = 0$, $\rho = 1$, $a > 0$, we recapture the Eneström-Kakeya Theorem 1.3 (see [13,14]).

(2) For $t = 1$ and $\lambda = n$, $\mu = 0$, $\rho = 1$, a is non-negative, we recapture the results of Joyal, Labella and Rahman [11].

(3) For $t = 1$ and $\lambda = n$, $\mu = k - 1$, $\rho = 1$, a is non-negative, we recapture the results of Aziz and Zargar [2].

(4) For $t = 1$ and $\lambda = n$, $\mu = k - 1$, $a \geq 0$, we recapture the results of Aziz and Zargar [3].

Remark 2.3. Finding the zeros of a polynomial is a long standing classical problem which has emerged as an interesting and fascinating area of research for Mathematicians and Engineers (see [7, 10]). Eneström-Kakeya result serves as a very strong tool for obtaining the region in the complex plane having all the zeros of a class of polynomial. The result has been employed to: analyze overflow oscillation of discrete-time dynamical system [15], investigate the properties of orthogonal wavelets [12], determine the asymptotic behavior of zeros of the Daubechies filter [10, 12].

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