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A NOTE ON EQUIVARIANT ETA INVARIANTS

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ABSTRACT. We prove the regulairty of equivariant eta functions near the origin. We also propose an equivariant version of the Cheeger–Chou index theorem on spaces with conelike singularities.

0. INTRODUCTION

Let M be an odd-dimensional compact Riemannian spin manifold with a fixed spin structure, D the Dirac operator on M. The η function associated to D is defined by [1]

$$(0.1) \qquad \eta(s, D) = \sum_{\lambda \neq 0} (\operatorname{sign} \lambda) \frac{\dim \Gamma(E_\lambda)}{|\lambda|^s},$$

where λ runs over the nonzero eigenvalues of D and $\Gamma(E_\lambda)$ is the eigenspace of λ.

If $T: M \to M$ is an isometry preserving the orientation and spin structure and $\widetilde{dT} D = D \widetilde{dT}$, where $\widetilde{dT}: \Gamma(S(M)) \to \Gamma(S(M))$ is the lift of $dT: \Gamma(TM) \to \Gamma(TM)$ to the spinors, then one can also define the equivariant η function [1, 5] by

$$(0.2) \qquad \eta_T(s, D) = \sum_{\lambda \neq 0} (\operatorname{sign} \lambda) \frac{\operatorname{Tr} \widetilde{dT}|_{\Gamma(E_\lambda)}}{|\lambda|^s}.$$

In the first two sections we prove some basic properties of equivariant η functions and in §3, we point out that a slight modification of Bismut–Cheeger [3] yields an equivariant index theorem for spaces with conelike singularities.

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Related work has also been done by J.-M. Bismut and J. Cheeger (unpublished).

1. Equivariant η functions

It is clear that (0.1) and (0.2) are only defined when $\operatorname{Re}(s)$ is large enough. Then by analytic continuation to the whole complex plane, we obtain the meromorphic functions $\eta(s,D)$ and $\eta_T(s,D)$ on $\mathbf{C}$. Of particular interest is their regularity at $s=0$.

Theorem 1.1. *For* $\operatorname{Re}(s)$ *large enough,*

$$\begin{aligned}\eta_T(s,D) &= \frac{1}{\Gamma((s+1)/2)}\int_0^\infty t^{(s-1)/2}\operatorname{tr}(\widetilde{dT}D\exp(-tD^2))\,dt\\ &= \frac{1}{\Gamma((s+1)/2)}\int_0^\infty t^{(s-1)/2}\,dt\int_M \operatorname{tr}(\widetilde{dT}_x D\exp(-tD^2)(x,Tx))\,dx,\end{aligned} \tag{1.2}$$

where dx *is the volume element associated to the metric, and* $D\exp(-tD^2)(x,y)$ *is the kernel of* $D\exp(-tD^2)$.

Proof. For any $\lambda\neq 0$, let $\varphi_{\lambda_1},\dots,\varphi_{\lambda_{i(\lambda)}}$ be an orthonormal basis of $\Gamma(E_\lambda)$, then it is standard that

$$D\exp(-tD^2)(x,y)=\sum_{\lambda\neq 0}\lambda e^{-t\lambda^2}\sum_{k=1}^{i(\lambda)}\varphi_{\lambda_k}(x)\langle\varphi_{\lambda_k}(y),\cdot\rangle. \tag{1.3}$$

Let $\{e_i(x)\}$ be an orthonormal basis of S_x, then

$$\widetilde{dT}_x D\exp(-tD^2)(x,Tx)e_i(Tx)=\sum_\lambda\lambda e^{-t\lambda^2}\sum_k\widetilde{dT}\varphi_{\lambda_k}(x)\langle\varphi_{\lambda_k}(Tx),e_i(Tx)\rangle.$$

So

$$\begin{aligned}&\operatorname{tr}\widetilde{dT}D\exp(-tD^2)(x,Tx)\\ &\qquad=\sum_\lambda\lambda e^{-t\lambda^2}\sum_k\sum_i\langle\widetilde{dT}\varphi_{\lambda_k}(x),e_i(Tx)\rangle\langle\varphi_{\lambda_k}(Tx),e_i(Tx)\rangle\\ &\qquad=\sum_\lambda\lambda e^{-t\lambda^2}\sum_k\langle\widetilde{dT}\varphi_{\lambda_k}(x),\varphi_{\lambda_k}(Tx)\rangle,\end{aligned}$$

and

$$\begin{aligned}\int_M\operatorname{tr}\widetilde{dT}D\exp(-tD^2)(x,Tx)\,dx&=\sum_\lambda\lambda e^{-t\lambda^2}\int_M\sum_k\langle\widetilde{dT}\varphi_{\lambda_k}(x),\varphi_{\lambda_k}(Tx)\rangle\,dx\\ &=\sum_\lambda\lambda e^{-t\lambda^2}\operatorname{tr}(\widetilde{dT}|_{\Gamma(E_\lambda)}).\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{\Gamma((s+1)/2)}&\int_0^\infty t^{(s-1)/2}\operatorname{tr}(\widetilde{dT}D\exp(-tD^2))\,dt\\&=\frac{1}{\Gamma((s+1)/2)}\int_0^\infty t^{(s-1)/2}\sum_\lambda \lambda e^{-t\lambda^2}\operatorname{tr}\widetilde{dT}|_{\Gamma(E_\lambda)}\,dt\\&=\sum_{\lambda\neq 0}(\operatorname{sign}\lambda)\frac{\operatorname{tr}\widetilde{dT}|_{\Gamma(E_\lambda)}}{|\lambda|^s}.\quad\square\end{aligned}$$

Corollary 1.4. *If T has no fixed points on M, then $\eta_T(s, D)$ is an entire function.*

Proof. We introduce an auxiliary Grassmann variable z as in [4]. Then we have

$$\exp -t(D^2 - zD) = \exp(-tD^2) + ztD\exp(-tD^2), \tag{1.5}$$

so

$$\exp(-t(D^2 - zD))(x, y) = \exp(-tD^2)(x, y) + ztD\exp(-tD^2)(x, y). \tag{1.6}$$

Since T has no fixed points, $d(x, Tx) > \delta$ for some constant $\delta > 0$. By standard results for elliptic operators, there exist positive constants C_i ($i = 1, 2, 3, 4$) such that as $t \to 0^+$,

$$\|\exp(-t(D^2 - zD))(x, Tx)\| \le \frac{C_1}{t^{n/2}}\exp(-C_2/t), \tag{1.7}$$

$$\|\exp(-tD^2)(x, Tx)\| \le \frac{C_3}{t^{n/2}}\exp(-C_4/t) \tag{1.8}$$

for all $x \in M$. Here we have assumed that z has "norm" $\|z\| = 1$. By (1.6)–(1.8), there are positive constants C_5, C_6 such that as $t \to 0^+$,

$$\|D\exp(-tD^2)(x, Tx)\| \le \frac{C_5}{t^{n/2+1}}\exp(-C_6/t). \tag{1.9}$$

Thus, $\forall s \in \mathbf{C}$

$$\begin{aligned}\lim_{t\to 0}&|t^{(s-1)/2}\operatorname{tr}(\widetilde{dT}D\exp(-tD^2))|\\&=\lim_{t\to 0}|t^{(s-1)/2}|\cdot\left|\int_M \operatorname{tr}\widetilde{dT}D\exp(-tD^2)(x, Tx)\,dx\right|\\&\le\lim_{t\to 0}|t^{(s-1)/2}|\cdot C_7\frac{C_5}{t^{n/2+1}}\exp(-C_6/t)\\&=0,\end{aligned}\tag{1.10}$$

where C_7 is a suitable positive constant.

Thus $\int_0^\infty t^{(s-1)/2}\operatorname{tr}(\widetilde{dT}D\exp(-tD^2))\,dt$ is an entire function. On the other hand, $1/\Gamma((s+1)/2)$ is also an entire function, (1.4) follows from Theorem 1.1. $\square$

Remark 1.11. In fact, (1.4) is a known result and can be proved directly without introducing the Grassmann variable z.

2. Regularity near $s = 0$

In this section, we prove the following theorem:

Theorem 2.1. *$\eta_T(s, D)$ is holomorphic on* $\mathrm{Re}(s) > -2$.

First note that it is sufficient to show that

$$\left(\frac{1}{t}\right)^{1/2} |\operatorname{tr} \widetilde{dTD} \exp(-tD^2)| \le C, \qquad \text{as } t \to 0^+ \tag{2.2}$$

for some constant $C > 0$. Because then the integration in (1.2) will be convergent absolutely and uniformly on the compact subsets in $\mathrm{Re}(s) > -2$, which implies the holomorphic property.

Now we begin to prove (2.2).

Since T is an isometry, the fixed point set F of T consists of components $F_1, \dots, F_k$, each of even codimension. If U is an open neighborhood of F, then by the argument in the proof of (1.4), it is easy to see that

$$\lim_{t\to 0} \left(\frac{1}{t}\right)^{1/2} \int_{M\setminus U} \operatorname{tr} \widetilde{dTD} \exp(-tD^2)(x, Tx)\, dx = 0, \tag{2.3}$$

so

$$\begin{aligned} \lim_{t\to 0} \left(\frac{1}{t}\right)^{1/2} \int_{M} \operatorname{tr} \widetilde{dTD} \exp(-tD^2)(x, Tx)\, dx \\ = \lim_{t\to 0} \left(\frac{1}{t}\right)^{1/2} \int_{U} \operatorname{tr} \widetilde{dTD} \exp(-tD^2)(x, Tx)\, dx. \end{aligned} \tag{2.4}$$

Thus we meet a local problem, and the situation is similar to what was considered in [8]. As in [8], we may assume $k = 1$ and $\operatorname{codim} F = 2n'$. Denote by $N(F)$ the normal bundle to F. We need only to prove that

$$\lim_{t\to 0} \left(\frac{1}{t}\right)^{1/2} \left| \int_F \int_{N_\xi(\varepsilon)} \operatorname{tr} \widetilde{dTD} \exp(-tD^2)(x, dTx)\, dN_\xi\, d\xi \right| \le C, \tag{2.5}$$

for some constant $C > 0$. Here $N_\xi(\varepsilon) = \{v \in N_\xi(F) \| \|v\| < \varepsilon\}$, is a sufficiently small neighborhood of the origin in the normal space $N_\xi(F)$, and we use the local trivialization of $\exp N_x(\varepsilon)$ to $N_x(\varepsilon)$.

We choose an orthonormal basis and associated coordinates as in [8]. And we define as in [9, 8],

$$\chi(x^\alpha D_x^\beta e^\gamma) = |\beta| - |\alpha| + |\gamma|, \tag{2.6}$$

for α, $\beta \in Z^n$, $\gamma \in (Z_2)^n$. We also define $\chi(z) = 1$. Then we have

$$\chi(z x^\alpha D_x^\beta e^\gamma) = 1 + |\beta| - |\alpha| + |\gamma|. \tag{2.7}$$

Now as in [6], set

$$h(x) = 1 + \frac{1}{2} z \sum_{i=1}^{n} x_i e_i . \tag{2.8}$$

Then it is trivial to verify that

$$\begin{aligned} h e_i h^{-1} &= e_i + x_i z , \\ &= e_i + (\chi = 0) , \end{aligned} \tag{2.9}$$

and

$$h(D^2 - zD)h^{-1} = D^2 + zu , \tag{2.10}$$

where $\chi(u) \le 0$ and u contains no z. Also write (2.10) as

$$D^2 - zD = h^{-1}(D^2 + zu)h . \tag{2.11}$$

Thus,

$$\exp(-t(D^2 - zD))(x , y) = h^{-1}(x) \exp(-t(D^2 + zu))(x , y) h(y) . \tag{2.11$'$}$$

By (1.6) and (2.11)$'$,

$$\begin{aligned} ztD \exp(-tD^2)(x , y) = h^{-1}(x) \exp(-t(D^2 + zu))(x , y) h(y) \\ - \exp(-tD^2)(x , y) . \end{aligned} \tag{2.12}$$

Then

$$\begin{aligned} zt\widetilde{dT} D \exp(-tD^2)(x , dTx) = \widetilde{dT} h^{-1}(x) \exp(-t(D^2 + zu))(x , dTx) h(dTx) \\ - \widetilde{dT} \exp(-tD^2)(x , dTx) . \end{aligned} \tag{2.13}$$

As in [9] (compare also with [4] or [7]), it is easy to see that

$$\exp(-t(D^2 + zu))(x , y) = \frac{e^{-d(x,y)^2/4t}}{(4\pi t)^{n/2}} \left(\sum_{i=0}^{[n/2]+2} (U_i + zV_i) t^i + o(t^{[n/2]+2}) \right) \tag{2.14}$$

$$\exp(-tD^2)(x , y) = \frac{e^{-d(x,y)^2/4t}}{(4\pi t)^{n/2}} \left(\sum_{i=0}^{[n/2]+2} U_i t^i + o(t^{[n/2]+2}) \right) , \tag{2.15}$$

where $\chi(U_i) \le 2i$, $\chi(V_i) \le 2(i-1)$ and U_i, V_i contain no z.

By (2.9) and (2.13)–(2.15), we get

$$\begin{aligned} &t\widetilde{dT} D \exp(-tD^2)(x , dTx) \\ &= \frac{e^{-d(x,dTx)^2/4t}}{(4\pi t)^{n/2}} \widetilde{dT} \left[\left(\sum_i ((dT - I)x)_i e_i \right) \cdot \left(\sum_{i=0}^{[n/2]+2} U_i t^i + o\left(t^{[n/2]+2}\right) \right) \right. \\ &\qquad \left. + \left(\sum_{i=0}^{[n/2]+2} W_i t^i + o\left(t^{[n/2]+2}\right) \right) \right] , \end{aligned} \tag{2.16}$$

where $\chi(W_i) \le 2(i-1)$.

Lemma 2.17. *Suppose $i \le [n/2]+2$. If W is an odd element and $\chi(W) \le 2i-2+2n'$, then*

$$\lim_{t\to 0}\left(\frac{1}{t}\right)^{3/2}\left|\int_{N_\xi(\varepsilon)} \frac{e^{-d(x,dTx)^2/4t}}{(4\pi t)^{n/2}} \operatorname{tr}(W(0;x))t^i\,dx\right| \le C_1, \tag{2.18}$$

for some constant $C_1 > 0$; where in the $W(y;x)$, y stands for tangential coordinates and x stands for normal coordinates.

Proof. We can assume that W is a monomial, then it can be written as

$$W = \varphi(0)x_{i_1}\cdots x_{i_k}\cdot e_1\cdots e_n. \tag{2.19}$$

Here $e_1\cdots e_n$ all appear because otherwise, $\operatorname{tr} W = 0$. Also note that we can assume that the x_i's in (2.19) are normal coordinates, for otherwise $\operatorname{tr} W(0;\cdot) = 0$.

(i) If $\chi(W) = 2i-2+2n'$, then $k = n+2-2n'-2i$. By making the change of variables $x = t^{1/2}b$, we get

$$\begin{aligned}
&\lim_{t\to 0}\left|\left(\frac{1}{t}\right)^{3/2}\int_{N_\xi(\varepsilon)} \frac{e^{-d(x,dTx)^2/4t}}{(4\pi t)^{n/2}} \operatorname{tr}(W)t^i\,dx\right| \\
&\le \lim_{t\to 0}\frac{C_2}{t^{3/2}}\left|\int_{N_\xi(\varepsilon/\sqrt{t})} \frac{e^{-\|(I-dT)b\|^2/4}}{(4\pi t)^{n/2}} t^{n/2+1-n'-i} b_{i_1}\cdots b_{i_k} t^i t^{n'}\,db\right| \\
&= 0
\end{aligned}$$

(because k is odd).

(ii) If $\chi(W) < 2i-2+2n'$, then it is clear that

$$\lim_{t\to 0}\left|\left(\frac{1}{t}\right)^{3/2}\int_{N_\xi(\varepsilon)} \frac{e^{-d(x,dTx)^2/4t}}{(4\pi t)^{n/2}} \operatorname{tr}(W)t^i\,dx\right| \le C_3. \quad \square$$

Now consider the terms in (2.16). First note that the simple argument in [8] shows that

$$\widetilde{dT} = \prod_{\alpha=n-2n'+1}^{n} e_\alpha\cdot f + g, \tag{2.20}$$

where $\chi(f) \le 0$ and $\chi(g) \le 2n'-2$.

On the otherhand, $dT|_{TF} = Id_{TF}$, so

$$\sum_i ((dT-I)x)_i e_i = \sum_{\alpha=n-2n'+1}^{n} ((dT-I)x)_\alpha e_\alpha. \tag{2.21}$$

By (2.21), (2.20),

$$\chi\left(\widetilde{dT}\cdot\sum_i((dT-I)x)_i e_i\right) \le 2n'-2. \tag{2.22}$$

Thus, all the monomial terms in (2.16) satisfy the condition of Lemma 2.17. So we get

$$\lim_{t\to 0}\left(\frac{1}{t}\right)^{1/2}\left|\int_{N_\xi(\varepsilon)}\operatorname{tr}(\widetilde{dT}D\exp(-tD^2)(x\,,dTx))\,dx\right| < C_4\,, \tag{2.23}$$

for some constant $C_4 > 0$. By combining this with (2.5), (2.2), the proof of Theorem 2.1 is completed. □

Remark 2.24. A more careful look at the proof allows us to write (2.2) in a more precise form:

Theorem 2.25. *If $T\colon M \to M$ is an isometry and $F = \cup F_i$ is its fixed point set, then there exist smooth functions φ_i on F_i such that*

$$\operatorname{tr}\widetilde{dT}D\exp(-tD^2) = \sum_i \int_{F_i}\varphi_i(x)\,dx\cdot t^{1/2} + o(t^{1/2})\,, \qquad t\searrow 0\,. \tag{2.26}$$

Remark 2.27. For $T = Id$, (2.1) and (2.25) were obtained by Bismut–Freed [4] who improved an observation of Atiyah, Patodi, and Singer [1].

3. An index theorem

Now that we have established the basic properties of equivariant η functions, an equivariant version of the index theorem of Cheeger and Chou (cf. [3]), can be obtained immediately by a slight modification of what was done in §1 of [3], where a new proof of the Cheeger–Chou theorem was given.

Here we just state the theorem and indicate briefly what should be modified. For notation and other details, we just refer to [3, §1].

Let Z denote a smooth connected compact manifold with smooth compact boundary ∂Z. Assume Z has even dimension, is oriented and spin. Define

$$C(\partial Z) = ((0\,,1]\times \partial Z)\cup\{\delta\}$$

and

$$Z' = Z\bigcup_{\partial Z} C(\partial Z)\,.$$

Introduce a metric $g^{Z',\varepsilon}$ on Z' as in [3].

Let $T\colon Z' \to Z'$ be an isometry, which is a product near ∂Z, i.e. it is a trivial extension of the isometry $T|_{\partial Z}\colon \partial Z \to \partial Z$ to $(0\,,1+\varepsilon]\times\partial Z$ in a tubular neighborhood of ∂Z. Assume further more that T has no fixed points on ∂Z. Let $D^{\varepsilon}_{\pm}$ be the Dirac operator on Z' associated to $g^{Z',\varepsilon}$. Assume $\widetilde{dT}D^{\varepsilon}_{\pm} = D^{\varepsilon}_{\pm}\widetilde{dT}$. Then we can define the Lefschetz number

$$L(T) = \operatorname{tr}\widetilde{dT}|_{\ker D^{\varepsilon}_{+}} - \operatorname{tr}\widetilde{dT}|_{\ker D^{\varepsilon}_{-}}\,. \tag{3.1}$$

Suppose as in [3] that

$$\ker D^{\partial Z} = 0\,, \tag{3.2}$$

then we can state the theorem as follows.

Theorem 3.3. *For ε sufficiently small,*

$$L(T) = \sum_i \int_{F_i} \widehat{A}(TF_i)(Pf(2\sin(\Omega/4\pi + \sqrt{-1}\Theta/2))(N(F_i)))^{-1}$$

$$- \tfrac{1}{2}\eta_T(0, D^{\partial Z}), \tag{3.3}$$

where the F_i are the components of the fixed point set of T in Z, and the integrand is the standard density in Lefschetz fixed point formulas (cf., e.g. [8] *or* [2]).

The proof is almost the same as what was done in [3, §1]. The first point is to note that here we just use

$$L(T) = \mathrm{tr}_s\, \widetilde{dT} \exp(-t(D^{\varepsilon})^2) \tag{3.5}$$

to replace the corresponding

$$\mathrm{ind}\, D^{\varepsilon}_{+} = \mathrm{tr}_s(\exp(-t(D^{\varepsilon})^2)) \tag{3.6}$$

in [3]. Then the proof is almost line by line the same as there. We have only to insert $\widetilde{dT}$ in all kernels, e.g. we write

$$\widetilde{dT} P^{\varepsilon}_s((r, x), (r, Tx)) \tag{3.7}$$

to replace

$$P^{\varepsilon}_s((r, x), (r, x)) \tag{3.8}$$

in [3]. The only difference worth mentioning is that the equivariant replacement of (1.44) in [3] can be proved easily by using the fact that T has no fixed points on ∂Z and the proof of Corollary 1.4 in this note.

Details are omitted.

Also note that the index theorem of Atiyah–Patodi–Singer and Donnelly [5] for G-manifolds with boundary can also be deduced from (3.3) by a simple argument, similar to what was done in §1, d) of [3].

Remark 3.9. The condition (3.2) and the assumption that T has no fixed points on the boundary ∂Z are not essential. Indeed, this has been treated by J.-M. Bismut and J. Cheeger (unpublished). Here we will not go into the analytical details.

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