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A Note on exact spherically symmetric interior solutions in higher dimensions†

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Abstract

A method of constructing a new family of solutions for higher dimensional perfect-fluid spheres from a known perfect fluid solution is presented. We consider perfect-fluid solutions satisfying the equation of state $p = k\rho$, $0 < k \leq 1$. A new family of higher dimensional interior solutions is obtained for each value of k from a Tolman-type higher dimensional perfect fluid solution. Also, we present a solution, in higher dimensions, that has spheroidal geometry in the hypersurfaces $t =$ constant.

1 Introduction

Many exact static, spherically symmetric solutions to the equations of general relativity, corresponding to perfect–fluid distributions, are available in the literature. A review of such solutions is given by Kramer *et al* [1]. Such solutions may be used to describe interior gravitational fields of relativistic stars in equilibrium in 4–dimensional spacetime.

In view of recent developments in superstring theory [2, 3] and supergravity theory, higher–dimensional physics has assumed a high measure of importance. This has motivated a number of authors to investigate the properties of the field equations in higher dimensions. A number of interior and exterior solutions of Einstein's equations in higher dimensions has already been derived by Yoshimura [4], Koikawa [5], Koikawa and Yoshimura [6], Myers and Perry [7] and Krori *et al* [8, 9]. Tikekar [10] has obtained a new class of higher–dimensional interior solutions of Einstein's equations which may be considered as the generalisation of the spheroidal Vaidya–Tikekar [11] solution for a superdense star. Recently Patel *et al* [12] have found a number of exact solutions in higher dimensions which include many previously known 4–dimensional interior solutions as particular cases.

The purpose of the present note is to report a method for obtaining new interior solutions in higher dimensions from a known perfect–fluid solution. We also present a new solution describing the interior gravitational field of a higher dimensional relativistic fluid sphere. This reduces to a particular case of of the Tolman VI solution [13]. In addition the solution, with a stiff equation of state, due to Buchdahl and Land [14] is regained. Finally we present a new exact solution of the field equations in higher dimensions that reduces in the 4–dimensional limit to the spheroidal solutions of Maharaj and Leach [15] and Tikekar [16]. These spheroidal solutions are important in the description of the gravitational interactions in the central core regions of neutron stars.

2 Basic equations and the Method

We consider the $(n+3)$-dimensional spherically symmetric static metric in the form

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 d\Omega^2 \tag{1}$$

where $d\Omega^2$ is the line element on a unit $(n+1)$ sphere and ν and λ are functions only of the radial co–ordinate r.

As we are interested in perfect fluid distributions, the energy–momentum tensor is taken to be

$$T_{ik} = (p+\rho) v_i v_k - p g_{ik}, \qquad v^i v_i = 1 \tag{2}$$

where ρ is the density, p is the pressure and v^i is the flow vector. It can be easily checked that the Einstein field equations

$$R_{ik} - \frac{1}{2} g_{ik} = -8\pi T_{ik} \tag{3}$$

with T_{ik} given by (2)) are equivalent to the following set

$$v^i = e^{-\nu/2} \delta^i_t \tag{4}$$

$$\nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}\nu'\lambda' - \frac{\nu'}{r} - \frac{n\lambda'}{r} + \frac{2n}{r^2}\left(e^{\lambda} - 1\right) = 0 \tag{5}$$

$$\frac{(n+1)e^{-\lambda}\lambda'}{2r} + \frac{n(n+1)}{2r^2}\left(1 - e^{-\lambda}\right) = 8\pi\rho \tag{6}$$

$$\frac{(n+1)e^{-\lambda}\nu'}{2r} + \frac{n(n+1)}{2r^2}\left(e^{-\lambda} - 1\right) = 8\pi p \tag{7}$$

Here and in what follows an overhead dash indicates differentiation with respect to the coordinate r. When $n=1$, the spacetime becomes 4–dimensional and the equations (5)–(7) reduce to the usual governing equations for the hydrostatic equilibrium of a static spherically symmetric relativistic star in four dimensions.

Now one can easily establish the following theorem:

Theorem: If λ_0 and ν_0 constitute a particular solution of the differential equation (5), then $\lambda_0 + \log f(r)$ and ν_0 is also a solution of the equation (5) where f is given by

$$\log\left(1 - \frac{1}{f}\right) = 4n \int \frac{e^{\lambda_0} dr}{r\,(r\nu_0' + 2n)} + \log C \tag{8}$$

and C is an arbitrary constant of integration.

Thus the above theorem provides us with a method for obtaining a new solution of (5) from a known particular solution of the equation (5). However one should remember that the integral occurring in (8) may not always be evaluated in terms of elementary functions from the given solutions of the equation (5). We shall now consider an application of this theorem.

3 The seed solution

We consider a linear density–pressure relationship in the form

$$p = k\rho, \qquad 0 < k \leq 1 \tag{9}$$

where k is a constant. If $k = 0$ then one can verify from the equations (5)–(7) and (9) that $\rho = 0$. This shows that there exists no higher dimensional solution describing the interior field of a dust sphere.

Substituting ρ and p from (6) and (7) in the relation (9) we have

$$\nu' = k\lambda' + \frac{n}{2}(1+k)\left(e^{\lambda} - 1\right) \tag{10}$$

Thus the functions ν and λ satisfy the two differential equations (5) and (10). The general solution of these two equations seems to be difficult to obtain. A simple and particular solution to the equations (5) and (10), for any value of k, is given by

$$e^{\lambda} = 1 + \frac{4k}{n(1+k)^2}, \qquad e^{\nu} = Ar^{4k/(1+k)} \tag{11}$$

where A is an arbitrary constant. Here the density is given by

$$8\pi\rho = \frac{2n(n+1)k}{r^2\left[4k + n(1+k)^2\right]} \tag{12}$$

The geometry of this particular solution is described by the line element

$$ds^2 = Ar^{4k/(1+k)}dt^2 - \left[1 + \frac{4k}{n(1+k)^2}\right] dr^2 - r^2 d\Omega^2 \tag{13}$$

The above family of solutions is not regular at the centre. Therefore these solutions can be considered unphysical with regard to the possibility of representing stellar interiors globally. However, they can be used locally, then representing only certain regions of the star. When $n = 1$, the solution (13) reduces to a particular case of the Tolman VI solution [13].

4 A new class of solutions

By applying the theorem of Sect. 2 to our exact analytical solution given by the equation (11), we have obtained

$$ds^2 = Ar^{4k/(1+k)}dt^2 - a\left[1 - cr^b\right]^{-1} dr^2 - r^2 d\Omega^2 \tag{14}$$

$$8\pi\rho = \frac{(n+1)}{2r^2}\left[\frac{n(a-1)}{a} + \left(\frac{b}{a} + \frac{n}{a}\right) cr^b\right] \tag{15}$$

$$8\pi p = \frac{(n+1)}{2r^2 a}\left[\frac{4k}{(1+k)}\left(1 - cr^b\right) - n\left(cr^b + a - 1\right)\right] \tag{16}$$

where c is an arbitrary constant and a and b are given by

$$a = 1 + \frac{4k}{n(1+k)^2}, \qquad b = \frac{2\left[n(1+k)^2 + 4k\right]}{(1+k)\left[n(1+k) + 2k\right]} \tag{17}$$

If c is positive, then equation (15) implies that ρ remains positive always for all $r > 0$. Thus the physical requirement $\rho > 0$ is satisfied.

The above new class of solutions is also singular at the centre $r = 0$ at which the pressure and the density become infinite. Clearly this is an undesirable feature of the solutions. However such solutions with this defect may be matched to a core solution which is regular at the centre. Let us assume that the sphere has the boundary $r = r_0$. At $r = r_0$, the following boundary conditions must be satisfied:

(1) The pressure p vanishes at the boundary, and

(2) the interior metric (14) should match with the higher dimensional Schwarzschild exterior metric

$$ds^2 = \left(1 - \frac{2m}{r^n}\right) dt^2 - \left(1 - \frac{2m}{r^n}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{18}$$

across the boundary.

Applying the above boundary conditions to our interior solution (14) we get

$$\frac{2m}{r_0^n} = \frac{4k(1+k)^2}{n(1+k)+4k}, \qquad c = \frac{4k^2 r_0^b}{(1+k)[n(1+k)+4k]},$$
$$A = \frac{n(1+k) r_0^{-4k/(1+k)}}{n(1+k)+4k} \tag{19}$$

Note that the constant c is always positive. From (19) it is clear that the mass parameter m is always positive. For regularity of the metric we must have the inequality

$$r_0^b < \frac{1}{c} \tag{20}$$

Also the positivity of the pressure p puts some more restrictions on the size of the higher dimensional fluid sphere.

Finally we remark that for the particular value $k = 1$ (a stiff equation of state for our old solution), we obtain

$$ds^2 = r^2 \left(dt^2 - d\Omega^2\right) - \left(1 + \frac{1}{n}\right) \left(1 - cr^2\right)^{-1} dr^2 \tag{21}$$

and

$$8\pi\rho = \frac{n}{2r^2} + \frac{n(n+2)}{2}c,$$

$$8\pi p = \frac{n}{2r^2} - \frac{n(n+2)}{2}c = \rho - n(n+2)c \quad (22)$$

As c is positive the physical requirements $\rho > 0$ and $\rho \geq p$ are satisfied. The physical requirement $p \geq 0$ implies $r^2 \leq \frac{1}{c(n+2)}$. The only undesirable feature of the solution is the irregularity at the centre. When $n = 1$, the solution (21) reduces to the irregular solution found by Buchdahl and Land [14] in their study of the most natural relativistic analogue of the classical incompressible sphere. Therefore the solution (21) can be regarded as a higher-dimensional generalisation of the Buchdahl and Land solution.

5 Conclusion

We have generated a method for finding solutions in higher dimensions from a known perfect fluid solution. This essentially involves completing the integration of (8); the integral, that defines the function f, may not always be expressible in closed form. A new class of solutions in higher dimensions, with a linear equation of state, was presented utilising this result. Our class has the advantage of containing, as special cases, the Tolman VI solution [13], and the Buchdahl and Land [14] solution which satisfies the stiff equation of state. The new solutions found were presented in Sect. 4. It is clear that other perfect fluid solutions may generate models in higher dimensions. We should choose potentials ν_0 and λ_0 such that the integration in (8) is possible. It is desirable to obtain solutions which are expressible in terms of elementary functions and special functions so that a physical analysis is made easier.

As a new example we are in a position to generate another solution utilising the central theorem of this paper. We obtain the line element

$$ds^2 = Bn^2\left[1 - \frac{n-3}{n}\frac{r^2}{R^2}\right]^3 dt^2 - \left[\frac{1 - r^2/R^2}{1-(1-3/n)r^2/R^2} - \right.$$

$$\left.\frac{Cr^2/R^2}{[1-(1-3/n)r^2/R^2][1-(1-9/n^2)r^2/R^2]^{(3-n)/(3+n)}}\right]^{-1} dr^2 - r^2 d\Omega^2 \quad (23)$$

as a new solution to the Einstein field equations (4)–(7). It is easy to show that (23) matches to the Schwarzschild exterior (18) in higher dimensions. In addition an analysis of the energy density ρ and the pressure p verifies that the model is well–behaved in the interior. If we set $n = 1$ in (23) then we obtain

$$ds^2 = B\left[1 + 2r^2/R^2\right]^3 dt^2 - \left[\frac{1 - r^2/R^2}{1 + 2r^2/R^2} - \right.$$

$$\left.\frac{Cr^2/R^2}{[1 + 2r^2/R^2]\sqrt{1 + 8r^2/R^2}}\right]^{-1} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad (24)$$

in four dimensions. The advantage of the solution (23) is its simplicity which enables us to study the physical features of the model. Also, it has a clear geometrical interpretation.

The solution (24) arises as a special case in the class of metrics found by Maharaj and Leach [15] in which the surfaces generated by t = constant are spheroidal. This class of solutions with spheroidal geometry is important in the description of gravitational processes in the central core regions of dense stars where the density is of the order of 2×10^{14} gm cm^{-3}. We have little information about the behaviour of matter or the explicit equation of state for superdense stars; the spheroidal geometry is applicable in the core regions because of highly nonlinear effects. For a detailed analysis of the physical behaviour of spheroidal stars in general relativity see Tikekar [16] and Vaidya and Tikekar [11]. Our explicit solution (23) allows us to study the gravitational behaviour of dense stars, with a spheroidal geometry, in higher dimensions.

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