A NOTE ON EQUATIONALLY COMPACT LATTICES

by

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Correction to "A Note on Equationally Compact Lattices"

~~There is a large gap in the proof of Theorem 1 where I stated that $b \in (a_1, a_2)_{\mathcal{L}}$ [line two of second paragraph of proof].~~

Please replace the original page 3 with the attached pages 3a and 3b.

~~(Remark: I can now show that statement (D) (and also $(D_{\mathfrak{m}})$ for any cardinal $\mathfrak{m}$) holds for the class $\underset{\sim}{L}$ of all lattices in case $\mathcal{L}$ is a finite chain.)~~

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therefore in $\mathcal{L}$; therefore, $\psi(b, g_k)$ holds in $\mathcal{L}$ for some k in $\{1, \ldots, r\}$ which implies that $g_k \in (a_1, b]_{\mathcal{L}} \cup [b, a_2)_{\mathcal{L}}$. This is a contradiction since $(a_1, b]_{\mathcal{L}} \cup [b, a_2)_{\mathcal{L}}$ does not meet A.

For a chain $\mathcal{L} = (B; \vee, \wedge)$ and a family $(\mathfrak{A}_s \mid s \in S)$ of lattices such that each lattice $\mathfrak{A}_s$ has a least element 0_s and a greatest element 1_s where $S \subseteq \{(a, b) \mid a \text{ is covered by } b \text{ in } \mathcal{L}\}$, $\chi(\mathcal{L}, (\mathfrak{A}_s \mid s \in S))$ denotes the lattice $\mathfrak{C}$ where C is $B \,\dot{\cup}\, \dot{\bigcup}(A_s - \{0_s, 1_s\} \mid s \in S)$ and the order in $\mathfrak{C}$ is defined by identifying $0_{(a,b)}$ with a and $1_{(a,b)}$ with b, and preserving the order in $\mathcal{L}$ and in each $\mathfrak{A}_s$.

THEOREM 2. For a complete chain $\mathcal{L}$ and family $(\mathfrak{A}_s \mid s \in S)$

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0. Introduction

The concept of equational compactness was introduced by Jan Mycielski [6]. (The definitions are given in section 1.) The main result of this note (Theorem 1) is the characterization of equationally compact lattices in $\underset{\sim}{W}$, the class of all lattices which do not contain an infinite anti-chain. Some results concerning the equational compactness of arbitrary lattices are also presented.

1. Preliminaries

The reference for lattice theory is Birkhoff [1] while the reference for universal algebra is Grätzer [3]. A lattice $\mathfrak{A}=(A; \vee, \wedge)$ is <u>lower continuous</u> (resp., <u>upper continuous</u>) if $\mathfrak{A}$ is complete and, for every downward (resp., upward) directed set C and <u>a</u> in $\mathfrak{A}$, $a \vee \bigwedge C = \bigwedge(a \vee c \mid c \in C)$ (resp., $a \wedge \bigvee C = \bigvee(a \wedge c \mid c \in C)$). (By Appendix II of Maeda [5], it suffices to consider only chains C in the previous definition.) A lattice is <u>continuous</u> if it is both lower and upper continuous. An <u>anti-chain</u> in a lattice is a set in which every two distinct elements are incomparable.

A (universal) algebra $\mathfrak{A}$ is <u>equationally compact</u> if **any** set of equations with constants in A that is finitely solvable in $\mathfrak{A}$ is solvable in $\mathfrak{A}$ (see [6] or [7]). An algebra $\mathfrak{A}$ is $\mathfrak{m}$-<u>variable equationally compact</u> (where $\mathfrak{m}$ is a cardinal) if **any** set of equations with constants in A in which at most $\mathfrak{m}$ variables appear is solvable in $\mathfrak{A}$ whenever it is finitely solvable in $\mathfrak{A}$. The algebra $\mathfrak{B}$ is an <u>elementary extension</u> of $\mathfrak{A}$ if any sentence with constants in A holds in $\mathfrak{A}$ if and only if it holds in $\mathfrak{B}$ (cf. Definition 38.4 of [3]). B. Weglorz [7] has shown that an algebra is equationally compact if and only if it is a retract of every elementary extension.

2. Equationally Compact Lattices

By a result of G. Grätzer and H. Lakser [4], every 1-variable equationally compact lattice is continuous. In particular, any equationally compact lattice is continuous; Theorem 1 shows that the converse holds for lattices in $\underset{\sim}{W}$, a result that is known for Boolean algebras (see [7]).

LEMMA. Let $\mathfrak{A}$ be a lower continuous lattice and $\mathfrak{B}$ be a (lattice) extension of $\mathfrak{A}$. The map $\phi : \mathfrak{B} \longrightarrow \mathfrak{A}$ defined by $\phi(b) = \inf_{\mathfrak{A}} \{a \in A \mid a \geq b\}$ is a join-homomorphism.

<u>Proof</u>. For b in B, let $U(b) = \{a \in A \mid a \geq b\}$. Let b_1 and b_2 be in B. If $U(b_1)$ is empty, then $\phi(b_1 \vee b_2) = \phi(b_1) \vee \phi(b_2)$ since $\inf_{\mathfrak{A}} \varnothing$ is the largest element of $\mathfrak{A}$. We now assume that $U(b_1)$ and $U(b_2)$ are nonempty. For a_1 in $U(b_1)$ and a_2 in $U(b_2)$, $\phi(b_1 \vee b_2) \leq a_1 \vee a_2$. Taking the infimum over all a_1 in $U(b_1)$ and then over all a_2 in $U(b_2)$, we obtain $\phi(b_1 \vee b_2) \leq \phi(b_1) \vee \phi(b_2)$. This completes the proof since the opposite inequality obviously holds.

THEOREM 1. A lattice in $\underset{\sim}{W}$ is equationally compact if and only if it is continuous.

<u>Proof</u>. Let $\mathfrak{A}$ be a continuous lattice in $\underset{\sim}{W}$ and $\mathfrak{B}$ be an elementary extension of $\mathfrak{A}$. We define ϕ and $\phi' : \mathfrak{B} \longrightarrow \mathfrak{A}$ by $\phi(b) = \inf_{\mathfrak{A}} U(b)$ and $\phi'(b) = \sup_{\mathfrak{A}} L(b)$ where $U(b) = \{a \in A \mid a \geq b\}$

and $L(b)=\{a\in A \mid a\leq b\}$. We will show that $\varphi=\varphi'$; then, by the lemma, φ is a lattice retraction. Therefore, $\mathfrak{A}$ is equationally compact by the result of E. Weglorz.

Suppose that $\varphi\neq\varphi'$; then, for some b in B-A, $a_1=\sup L(b)\neq \inf U(b)=a_2$ and $b\in(a_1,a_2)_{\mathfrak{B}}=\{x\in B \mid a_1<x<a_2\}$. For any interval I , the statement "$x\in I$" is equivalent to a lattice formula involving only x and the end points of I; each statement in quotation marks that follows is easily seen to be equivalent to a lattice formula with constants in A. Let $\psi(x,y)$ be the formula "$x\in(a_1,y]\cup[y,a_2)$ and $y\in(a_1,a_2)$". Since "$(\exists x)\ x\in(a_1,a_2)$" holds in $\mathfrak{B}$, $(a_1,a_2)_{\mathfrak{B}}$ meets A. Let $\{c_1,c_2,\ldots,c_n\}$ be a maximal (with respect to inclusion) anti-chain in $(a_1,a_2)_{\mathfrak{B}}\cap A$. The sentence "$(\forall x)\ x\in(a_1,a_2)\Longrightarrow\psi(x,c_1)$ or ... or $\psi(x,c_n)$" holds in $\mathfrak{A}$ and therefore holds in $\mathfrak{B}$. Thus, $\psi(b,c_i)$ holds in $\mathfrak{B}$ for some i in $\{1,\ldots,n\}$ which implies that $c_i\in(a_1,b]_{\mathfrak{B}}\cup[b,a_2)_{\mathfrak{B}}$. This is a contradiction since $(a_1,b]_{\mathfrak{B}}\cup[b,a_2)_{\mathfrak{B}}$ does not meet A.

For a chain $\mathfrak{B}=(B;\vee,\wedge)$ and a family $(\mathfrak{A}_s \mid s\in S)$ of lattices such that each lattice $\mathfrak{A}_s$ has a least element 0_s and a greatest element 1_s where $S\subseteq\{(a,b) \mid a \text{ is covered by } b \text{ in } \mathfrak{B}\}$, $\chi(\mathfrak{B},(\mathfrak{A}_s \mid s\in S))$ denotes the lattice $\mathfrak{C}$ where C is $B\,\dot{\cup}\,\dot{\bigcup}(A_s-\{0_s,1_s\} \mid s\in S)$ and the order in $\mathfrak{C}$ is defined by identifying $0_{(a,b)}$ with a and $1_{(a,b)}$ with b , and preserving the order in $\mathfrak{B}$ and in each $\mathfrak{A}_s$.

THEOREM 2. For a complete chain $\mathfrak{B}$ and family $(\mathfrak{A}_s \mid s\in S)$

of continuous lattices with $S \subseteq \{(a,b) \mid a \text{ is covered by } b \text{ in } \mathcal{B}\}$, $\mathcal{L} = \chi(\mathcal{B}, (\mathcal{A}_s \mid s \in S))$ is continuous. Moreover, $\mathcal{L}$ is in the smallest non-trivial equational class containing every $\mathcal{A}_s$ for s in S.

Proof. We can assume that S is nonempty and each $\mathcal{A}_s$ is non-trivial. For $s=(a,b)$ in S and c in $A_s - \{0_s, 1_s\}$, we define $\lambda(c)=a$; otherwise, $\lambda(c)=c$ for c in B. Let T be an ward directed subset of $\mathcal{L}$ and $c=\sup_{\mathcal{B}}(\lambda(t) \mid t \in T)$. It is easily shown that

$$\sup_{\mathcal{L}} T = \begin{cases} c, & \text{if } (c,d) \notin S \text{ for any } d \in B; \\ \sup_{\mathcal{A}_s}(c \vee t \mid t \in T), & \text{where } s=(c,d) \in S. \end{cases}$$

Let a be in $\mathcal{L}$. We now show that $a \wedge \bigvee T \leq \bigvee(a \wedge t \mid t \in T)$ in $\mathcal{L}$. We first suppose that $\sup_{\mathcal{L}} T = c$. We can assume that $a \ngeq c$; thus, $a < c$ and the result follows since $a \geq \lambda(t)$ cannot hold for every t in T. We now suppose that $\sup_{\mathcal{L}} T = \sup_{\mathcal{A}_s}(c \vee t \mid t \in T)$ where $s=(c,d) \in S$. We can assume that $a \geq c$ since otherwise $a < \lambda(t)$ for some t in T. Then, $a \wedge \sup_{\mathcal{L}} T = (a \wedge d) \wedge \sup_{\mathcal{A}_s}(T \cap A_s)$ $= \sup_{\mathcal{A}_s}(a \wedge d \wedge t \mid t \in T \cap A_s) = \sup_{\mathcal{A}_s}(a \wedge t \mid t \in T \cap A_s)$ $= \sup_{\mathcal{L}}(a \wedge t \mid t \in T)$. By duality, this completes the proof of the first statement of the theorem.

The second statement is true if the chain $\mathcal{B}$ is finite. Indeed, in this case we can suppose that $S = \{(k, k+1) \mid k = 1, 2, \ldots, n-1\}$ where $B = \{1, 2, \ldots, n\}$ with the usual order. $\mathcal{L}$ is then isomorphic to the sublattice

$$\bigcup_{k=1}^{n-1} \{1_{(1,2)}\} \times \cdots \times \{1_{(k-1,k)}\} \times \mathcal{A}_{(k,k+1)} \times \{0_{(k+1,k+2)}\} \times \cdots \times \{0_{(n-1,n)}\}$$

of $\mathcal{A}_{(1,2)} \times \cdots \times \mathcal{A}_{(n-1,n)}$; therefore $\mathcal{L}$ is in the equational class generated by $\{\mathcal{A}_s \mid s \in S\}$. We now consider the general case. Let $p(x_1,\ldots,x_n)=q(x_1,\ldots,x_n)$ be an identity that holds in each $\mathfrak{A}_s$ for s in S. If $a_1,\ldots,a_n \in C$, then the sublattice of $\mathfrak{L}$ generated by $\{a_1,\ldots a_n\}$ is isomorphic to a sublattice of $\chi(\mathfrak{B}',(\mathfrak{A}_t \mid t\in T))$ for a finite subchain $\mathfrak{B}'$ of $\mathfrak{B}$ and $T\subseteq S$; therefore, $p(a_1,\ldots,a_n)$ $=q(a_1,\ldots,a_n)$. Thus the identity $p=q$ holds in $\mathfrak{L}$.

THEOREM 3. Any continuous distributive lattice (i.e.,infinitely distributive complete lattice) is 1-variable equationally compact.

<u>Proof</u>. Let $\mathfrak{A}$ be an infinitely distributive complete lattice and Σ be a set of lattice equations in one variable x with constants in A that is finitely solvable in $\mathfrak{A}$. In a Boolean algebra $\mathfrak{B}$, the nonempty solution set in $\mathfrak{B}$ of a lattice equation in one variable with constants in B is a closed interval in $\mathfrak{B}$. Since $\mathfrak{A}$ can be (lattice) embedded in a Boolean algebra, $S(\Psi)$, the solution set in $\mathfrak{A}$ of the equation Ψ in Σ, is a convex set. We can suppose that Ψ is $(a\wedge x)\vee b = (c\wedge x)\vee d$ with $a,b,c,d\in A$; an easy calculation shows that $\sup S(\Psi)$ satisfies Ψ. Therefore, $S(\Psi)$ is a closed interval in $\mathfrak{A}$. Since $\mathfrak{A}$ is complete, the interval topology makes A a compact topological space and therefore $\bigcap(S(\Psi)\mid\Psi\in\Sigma)\neq\emptyset$ since $(S(\Psi)\mid\Psi\in\Sigma)$ is a family of closed sets with the finite intersection property. This means that Σ is solvable.

COROLLARY. Let $\mathfrak{A}$ be an infinitely continuous complete lattice and $\mathfrak{B}=(B;\vee,\wedge,{}',0,1)$ be a Boolean algebra. If there is a (lattice) embedding $\phi:\mathfrak{A}\longrightarrow(B;\vee,\wedge)$ such that $\phi(A)$ generates $\mathfrak{B}$, then ϕ is a complete embedding.

Proof. We can assume that $\mathfrak{A}$ is a sublattice of $\mathfrak{B}$ such that A generates $\mathfrak{B}$. For $S \subseteq A$, let $u = \sup_{\mathfrak{A}} S$ and c in $\mathfrak{B}$ be an upper bound of S. We can write c as $c=(a_1 \vee b_1') \wedge \ldots \wedge (a_n \vee b_n')$ with $a_i, b_i \in A$ for all i. For fixed i, the set Σ_i of equations $\{x \wedge b_i \leq a_i \wedge b_i\} \cup \{s \leq x \leq u \mid s \in S\}$ is finitely solvable in $\mathfrak{A}$. Since Σ_i is equivalent to a set of lattice equations in one variable with constants in A, Σ_i is solvable in the 1-variable equationally compact lattice $\mathfrak{A}$; the solution can only be x=u. Therefore, $u \wedge b_i \leq a_i \wedge b_i$ which is equivalent to $u \leq a_i \vee b_i'$. It follows that $u \leq c$ and thus that $u = \sup_{\mathfrak{B}} S$, proving the corollary.

Since every distributive lattice can be embedded in a Boolean algebra and the completion by cuts of a Boolean algebra is a Boolean algebra, the corollary yields the result that every infinitely distributive complete lattice $\mathfrak{A}$ can be completely embedded in a complete Boolean algebra; this result, without requiring $\mathfrak{A}$ to be complete, has been proved by Nenosuke Funayama [2].

The continuous modular lattice consisting of the infinite anti-chain $\{a_n \mid n < \omega\}$ together with 0 and 1 is not 1-variable equationally compact since the set of equations $\{a_n \vee x = 1,\ a_n \wedge x = 0 \mid n < \omega\}$ is not solvable.

We label the following statements for a class $\underset{\sim}{K}$ of lattices:

(A) For $\mathfrak{A} \in \underset{\sim}{K}$, $\mathfrak{A}$ is equationally compact if and only if $\mathfrak{A}$ is continuous.

(B) For $\mathfrak{A} \in \underset{\sim}{K}$, $\mathfrak{A}$ is 1-variable equationally compact if and only if $\mathfrak{A}$ is continuous.

(C) For $\mathfrak{A} \in \underset{\sim}{K}$, $\mathfrak{A}$ is equationally compact if and only if $\mathfrak{A}$ is 1-variable equationally compact.

(D) If $\mathfrak{L}$ is a complete chain and $(\mathfrak{A}_s \mid s \in S)$ is a family of equationally compact algebras of $\underset{\sim}{K}$ where $S \subseteq \{(a,b) \mid a \text{ is covered by } b \text{ in } \mathfrak{L}\}$, then $\chi(\mathfrak{L}, (\mathfrak{A}_s \mid s \in S))$ is equationally compact.

(D_1) If $\mathfrak{L}$ is a complete chain and $(\mathfrak{A}_s \mid s \in S)$ is a family of 1-variable equationally compact algebras of $\underset{\sim}{K}$ where $S \subseteq \{(a,b) \mid a \text{ is covered by } b \text{ in } \mathfrak{L}\}$, then $\chi(\mathfrak{L}, (\mathfrak{A}_s \mid s \in S))$ is 1-variable equationally compact.

Since the "only if" implication always holds in (A), (B), and (C), (A) holds if and only if both (B) and (C) hold; therefore, (A) holds for the class of distributive lattices if and only if (C) holds. For a class $\underset{\sim}{K}$ for which $\chi(\mathfrak{L}, (\mathfrak{A}_s \mid s \in S)) \in \underset{\sim}{K}$ whenever $\{\mathfrak{A}_s \mid s \in S\} \subseteq \underset{\sim}{K}$, it follows from Theorem 2 that (D) (resp., (D_1)) holds whenever (A) (resp., (B)) holds; in particular, (D) and (D_1) hold for $\underset{\sim}{W}$.

Boolean algebras

Let $\underset{\sim}{B}$ be the class of Λ, considered as lattices; $\underset{\sim}{D}$ be the class of distributive lattices; $\underset{\sim}{M}$ be the class of modular lattices; and $\underset{\sim}{L}$ be the class of all lattices. The following table summarizes the preceeding results.

	(A)	(B)	(C)	(D)	(D_1)
$\underset{\sim}{W}$	yes	yes	yes	yes	yes
$\underset{\sim}{B}$	yes	yes	yes	?	yes
$\underset{\sim}{D}$	?	yes	?	?	yes
$\underset{\sim}{M}$	no	no	?	?	?
$\underset{\sim}{L}$	no	no	?	?	?

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