

## A NOTE ON EXISTENCE OF ENVELOPES AND COVERS

JIANLONG CHEN AND NANQING DING

We prove the following results for a ring  $R$ . (a) If  $\mathcal{C}$  is a class of right  $R$ -modules closed under direct summands and isomorphisms, then every right  $R$ -module has an epic  $\mathcal{C}$ -envelope if and only if  $\mathcal{C}$  is closed under direct products and submodules. (b) If  $R$  is left  $T$ -coherent and pure injective as a right  $R$ -module, then every  $T$ -finitely presented right  $R$ -module has a  $T$ -flat envelope. (c) Let  $R$  be a left  $T$ -coherent ring and injective right  $R$ -modules be  $T$ -flat. If every finitely presented left  $R$ -module has a flat envelope, then every  $T$ -finitely presented right  $R$ -module has a projective cover.

### 1. INTRODUCTION

Throughout this paper,  $R$  will denote an associative ring with identity and all modules will be unitary.

If  $R$  is a ring and  $\mathcal{C}$  a subclass of the category  $\text{Mod}_R$  of right  $R$ -modules, by a  $\mathcal{C}$ -pre-envelope of a right  $R$ -module  $M$  we mean a homomorphism  $\phi: M \rightarrow F$  with  $F \in \mathcal{C}$  such that for any homomorphism  $f: M \rightarrow F'$  where  $F' \in \mathcal{C}$  there is a homomorphism  $g: F \rightarrow F'$  such that  $g\phi = f$ . If, furthermore, when  $F' = F$  and  $f = \phi$ , the only such  $g$  are automorphisms of  $F$ , then  $\phi$  is called a  $\mathcal{C}$ -envelope of  $M$ . If  $\mathcal{C}$  is the class of injective modules, then we get the usual injective envelopes. For  $\mathcal{C}$  some familiar class of modules, say the class of flat (respectively finitely projective, projective) modules,  $\mathcal{C}$ -envelopes will simply be called flat (respectively finitely projective, projective) envelopes. If envelopes exist they are unique up to isomorphism.  $\mathcal{C}$ -(pre)covers can be defined dually.

The question on the existence of envelopes and covers has been studied by many authors (see for example, [1, 2, 6, 7, 9, 11]). In this paper, we first show that if  $\mathcal{C}$  is a class of right  $R$ -modules closed under direct summands and isomorphisms, then every right  $R$ -module has an epic  $\mathcal{C}$ -envelope if and only if  $\mathcal{C}$  is closed under direct products and submodules (Theorem 2). It is not known over which rings every module has an  $FP$ -injective cover. But, as an immediate consequence of the dual of Theorem 2, we

---

Received 7 December 1995

This work was carried out during a visit of the first author to the Academia Sinica, China and a visit of the second author to the University of Tübingen, Germany. The authors would like to thank the Institutes of Mathematics in both institutions for their kind hospitality.

---

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9729/96 \$A2.00+0.00.

have that a ring  $R$  is right semihereditary if and only if every right  $R$ -module has a monic  $FP$ -injective cover. Then, by a result of Gruson and Jensen [12], it is shown that if  $R$  is left  $\mathcal{T}$ -coherent and pure injective as a right  $R$ -module, then every  $\mathcal{T}$ -finitely presented right  $R$ -module has a  $\mathcal{T}$ -flat envelope (Theorem 10). Finally, let  $R$  be a left  $\mathcal{T}$ -coherent ring and let injective right  $R$ -modules be  $\mathcal{T}$ -flat, we prove that if every finitely presented left  $R$ -module has a flat envelope, then every  $\mathcal{T}$ -finitely presented right  $R$ -module has a projective cover (Theorem 14). In particular, some known results are obtained as corollaries of the main results of this paper.

## 2. PRELIMINARIES

In this section we recall some known notions and facts which we need in the later section.

(1) Finite (local) projectivity. An  $R$ -module  $M$  is called finitely (respectively locally) projective [3, 14] if, for any finitely generated submodule  $M_0$  of  $M$ , there exist a finitely generated free module  $F$  and homomorphisms  $f: M_0 \rightarrow F$  (respectively  $f: M \rightarrow F$ ) and  $g: F \rightarrow M$  such that  $g(f(x)) = x$  for all  $x \in M_0$ . Finitely projective modules were called  $f$ -projective in [14]. In general, projective  $\Rightarrow$  locally projective  $\Rightarrow$  finitely projective  $\Rightarrow$  flat, but no two of these concepts are equivalent.

(2) Relative flatness. In [6], we defined the concept of flat modules with respect to an arbitrary torsion theory. Here we recall this definition in a more general setting. Let  $\mathcal{T}$  be a subclass of  $\text{Mod}_R$  with  $0 \in \mathcal{T}$ . A right  $R$ -module  $M$  is said to be  $\mathcal{T}$ -finitely generated if  $M/M' \in \mathcal{T}$  for some finitely generated submodule  $M'$  of  $M$ .  $M$  is said to be  $\mathcal{T}$ -finitely presented if there is an exact sequence  $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$  with  $F$  finitely generated free and  $K$   $\mathcal{T}$ -finitely generated.  $M$  is called  $\mathcal{T}$ -flat if every homomorphism from a  $\mathcal{T}$ -finitely presented  $R$ -module to  $M$  can be factored through a finitely generated free module, that is, for any  $\mathcal{T}$ -finitely presented  $R$ -module  $P$  and any homomorphism  $f: P \rightarrow M$ , there exist a finitely generated free module  $F$  and homomorphisms  $g: P \rightarrow F$  and  $h: F \rightarrow M$  such that  $f = hg$ . It is clear that every finitely projective module is  $\mathcal{T}$ -flat, and every  $\mathcal{T}$ -flat module is flat. If  $\mathcal{T} = \{0\}$  (respectively  $\text{Mod}_R$ ), then an  $R$ -module  $M$  is  $\mathcal{T}$ -flat if and only if  $M$  is flat (respectively finitely projective).

(3) Coherent rings. A ring  $R$  is said to be left coherent if every finitely generated left ideal of  $R$  is finitely presented, or equivalently, any direct product of copies of  $R$  is a flat right  $R$ -module.  $R$  is called left  $\Pi$ -coherent [4, 14] if every finitely generated torsionless left  $R$ -module is finitely presented, or equivalently, any direct product of copies of  $R$  is a finitely projective right  $R$ -module [14].  $R$  is called strongly left coherent [18] if any direct product of copies of  $R$  is a locally projective right  $R$ -module. Note that  $\Pi$ -coherent rings were called strongly coherent rings by Jones [14].  $R$  is called left

$\mathcal{T}$ -coherent [6] if any direct product of copies of  $R$  is a  $\mathcal{T}$ -flat right  $R$ -module. Clearly, if  $\mathcal{T} = \{0\}$  (respectively  $\text{Mod}_R$ ), then  $R$  is left coherent (respectively  $\Pi$ -coherent).

### 3. MAIN RESULTS

We start with

**LEMMA 1.** *Let  $\mathcal{C}$  be a class of right  $R$ -modules closed under direct summands. If every right  $R$ -module has a  $\mathcal{C}$ -pre-envelope, then  $\mathcal{C}$  is closed under direct products.*

**PROOF:** For any family  $\{F_i\}_{i \in I} \subseteq \mathcal{C}$ ,  $\prod F_i$  has a  $\mathcal{C}$ -pre-envelope  $\phi: \prod F_i \rightarrow F$  by hypothesis. Let  $p_i: \prod F_i \rightarrow F_i$  be the projection. Then there exists  $\psi_i: F \rightarrow F_i$  such that  $\psi_i \phi = p_i$ ,  $i \in I$ . Define  $\psi: F \rightarrow \prod F_i$  by  $\psi(x) = (\psi_i(x))$  for  $x \in F$ . For any  $(x_i) \in \prod F_i$ , let  $\phi((x_i)) = x$ , then

$$x_i = p_i((x_i)) = \psi_i \phi((x_i)) = \psi_i(x),$$

and hence

$$\psi \phi((x_i)) = \psi(x) = (\psi_i(x)) = (x_i),$$

that is,  $\psi \phi = 1$ . Thus  $\prod F_i$  is a direct summand of  $F$ , and so  $\prod F_i \in \mathcal{C}$  by assumption. □

**THEOREM 2.** *Let  $\mathcal{C}$  be a class of right  $R$ -modules closed under direct summands and isomorphisms. Then the following are equivalent.*

- (1) *Every right  $R$ -module has an epic  $\mathcal{C}$ -envelope.*
- (2)  *$\mathcal{C}$  is closed under direct products and submodules.*

**PROOF:** (1)  $\Rightarrow$  (2).  $\mathcal{C}$  is closed under direct products by Lemma 1. For any submodule  $K$  of a right  $R$ -module  $N \in \mathcal{C}$ , since  $K$  has an epic  $\mathcal{C}$ -envelope  $f: K \rightarrow F$ , there is a homomorphism  $g: F \rightarrow N$  such that  $gf = i$ , where  $i: K \rightarrow N$  is the inclusion map. Thus  $f$  is monic, and so  $K \cong F \in \mathcal{C}$ .

(2)  $\Rightarrow$  (1). Let  $X$  be any right  $R$ -module and let  $\{N_i\}_{i \in I}$  be the family of all the submodules of  $X$  such that  $X/N_i \in \mathcal{C}$ . Let  $\mathcal{C}(X) = X / \bigcap_{i \in I} N_i$  and  $\pi: X \rightarrow \mathcal{C}(X)$

be the quotient map. Define  $\lambda: \mathcal{C}(X) \rightarrow \prod_{i \in I} X/N_i$  via  $\lambda \left( x + \bigcap_{i \in I} N_i \right) = (x + N_i)$  for  $x \in X$ . Then  $\lambda$  is a monomorphism. Since  $X/N_i \in \mathcal{C}$ ,  $\prod_{i \in I} X/N_i \in \mathcal{C}$ . So  $\mathcal{C}(X) \in \mathcal{C}$ .

For any  $F \in \mathcal{C}$  and any  $\phi: X \rightarrow F$ ,  $X/\text{Ker}(\phi) \in \mathcal{C}$  since  $X/\text{Ker}(\phi) \cong \text{Im}(\phi) \subseteq F$ , and hence  $\text{Ker}(\phi) = N_\alpha$  for some  $\alpha \in I$ . Now define  $\xi: \mathcal{C}(X) \rightarrow F$  such that

$\xi \left( x + \bigcap_{i \in I} N_i \right) = \phi(x)$  for  $x \in X$ . Then  $\xi$  is well-defined (for  $\text{Ker}(\phi) = N_\alpha$ ), and

$\xi \pi = \phi$ . This shows that  $\pi$  is a  $\mathcal{C}$ -pre-envelope. Since  $\pi$  is epic,  $\pi$  is a  $\mathcal{C}$ -envelope. This completes the proof. □

As applications, we list some corollaries of the Theorem 2 above.

Let  $\mathcal{C}$  be the class of  $\mathcal{T}$ -flat right  $R$ -modules. Clearly,  $\mathcal{C}$  is closed under isomorphisms and direct summands. By [6, Proposition 3.4],  $\mathcal{C}$  is closed under direct products if and only if  $R$  is left  $\mathcal{T}$ -coherent. Thus we have

**COROLLARY 3.** ([6, Theorem 5.1].) *The following are equivalent for a ring  $R$ .*

- (1)  *$R$  is left  $\mathcal{T}$ -coherent and every submodule of a  $\mathcal{T}$ -flat right  $R$ -module is  $\mathcal{T}$ -flat.*
- (2) *Every right  $R$ -module has an epic  $\mathcal{T}$ -flat envelope.*

Recall that a ring  $R$  is strongly left coherent if and only if any direct product of locally projective right  $R$ -modules is locally projective [18]. So we obtain

**COROLLARY 4.** ([6, Proposition 5.4].) *The following are equivalent for a ring  $R$ .*

- (1)  *$R$  is strongly left coherent and submodules of locally projective right  $R$ -modules are locally projective.*
- (2) *Every right  $R$ -module has an epic locally projective envelope.*

It is well known that  $R$  is left coherent and right perfect if and only if every direct product of projective right  $R$ -modules is projective. Therefore we get

**COROLLARY 5.** ([6, Proposition 5.5].) *The following are equivalent for any ring  $R$ .*

- (1)  *$R$  is left semihereditary and right perfect.*
- (2)  *$R$  is left coherent, right perfect and right hereditary.*
- (3) *Every right  $R$ -module has an epic projective envelope.*

PROOF: (1)  $\Leftrightarrow$  (2) is easy.

(2)  $\Leftrightarrow$  (3) by Theorem 2. □

Now we state the dual of Theorem 2.

**THEOREM 6.** ([11, Proposition 4].) *Let  $\mathcal{C}$  be a class of right  $R$ -modules closed under direct summands and isomorphisms. Then the following are equivalent.*

- (1) *Every right  $R$ -module has a monic  $\mathcal{C}$ -cover.*
- (2)  *$\mathcal{C}$  is closed under direct sums and homomorphic images.*

Let  $\mathcal{C}$  be the class of injective right  $R$ -modules in Theorem 6. We obtain

**COROLLARY 7.** *The following are equivalent for any ring  $R$ .*

- (1) *Every right  $R$ -module has a monic injective cover.*
- (2)  *$R$  is right Noetherian and right hereditary.*

REMARK 1. We note that Corollary 7 above appeared in [8, Corollary 3.4].

Recall that an  $R$ -module  $M$  is called  $FP$ -injective (or absolutely pure) [15, 16] if  $\text{Ext}_R^1(N, M) = 0$  for all finitely presented  $R$ -modules  $N$ . It is well known that the class of  $FP$ -injective modules over any ring is closed under direct sums [15, Corollary], and a ring  $R$  is right semihereditary if and only if the class of  $FP$ -injective right  $R$ -modules is closed under homomorphic images [16, Theorem 2]. Thus, if  $\mathcal{C}$  is the class of  $FP$ -injective modules in Theorem 6, one gets the following corollary which characterises semihereditary rings in terms of  $FP$ -injective covers.

**COROLLARY 8.** *The following are equivalent for any ring  $R$ .*

- (1) *Every right  $R$ -module has a monic  $FP$ -injective cover.*
- (2)  *$R$  is right semihereditary.*

In order to prove the next main result, we need a result of Gruson and Jensen [12]. In the notation of [12], if  $Pf(R)$  denotes the full subcategory of the category of left  $R$ -modules whose objects are the finitely presented left  $R$ -modules and  $D(R)$  is the Grothendieck category of additive functors from  $Pf(R)$  to Abelian groups, we have the following characterisation of the injective objects in this category.

**LEMMA 9.** ([12, Proposition 1.2].) *An object  $F$  of  $D(R)$  is injective if and only if  $F$  is naturally equivalent to a functor  $E \otimes -$  with  $E$  a right pure injective module.*

**THEOREM 10.** *Let  $R$  be left  $T$ -coherent and pure injective as a right  $R$ -module. Then every  $T$ -finitely presented right  $R$ -module has a  $T$ -flat (or projective) envelope.*

PROOF: Let  $M$  be a  $T$ -finitely presented right  $R$ -module, then  $M$  has a  $T$ -flat pre-envelope  $\phi: M \rightarrow F'$  by [6, Theorem 3.10]. In view of the  $T$ -flatness of  $F'$ , there exist a finitely generated free right  $R$ -module  $F$  and homomorphisms  $f: M \rightarrow F$  and  $f': F \rightarrow F'$  such that  $\phi = f'f$ . It is easy to see that  $f: M \rightarrow F$  is a  $T$ -flat pre-envelope of  $M$ . Since  $R$  is pure injective as a right  $R$ -module,  $F$  is a right pure injective module, and hence  $F \otimes -$  is an injective object of  $D(R)$  by Lemma 9. To prove that  $M$  has a  $T$ -flat envelope, we use an argument similar to that in [2, Theorem 3.3]. If  $E \otimes -$  is the injective hull of the image functor  $G = \text{Im}(f \otimes -)$ , then  $E \cong E \otimes R$  is a direct summand of  $F \cong F \otimes R$  and so it is finitely generated projective (obviously, it is  $T$ -flat). Moreover,  $f$  factors through  $M \rightarrow G(R) \rightarrow E$ , from which we obtain that if  $g$  is the above composition, it is a  $T$ -flat pre-envelope of  $M$ . Now, each endomorphism  $h$  of  $E$  such that  $hg = g$  induces an endomorphism  $h \otimes -$  of  $E \otimes -$  in  $D(R)$ , whose restriction to  $G$  is the canonical inclusion of  $G$  in its injective hull and so  $h \otimes -$  is an isomorphism in  $D(R)$ . In particular,  $h$  is an isomorphism, which proves that  $g: M \rightarrow E$  is a  $T$ -flat envelope of  $M$ . Since  $E$  is projective,  $g: M \rightarrow E$  is also a projective envelope.  $\square$

Since injective modules are always pure injective, we have

**COROLLARY 11.** *If  $R$  is left  $\mathcal{T}$ -coherent and right self-injective, then every  $\mathcal{T}$ -finitely presented right  $R$ -module has a  $\mathcal{T}$ -flat (or projective) envelope.*

By specialising Theorem 10 to the case  $\mathcal{T} = \{0\}$ , we obtain the following result of [9] immediately as a corollary.

**COROLLARY 12.** ([9, Corollary 2.4].) *If  $R$  is left coherent and pure injective as a right  $R$ -module, then every finitely presented right  $R$ -module has a flat (or projective) envelope.*

Let  $\mathcal{T} = \text{Mod}_R$  in Theorem 10. One gets

**COROLLARY 13.** *If  $R$  is left  $\Pi$ -coherent and pure injective as a right  $R$ -module, then every finitely generated right  $R$ -module has a (finitely) projective envelope.*

Next we consider when every  $\mathcal{T}$ -finitely presented right  $R$ -module has a projective cover.

**THEOREM 14.** *Let  $R$  be a left  $\mathcal{T}$ -coherent ring and let injective right  $R$ -modules be  $\mathcal{T}$ -flat. If every finitely presented left  $R$ -module has a flat envelope, then every  $\mathcal{T}$ -finitely presented right  $R$ -module has a projective cover.*

**PROOF:** Let  $M$  be a  $\mathcal{T}$ -finitely presented right  $R$ -module, then  $M^*$  is finitely presented by [6, Proposition 3.4] since  $R$  is left  $\mathcal{T}$ -coherent. From the hypothesis that every finitely presented left  $R$ -module has a flat envelope we know that  $R$  is right coherent by [1, Proposition 2] and  $M^*$  has a flat envelope, and so  $M^{**}$  has a projective cover by [1, Proposition 1]. Next we shall show that  $M$  is reflexive, and hence  $M$  has a projective cover. In fact, let  $F \rightarrow M \rightarrow 0$  be exact with  $F$  finitely generated free. Then we get an exact sequence  $0 \rightarrow M^* \rightarrow F^* \rightarrow N \rightarrow 0$ , where  $N = F^*/M^*$ . Since  $M^*$  is finitely generated,  $N$  is a finitely presented left  $R$ -module. By assumption, every injective right  $R$ -module is flat, and so  $R$  is left  $FP$ -injective by [13, Theorem 3.3]. Therefore we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccc} F & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \\ & & F^{**} & \longrightarrow & M^{**} \longrightarrow 0, \end{array}$$

and so  $M \rightarrow M^{**}$  is an epimorphism. On the other hand, since the injective envelope  $E(M)$  of  $M$  is  $\mathcal{T}$ -flat, the inclusion map  $i: M \rightarrow E(M)$  can be factored through a finitely generated free module. Thus  $M$  can be embedded in a free module, and so  $M$  is torsionless, that is,  $M \rightarrow M^{**}$  is a monomorphism. Consequently  $M$  is reflexive. The proof is complete. □

We recall that  $R$  is said to be semiregular [17] if each finitely presented right (or left)  $R$ -module has a projective cover.  $R$  is called right  $IF$  [13] if every injective right  $R$ -module is flat. Let  $\mathcal{T} = \{0\}$  in Theorem 14. We have

**COROLLARY 15.** *Let  $R$  be left coherent and right  $IF$ . If every finitely presented left  $R$ -module has a flat envelope, then  $R$  is semiregular.*

**COROLLARY 16.** *If  $R$  is a commutative  $IF$  ring, then  $R$  is semiregular if and only if every finitely presented  $R$ -module has a flat envelope.*

**PROOF:** Since  $R$  is commutative  $IF$ ,  $R$  is coherent by [5, Corollary 3.14]. Thus the necessity is clear by [1, Corollary 3], and the sufficiency follows from Corollary 15.  $\square$

**REMARK 2.** We note that the Corollary 16 above was obtained in [1] where the authors gave an  $IF$  ring without sufficient flat envelopes, even for finitely presented modules (see [1, p.125–126]).

Recall that a ring  $R$  is semiperfect if every finitely generated right (or left)  $R$ -module has a projective cover.  $R$  is called right  $FGF$  [10] if every finitely generated right  $R$ -module embeds in a free right  $R$ -module.  $R$  is right  $FGF$  if and only if every injective right  $R$ -module is finitely projective [14, Theorem 2.10]. If  $\mathcal{T} = \text{Mod}_R$  in Theorem 14, we obtain

**COROLLARY 17.** *Let  $R$  be left  $\Pi$ -coherent and right  $FGF$ . If every finitely presented left  $R$ -module has a flat envelope, then  $R$  is semiperfect.*

#### REFERENCES

- [1] J. Asensio Mayor and J. Martinez Hernandez, 'Flat envelopes in commutative rings', *Israel J. Math.* **62** (1988), 123–128.
- [2] J. Asensio Mayor and J. Martinez Hernandez, 'On flat and projective envelopes', *J. Algebra* **160** (1993), 434–440.
- [3] G. Azumaya, 'Finite splitness and finite projectivity', *J. Algebra* **106** (1987), 114–134.
- [4] V. Camillo, 'Coherence for polynomial rings', *J. Algebra* **132** (1990), 72–76.
- [5] N.Q. Ding and J.L. Chen, 'The flat dimensions of injective modules', *Manuscripta Math.* **78** (1993), 165–177.
- [6] N.Q. Ding and J.L. Chen, 'Relative coherence and preenvelopes', *Manuscripta Math.* **81** (1993), 243–262.
- [7] E.E. Enochs, 'Injective and flat covers, envelopes and resolvents', *Israel J. Math.* **39** (1981), 189–209.
- [8] E.E. Enochs and O. Jenda, 'Resolvents and dimensions of modules and rings', *Arch. Math.* **56** (1991), 528–532.
- [9] E.E. Enochs, O. Jenda and J. Xu, 'The existence of envelopes', *Rend. Sem. Mat. Univ. Padova* **90** (1993), 45–51.
- [10] C. Faith, 'Embedding torsionless modules in projectives', *Publ. Mat.* **34** (1990), 379–387.
- [11] J.R. García Rozas and B. Torrecillas, 'Relative injective covers', *Comm. Algebra* **22** (1994), 2925–2940.

- [12] L. Gruson and C.U. Jensen, 'Dimensions cohomologiques reliés aux foncteurs  $\varprojlim^{(i)}$ ', in *Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin*, Lecture Notes in Math. **867** (Springer-Verlag, Berlin, Heidelberg, New York, 1981), pp. 234–294.
- [13] S. Jain, 'Flat and  $FP$ -injectivity', *Proc. Amer. Math. Soc.* **41** (1973), 437–442.
- [14] M.F. Jones, 'Flatness and  $f$ -projectivity of torsion free modules and injective modules', in *Advances in Non-Commutative Ring Theory*, Lecture Notes in Math. **951** (Springer-Verlag, Berlin, Heidelberg, New York, 1982), pp. 94–116.
- [15] B. Maddox, 'Absolutely pure modules', *Proc. Amer. Math. Soc.* **18** (1967), 155–158.
- [16] C. Megibben, 'Absolutely pure modules', *Proc. Amer. Math. Soc.* **26** (1970), 561–566.
- [17] W.K. Nicholson, 'Semiregular modules and rings', *Canad. J. Math.* **28** (1976), 1105–1120.
- [18] B. Zimmermann-Huisgen, 'Pure submodules of direct products of free modules', *Math. Ann.* **224** (1976), 233–245.

Department of Mathematics and Mechanics  
Southeast University  
Nanjing 210096  
People's Republic of China

Department of Mathematics  
Nanjing University  
Nanjing 210093  
People's Republic of China