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A Note on f-biharmonic Legendre Curves in $\mathcal{S}$-space Forms

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ABSTRACT

In this paper, we study f-biharmonic Legendre curves in $\mathcal{S}$-space forms. Our aim is to find curvature conditions for these curves and determine their types, i.e., a geodesic, a circle, a helix or a Frenet curve of osculating order r with specific curvature equations. We also give a proper example of f-biharmonic Legendre curves in the $\mathcal{S}$-space form $\mathbb{R}^{2m+s}(-3s)$, with $m = 2$ and $s = 2$.

Keywords: *$\mathcal{S}-$space form; Legendre curve; f-biharmonic curve; Frenet curve.*
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1. Introduction

Let us consider a smooth map $\phi : (M, g) \to (N, h)$, where (M, g) and (N, h) are Riemannian manifolds. If ϕ is a critical point of the f-bienergy functional

$$E_{2,f}(\phi) = \frac{1}{2}\int_M f\,|\tau(\phi)|^2\, \upsilon_g,$$

then it is called an *f-biharmonic map*. Here, $f \in C(M, \mathbb{R})$, υ_g is the volume element and $\tau(\phi)$ is *the first tension field of* ϕ defined as $\tau(\phi) = trace\nabla d\phi$, (for further details, please refer to [15]). Using this definition, Y. L. Ou calculated f-biharmonic equation given by (3.2) in Section 3, which gives opportunity to study f-biharmonic curves in a variety of manifolds. The present author and Cihan Özgür studied f-biharmonic Legendre curves in Sasakian space forms in [11]. This paper generalizes these results to $\mathcal{S}$-space forms.

The paper is organised as follows. In Section 2, we give fundamentals of $\mathcal{S}$-manifolds. We give main results in Section 3, considering four different cases. At the end of this last section, we give a non-trivial example in $\mathbb{R}^6(-6)$, which satisfies our results.

2. $\mathcal{S}$-space forms

Let (M, g) be a $(2m + s)$-dimensional *framed metric manifold* [21] with a *framed metric structure* $(\varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, ..., s\}$, that is, φ is a $(1, 1)$ tensor field defining a *φ-structure of rank* $2m$; $\xi_1, ..., \xi_s$ are vector fields; $\eta^1, ..., \eta^s$ are 1-forms and g is a Riemannian metric on M such that for all $X, Y \in TM$ and $\alpha, \beta \in \{1, ..., s\}$,

$$\varphi^2 X = -X + \sum_{\alpha=1}^{s} \eta^\alpha(X)\xi_\alpha, \quad \eta^\alpha(\xi_\beta) = \delta_{\alpha\beta}, \quad \varphi(\xi_\alpha) = 0, \quad \eta^\alpha \circ \varphi = 0 \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{\alpha=1}^{s} \eta^\alpha(X)\eta^\alpha(Y), \tag{2.2}$$

$$d\eta^\alpha(X, Y) = g(X, \varphi Y) = -d\eta^\alpha(Y, X), \quad \eta^\alpha(X) = g(X, \xi). \tag{2.3}$$

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$(M^{2m+s},\varphi,\xi_\alpha,\eta^\alpha,g)$ is also called *framed φ-manifold* [16] or *almost r-contact metric manifold* [20]. If the Nijenhuis tensor of φ equals $-2d\eta^\alpha\otimes\xi_\alpha$ for all $\alpha\in\{1,...,s\}$, then $(\varphi,\xi_\alpha,\eta^\alpha,g)$ is called *$\mathcal{S}$-structure* [1].

For $s=1$, a framed metric structure becomes an almost contact metric structure and an $\mathcal{S}$-structure becomes a Sasakian structure. If a framed metric structure on M is an $\mathcal{S}$-structure, then we have [1]:

$$(\nabla_X\varphi)Y=\sum_{\alpha=1}^{s}\left\{g(\varphi X,\varphi Y)\xi_\alpha+\eta^\alpha(Y)\varphi^2X\right\}, \tag{2.4}$$

$$\nabla\xi_\alpha=-\varphi,\ \alpha\in\{1,...,s\}. \tag{2.5}$$

In Sasakian case ($s=1$), (2.5) can directly be calculated from (2.4) .

A *plane section in T_pM* is a φ-section if there exist a vector $X\in T_pM$ orthogonal to $\xi_1,...,\xi_s$ such that $\{X,\varphi X\}$ span the section. The sectional curvature of a *φ-section* is called *φ-sectional curvature*. In an $\mathcal{S}$-manifold of constant φ-sectional curvature, the *curvature tensor R of M* is calculated as

$$\begin{aligned}R(X,Y)Z=\sum_{\alpha,\beta}&\{\eta^\alpha(X)\eta^\beta(Z)\varphi^2Y-\eta^\alpha(Y)\eta^\beta(Z)\varphi^2X\\ &-g(\varphi X,\varphi Z)\eta^\alpha(Y)\xi_\beta+g(\varphi Y,\varphi Z)\eta^\alpha(X)\xi_\beta\}\\ &+\tfrac{c+3s}{4}\left\{-g(\varphi Y,\varphi Z)\varphi^2X+g(\varphi X,\varphi Z)\varphi^2Y\right\}\\ &\tfrac{c-s}{4}\left\{g(X,\varphi Z)\varphi Y-g(Y,\varphi Z)\varphi X+2g(X,\varphi Y)\varphi Z\right\},\end{aligned} \tag{2.6}$$

for all $X,Y,Z\in TM$ [3]. An $\mathcal{S}$-manifold of constant φ-sectional curvature c is called an *$\mathcal{S}$-space form* and it is denoted by $M(c)$. For $s=1$, an $\mathcal{S}$-space form transforms into a Sasakian space form [2].

A submanifold of an $\mathcal{S}$-manifold is called an *integral submanifold* if $\eta^\alpha(X)=0$, $\alpha=1,...,s$, for every tangent vector X [14]. A 1-dimensional integral submanifold of an $\mathcal{S}$-space form $(M^{2m+s},\varphi,\xi_\alpha,\eta^\alpha,g)$ is called a *Legendre curve of M*. Equally, a curve $\gamma:I\to M=(M^{2m+s},\varphi,\xi_\alpha,\eta^\alpha,g)$ is called a Legendre curve if $\eta^\alpha(T)=0$, for every $\alpha=1,...s$, where T denotes the tangent vector field of γ.

3. f-biharmonic Legendre curves in $\mathcal{S}$-space forms

Let us consider an arc-length curve $\gamma:I\to M$ in an n-dimensional Riemannian manifold (M,g). If there exists orthonormal vector fields $E_1,E_2,...,E_r$ along γ satisfying

$$\begin{aligned}E_1&=\gamma'=T,\\ \nabla_TE_1&=\kappa_1E_2,\\ \nabla_TE_2&=-\kappa_1E_1+\kappa_2E_3,\\ &\ \ \cdots\\ \nabla_TE_r&=-\kappa_{r-1}E_{r-1},\end{aligned} \tag{3.1}$$

then γ is called a *Frenet curve of osculating order r*, where $\kappa_1,...,\kappa_{r-1}$ are positive functions on I and $1\le r\le n$.

A Frenet curve of osculating order 1 is a called *geodesic*. A Frenet curve of osculating order 2 is a *circle* if κ_1 is a non-zero positive constant. A Frenet curve of osculating order $r\geq 3$ is called a *helix of order r*, when $\kappa_1,...,\kappa_{r-1}$ are non-zero positive constants; a helix of order 3 is simply called a *helix*.

An arclength parametrized curve $\gamma:(a,b)\to(M,g)$ is called an f-biharmonic curve with a function $f:(a,b)\to(0,\infty)$ if the following equation is satisfied [17]:

$$f(\nabla_T\nabla_T\nabla_TT-R(T,\nabla_TT)T)+2f'\nabla_T\nabla_TT+f''\nabla_TT=0. \tag{3.2}$$

Now let $(M^{2m+s},\varphi,\xi_\alpha,\eta^\alpha,g)$ be an $\mathcal{S}$-space form and $\gamma:I\to M$ a Legendre Frenet curve of osculating order r. If we differentiate

$$\eta^\alpha(T)=0 \tag{3.3}$$

and use (3.1), we find

$$\eta^\alpha(E_2)=0,\ \alpha\in\{1,...,s\}. \tag{3.4}$$

Using equations (2.1), (2.2), (2.3), (2.6), (3.1) and (3.4), we calculate

$$\nabla_T\nabla_TT=-\kappa_1^2E_1+\kappa_1'E_2+\kappa_1\kappa_2E_3,$$

$$\begin{aligned}\nabla_T\nabla_T\nabla_T T &= -3\kappa_1\kappa_1' E_1 + \left(\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2\right) E_2 \\ &\quad + \left(2\kappa_1'\kappa_2 + \kappa_1\kappa_2'\right) E_3 + \kappa_1\kappa_2\kappa_3 E_4,\end{aligned}$$

$$R(T,\nabla_T T)T = -\kappa_1\frac{(c+3s)}{4}E_2 - 3\kappa_1\frac{(c-s)}{4}g(\varphi T, E_2)\varphi T,$$

(see [19]). If the left-hand side of (3.2) is denoted by $f.\tau_3$, we find that

$$\begin{aligned}\tau_3 &= \nabla_T\nabla_T\nabla_T T - R(T,\nabla_T T)T + 2\frac{f'}{f}\nabla_T\nabla_T T + \frac{f''}{f}\nabla_T T \\ &= \left(-3\kappa_1\kappa_1' - 2\kappa_1^2\frac{f'}{f}\right)E_1 \\ &\quad + \left(\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2 + \kappa_1\frac{(c+3s)}{4} + 2\kappa_1'\frac{f'}{f} + \kappa_1\frac{f''}{f}\right)E_2 \\ &\quad + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2' + 2\kappa_1\kappa_2\frac{f'}{f})E_3 + \kappa_1\kappa_2\kappa_3 E_4 \\ &\quad + 3\kappa_1\frac{(c-s)}{4}g(\varphi T, E_2)\varphi T.\end{aligned} \tag{3.5}$$

Let $k = \min\{r,4\}$. From (3.5), the curve γ is f-biharmonic if and only if $\tau_3 = 0$, i.e.,

(1) $c = s$ or $\varphi T \perp E_2$ or $\varphi T \in span\{E_2, ..., E_k\}$; and

(2) $g(\tau_3, E_i) = 0$, for all $i = \overline{1,k}$.

Thus, we can state the following main theorem:

Theorem 3.1. *Let γ be a non-geodesic Legendre Frenet curve of osculating order r in an $\mathcal{S}$-space form $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, ..., s\}$ and $k = \min\{r,4\}$. Then γ is f- biharmonic if and only if*

(1) $c = s$ or $\varphi T \perp E_2$ or $\varphi T \in span\{E_2, ..., E_k\}$; and

(2) the first k of the following equations are satisfied (replacing $\kappa_k = 0$):

$$\begin{gathered}3\kappa_1' + 2\kappa_1\tfrac{f'}{f} = 0, \\ \kappa_1^2 + \kappa_2^2 = \tfrac{c+3s}{4} + \tfrac{3(c-s)}{4}\left[g(\varphi T, E_2)\right]^2 + \tfrac{\kappa_1''}{\kappa_1} + \tfrac{f''}{f} + 2\tfrac{\kappa_1'}{\kappa_1}\tfrac{f'}{f}, \\ \kappa_2' + \tfrac{3(c-s)}{4}g(\varphi T, E_2)g(\varphi T, E_3) + 2\kappa_2\tfrac{f'}{f} + 2\kappa_2\tfrac{\kappa_1'}{\kappa_1} = 0, \\ \kappa_2\kappa_3 + \tfrac{3(c-s)}{4}g(\varphi T, E_2)g(\varphi T, E_4) = 0.\end{gathered}$$

From Theorem 3.1, one can easily see that a curve γ with constant geodesic curvature κ_1 is f-biharmonic if and only if it is biharmonic. Since we studied biharmonic curves in $\mathcal{S}$-space forms in [19], we study curves with non-constant κ_1 in this paper. We call non-biharmonic f-biharmonic curves *proper f-biharmonic*.

Now we investigate results of Theorem 3.1 in four cases.

Case I. $c = s$.

In this case γ is proper biharmonic if and only if

$$\begin{gathered}3\kappa_1' + 2\kappa_1\tfrac{f'}{f} = 0, \\ \kappa_1^2 + \kappa_2^2 = s + \tfrac{\kappa_1''}{\kappa_1} + \tfrac{f''}{f} + 2\tfrac{\kappa_1'}{\kappa_1}\tfrac{f'}{f}, \\ \kappa_2' + 2\kappa_2\tfrac{f'}{f} + 2\kappa_2\tfrac{\kappa_1'}{\kappa_1} = 0, \\ \kappa_2\kappa_3 = 0.\end{gathered} \tag{3.6}$$

Theorem 3.2. *Let γ be a Legendre Frenet curve in an $\mathcal{S}$-space form $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, ..., s\}$, $c = s$ and $(2m+s) > 3$. Then γ is proper f-biharmonic if and only if either*

(i) γ is of osculating order $r = 2$ with $f = c_1\kappa_1^{-3/2}$ and κ_1 satisfies

$$t \pm \frac{1}{2\sqrt{s}}\arctan\left(\frac{2s + c_3\kappa_1}{2\sqrt{s}\sqrt{-\kappa_1^2 - c_3\kappa_1 - s}}\right) + c_4 = 0, \tag{3.7}$$

where $c_1 > 0$, $c_3 < -2\sqrt{s}$ and c_4 are arbitrary constants, t is the arc-length parameter and

$$\frac{1}{2}(-\sqrt{c_3^2 - 4s} - c_3) < \kappa_1(t) < \frac{1}{2}(\sqrt{c_3^2 - 4s} - c_3); \text{ or} \tag{3.8}$$

(ii) *γ is of osculating order $r=3$ with $f=c_1\kappa_1^{-3/2}$, $\frac{\kappa_2}{\kappa_1}=c_2$ and κ_1 satisfies*

$$t \pm \frac{1}{2\sqrt{s}} \arctan\left(\frac{2s+c_3\kappa_1}{2\sqrt{s}\sqrt{-(1+c_2^2)\kappa_1^2-c_3\kappa_1-s}}\right)+c_4=0, \quad (3.9)$$

where $c_1>0$, $c_2>0$, $c_3<-2\sqrt{s(1+c_2^2)}$ and c_4 are arbitrary constants, t is the arc-length parameter and

$$\frac{1}{2(1+c_2^2)}(-\sqrt{c_3^2-4s(1+c_2^2)}-c_3)<\kappa_1(t)<\frac{1}{2(1+c_2^2)}(\sqrt{c_3^2-4s(1+c_2^2)}-c_3). \quad (3.10)$$

Proof. From the first equation of (3.6), it is easy to see that $f=c_1\kappa_1^{-3/2}$ for an arbitrary constant $c_1>0$. So, we find

$$\frac{f'}{f}=\frac{-3}{2}\frac{\kappa_1'}{\kappa_1},\ \frac{f''}{f}=\frac{15}{4}\left(\frac{\kappa_1'}{\kappa_1}\right)^2-\frac{3}{2}\frac{\kappa_1''}{\kappa_1}. \quad (3.11)$$

If $\kappa_2=0$, then γ is of osculating order $r=2$ and the first two of equations (3.6) must be satisfied. Hence the second equation and (3.11) give us the ODE

$$3(\kappa_1')^2-2\kappa_1\kappa_1''=4\kappa_1^2(\kappa_1^2-s). \quad (3.12)$$

Let $\kappa_1=\kappa_1(t)$, where t denotes the arc-length parameter. If we solve (3.12) considering s is a positive integer, we find (3.7). Since (3.7) must be well-defined, $-\kappa_1^2-c_3\kappa_1-s>0$. Since $\kappa_1>0$, we have $c_3<-2\sqrt{s}$ and (3.8).

If $\kappa_2=constant\neq 0$, we find f is a constant. Hence γ is not proper f-biharmonic in this case. Let $\kappa_2\neq constant$. From the fourth equation, we have $\kappa_3=0$. So, γ is of osculating order $r=3$. The third equation of (3.6) gives us $\frac{\kappa_2}{\kappa_1}=c_2$, where $c_2>0$ is a constant. If we write these equations in the second equation of (3.6), we have the ODE

$$3(\kappa_1')^2-2\kappa_1\kappa_1''=4\kappa_1^2[(1+c_2^2)\kappa_1^2-s]$$

which has the general solution (3.9) under the condition $c_3<-2\sqrt{s(1+c_2^2)}$ and (3.10) must be satisfied. □

If we take $s=1$, we obtain Theorem 3.2 in [11].

Remark 3.1. If $2m+s=3$, then $m=s=1$. So M is a 3-dimensional Sasakian space form. Since a Legendre curve in a Sasakian 3-manifold has torsion 1 (see [2]), we can write $\kappa_1>0$ and $\kappa_2=1$. The first and the third equations of (3.6) give us f is a constant. Hence γ cannot be proper f-biharmonic. Previously, in [19], we claimed that γ cannot be proper biharmonic either.

Case II. $c\neq s$, $\varphi T\perp E_2$.

In this case, $g(\varphi T,E_2)=0$. From Theorem 3.1, we obtain

$$\begin{aligned} 3\kappa_1'+2\kappa_1\tfrac{f'}{f}&=0,\\ \kappa_1^2+\kappa_2^2&=\tfrac{c+3s}{4}+\tfrac{\kappa_1''}{\kappa_1}+\tfrac{f''}{f}+2\tfrac{\kappa_1'}{\kappa_1}\tfrac{f'}{f},\\ \kappa_2'+2\kappa_2\tfrac{f'}{f}+2\kappa_2\tfrac{\kappa_1'}{\kappa_1}&=0,\\ \kappa_2\kappa_3&=0. \end{aligned} \quad (3.13)$$

Firstly, we need the following proposition:

Proposition 3.1. *[19] Let γ be a Legendre Frenet curve of osculating order 3 in an $\mathcal{S}$-space form $(M^{2m+s},\varphi,\xi_\alpha,\eta^\alpha,g)$, $\alpha\in\{1,...,s\}$ and $\varphi T\perp E_2$. Then $\{T=E_1,E_2,E_3,\varphi T,\nabla_T\varphi T,\xi_1,...,\xi_s\}$ is linearly independent at any point of γ. Therefore $m\geq 3$.*

Now we have the following Theorem:

Theorem 3.3. *Let γ be a Legendre Frenet curve in an $\mathcal{S}$-space form $(M^{2m+s},\varphi,\xi_\alpha,\eta^\alpha,g)$, $\alpha\in\{1,...,s\}$, $c\neq s$ and $\varphi T\perp E_2$. Then γ is proper biharmonic if and only if*

(1) *γ is of osculating order $r=2$ with $f=c_1\kappa_1^{-3/2}$, $m\geq 2$, $\{T=E_1,E_2,\varphi T,\nabla_T\varphi T,\xi_1,...,\xi_s\}$ is linearly independent and*

(a) *if $c>-3s$, then κ_1 satisfies*

$$t \pm \frac{1}{\sqrt{c+3s}} \arctan\left(\frac{c+3s+2c_3\kappa_1}{\sqrt{c+3s}\sqrt{-4\kappa_1^2-4c_3\kappa_1-c-3s}}\right)+c_4=0,$$

(b) *if* $c=-3s$*, then* κ_1 *satisfies*

$$t\pm\frac{\sqrt{-\kappa_1(\kappa_1+c_3)}}{c_3\kappa_1}+c_4=0,$$

(c) *if* $c<-3s$*, then* κ_1 *satisfies*

$$t\pm\frac{1}{\sqrt{-c-3s}}\ln\left(\frac{c+3s+2c_3\kappa_1-\sqrt{-c-3s}\sqrt{-4\kappa_1^2-4c_3\kappa_1-c-3s}}{(c+3s)\kappa_1}\right)+c_4=0;\ or$$

(2) γ *is of osculating order* $r=3$ *with* $f=c_1\kappa_1^{-3/2}$, $\frac{\kappa_2}{\kappa_1}=c_2=constant>0$, $m\geq 3$, $\{T=E_1,E_2,E_3,\varphi T,\nabla_T\varphi T,\xi_1,...,\xi_s\}$ *is linearly independent and*

(a) *if* $c>-3s$*, then* κ_1 *satisfies*

$$t\pm\frac{1}{\sqrt{c+3s}}\arctan\left(\frac{c+3s+2c_3\kappa_1}{\sqrt{c+3s}\sqrt{-4(1+c_2^2)\kappa_1^2-4c_3\kappa_1-c-3s}}\right)+c_4=0,$$

(b) *if* $c=-3s$*, then* κ_1 *satisfies*

$$t\pm\frac{\sqrt{-\kappa_1\left[(1+c_2^2)\kappa_1+c_3\right]}}{c_3\kappa_1}+c_4=0,$$

(c) *if* $c<-3s$*, then* κ_1 *satisfies*

$$t\pm\frac{1}{\sqrt{-c-3s}}\ln\left(\frac{c+3s+2c_3\kappa_1-\sqrt{-c-3s}\sqrt{-4(1+c_2^2)\kappa_1^2-4c_3\kappa_1-c-3s}}{(c+3s)\kappa_1}\right)+c_4=0,$$

where $c_1>0$, $c_2>0$, c_3 *and* c_4 *are convenient arbitrary constants,* t *is the arc-length parameter* $\kappa_1(t)$ *is in convenient open interval.*

Proof. The proof is similar to the proof of Theorem 3.2. □

Case III. $c\neq s$, $\varphi T\parallel E_2$.

In this case, $\varphi T=\pm E_2, g(\varphi T,E_2)=\pm 1$, $g(\varphi T,E_3)=g(\pm E_2,E_3)=0$ and $g(\varphi T,E_4)=g(\pm E_2,E_4)=0$. From Theorem 3.1, γ is biharmonic if and only if

$$\begin{aligned}3\kappa_1'+2\kappa_1\tfrac{f'}{f}&=0,\\ \kappa_1^2+\kappa_2^2&=c+\tfrac{\kappa_1''}{\kappa_1}+\tfrac{f''}{f}+2\tfrac{\kappa_1'}{\kappa_1}\tfrac{f'}{f},\\ \kappa_2'+2\kappa_2\tfrac{f'}{f}+2\kappa_2\tfrac{\kappa_1'}{\kappa_1}&=0,\\ \kappa_2\kappa_3&=0.\end{aligned}\tag{3.14}$$

In [19], we have proved that $\kappa_2=\sqrt{s}$, that is, κ_2 is a constant. Then, the first and the third equations of (3.14) give us f is a constant. Hence, we give the following result:

Theorem 3.4. *There does not exist any proper* f*-biharmonic Legendre curve in an* $\mathcal{S}$*-space form* $(M^{2m+s},\varphi,\xi_\alpha,\eta^\alpha,g)$, $\alpha\in\{1,...,s\}$ *with* $c\neq s$ *and* $\varphi T\parallel E_2$.

Case IV. $c\neq s$ and $g(\varphi T,E_2)$ is not constant 0, 1 or -1.

In this final case, let $(M^{2m+s},\varphi,\xi_\alpha,\eta^\alpha,g)$ be an $\mathcal{S}$-space form, $\alpha\in\{1,...,s\}$ and $\gamma:I\to M$ a Legendre curve of osculating order r, where $4\leq r\leq 2m+s$ and $m\geq 2$. If γ is biharmonic, then $\varphi T\in span\{E_2,E_3,E_4\}$. Let $\theta(t)$ denote the angle function between φT and E_2, that is, $g(\varphi T,E_2)=\cos\theta(t)$. If we differentiate $g(\varphi T,E_2)$ along γ and use equations (2.1), (2.3), (3.1) and (2.4), we get

$$\begin{aligned}-\theta'(t)\sin\theta(t)&=\nabla_T g(\varphi T,E_2)=g(\nabla_T\varphi T,E_2)+g(\varphi T,\nabla_T E_2)\\&=g(\sum_{\alpha=1}^{s}\xi_\alpha+\kappa_1\varphi E_2,E_2)+g(\varphi T,-\kappa_1 T+\kappa_2 E_3)\\&=\kappa_2 g(\varphi T,E_3).\end{aligned}\tag{3.15}$$

If we write $\varphi T = g(\varphi T, E_2)E_2 + g(\varphi T, E_3)E_3 + g(\varphi T, E_4)E_4$, Theorem 3.1 gives us

$$3\kappa_1' + 2\kappa_1 \frac{f'}{f} = 0, \tag{3.16}$$

$$\kappa_1^2 + \kappa_2^2 = \frac{c+3s}{4} + \frac{3(c-s)}{4}\cos^2\theta + \frac{\kappa_1''}{\kappa_1} + \frac{f''}{f} + 2\frac{\kappa_1'}{\kappa_1}\frac{f'}{f}, \tag{3.17}$$

$$\kappa_2' + \frac{3(c-s)}{4}\cos\theta g(\varphi T, E_3) + 2\kappa_2\frac{f'}{f} + 2\kappa_2\frac{\kappa_1'}{\kappa_1} = 0, \tag{3.18}$$

$$\kappa_2\kappa_3 + \frac{3(c-s)}{4}\cos\theta g(\varphi T, E_4) = 0. \tag{3.19}$$

If we put (3.11) in (3.17) and (3.18) respectively, we find

$$\kappa_1^2 + \kappa_2^2 = \frac{c+3s}{4} + \frac{3(c-s)}{4}\cos^2\theta - \frac{\kappa_1''}{2\kappa_1} + \frac{3}{4}\left(\frac{\kappa_1'}{\kappa_1}\right)^2, \tag{3.20}$$

$$\kappa_2' - \frac{\kappa_1'}{\kappa_1}\kappa_2 + \frac{3(c-s)}{4}\cos\theta g(\varphi T, E_3) = 0. \tag{3.21}$$

If we multiply (3.21) with $2\kappa_2$ and use (3.15), we obtain

$$2\kappa_2\kappa_2' - 2\frac{\kappa_1'}{\kappa_1}\kappa_2^2 + \frac{3(c-s)}{4}(-2\theta'\cos\theta\sin\theta) = 0. \tag{3.22}$$

Let us denote $\upsilon(t) = \kappa_2^2(t)$, where t is the arc-length parameter. Then (3.22) turns into

$$\upsilon' - 2\frac{\kappa_1'}{\kappa_1}\upsilon = -\frac{3(c-s)}{4}(-2\theta'\cos\theta\sin\theta), \tag{3.23}$$

which is a linear ODE. If we solve (3.23), we get the following results:

$i)$ If θ is a constant, then

$$\frac{\kappa_2}{\kappa_1} = c_2, \tag{3.24}$$

where $c_2 > 0$ is an arbitrary constant. From (3.15) and (3.25), we find $g(\varphi T, E_3) = 0$. Since $\|\varphi T\| = 1$ and $\varphi T = \cos\theta E_2 + g(\varphi T, E_4)E_4$, we obtain $g(\varphi T, E_4) = \sin\theta$. By the use of (3.17) and (3.24), we have

$$3(\kappa_1')^2 - 2\kappa_1\kappa_1'' = 4\kappa_1^2[(1+c_2^2)\kappa_1^2 - \frac{c+3s+3(c-s)\cos^2\theta}{4}].$$

$ii)$ If $\theta = \theta(t)$ is a non-constant function, then

$$\kappa_2^2 = -\frac{3(c-s)}{4}\cos^2\theta + \lambda(t).\kappa_1^2, \tag{3.25}$$

where

$$\lambda(t) = -\frac{3(c-s)}{2}\int \frac{\cos^2\theta\kappa_1'}{\kappa_1^3}dt. \tag{3.26}$$

If we write (3.25) in (3.20), we find

$$[1+\lambda(t)].\kappa_1^2 = \frac{c+3s}{4} + \frac{3(c-s)}{2}\cos^2\theta - \frac{\kappa_1''}{2\kappa_1} + \frac{3}{4}\left(\frac{\kappa_1'}{\kappa_1}\right)^2.$$

Hence, we can state the following final theorem of the paper:

Theorem 3.5. *Let $\gamma : I \to M$ be a Legendre curve of osculating order r in an $\mathcal{S}$-space form $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, ..., s\}$, where $r \geq 4, m \geq 2, c \neq s$, $g(\varphi T, E_2) = \cos\theta(t)$ is not constant $0, 1$ or -1. Then γ is proper f-biharmonic if and only if $f = c_1\kappa_1^{-3/2}$ and*

(i) if θ is a constant,

$$\frac{\kappa_2}{\kappa_1} = c_2,$$

$$3(\kappa_1')^2 - 2\kappa_1\kappa_1'' = 4\kappa_1^2[(1+c_2^2)\kappa_1^2 - \frac{c+3s+3(c-s)\cos^2\theta}{4}],$$

$$\kappa_2\kappa_3 = \pm\frac{3(c-s)\sin 2\theta}{8},$$

(ii) *if* θ *is a non-constant function,*

$$\kappa_2^2 = -\frac{3(c-s)}{4}\cos^2\theta + \lambda(t).\kappa_1^2,$$

$$3(\kappa_1')^2 - 2\kappa_1\kappa_1'' = 4\kappa_1^2[(1+\lambda(t))\kappa_1^2 - \frac{c+3s+3(c-s)\cos^2\theta}{4}],$$

$$\kappa_2\kappa_3 = \pm\frac{3(c-s)\sin 2\theta\sin w}{8},$$

where c_1 *and* c_2 *are positive constants,* $\varphi T = \cos\theta E_2 \pm \sin\theta\cos w E_3 \pm \sin\theta\sin w E_4$, w *is the angle function between* E_3 *and the orthogonal projection of* φT *onto span* $\{E_3, E_4\}$ *.* w *is related to* θ *by* $\cos w = \frac{-\theta'}{\kappa_2}$ *and* $\lambda(t)$ *is given by*

$$\lambda(t) = -\frac{3(c-s)}{2}\int\frac{\cos^2\theta\kappa_1'}{\kappa_1^3}dt.$$

In case θ is a constant, we can give the following direct corollary of Theorem 3.5:

Corollary 3.1. *Let* $\gamma : I \to M$ *be a Legendre curve of osculating order* r *in an* $\mathcal{S}$*-space form* $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$*,* $\alpha \in \{1, ..., s\}$ *, where* $r \geq 4$*,* $m \geq 2$*,* $c \neq s$ *,* $g(\varphi T, E_2) = \cos\theta$ *is a constant and* $\theta \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$*. Then* γ *is proper* f*-biharmonic if and only if* $f = c_1\kappa_1^{-3/2}$ *,* $\frac{\kappa_2}{\kappa_1} = c_2 = constant > 0$ *and*
(i) *if* $a > 0$*, then* κ_1 *satisfies*

$$t \pm \frac{1}{2\sqrt{a}}\arctan\left(\frac{1}{2\sqrt{a}}\frac{2a + c_3\kappa_1}{\sqrt{c+3s}\sqrt{-(1+c_2^2)\kappa_1^2 - c_3\kappa_1 - a}}\right) + c_4 = 0,$$

(ii) *if* $a = 0$*, then* κ_1 *satisfies*

$$t \pm \frac{\sqrt{-\kappa_1\left[(1+c_2^2)\kappa_1 + c_3\right]}}{c_3\kappa_1} + c_4 = 0,$$

(iii) *if* $a < 0$*, then* κ_1 *satisfies*

$$t \pm \frac{1}{2\sqrt{-a}}\ln\left(\frac{2a + c_3\kappa_1 - 2\sqrt{-a}\sqrt{-(1+c_2^2)\kappa_1^2 - c_3\kappa_1 - a}}{2a\kappa_1}\right) + c_4 = 0,$$

where $a = c + 3s + 3(c-s)\cos^2\theta$*,* $\varphi T = \cos\theta E_2 \pm \sin\theta E_4$*,* $c_1 > 0$*,* $c_2 > 0$*,* c_3 *and* c_4 *are convenient arbitrary constants,* t *is the arc-length parameter and* $\kappa_1(t)$ *is in convenient open interval.*

At the end of this section, let us give an example of an f-biharmonic Legendre curve in the very well known $\mathcal{S}$-space form $\mathbb{R}^{2m+s}(-3s)$ (see [12]), where we take $m = 2$ and $s = 2$.

Example 3.1. Let us consider the curve $\gamma : I \to \mathbb{R}^6(-6)$,

$$\gamma(t) = (a_1, a_2, 2arcsinh(t), 2\sqrt{1+t^2}, a_3, a_4),$$

where a_i $(i = \overline{1,4})$ are real constants. After calculations, we find that γ is a Legendre curve of osculating order 2, t is the arc-length parameter,

$$\kappa_1 = \frac{1}{1+t^2},\ \kappa_2 = 0,\ \varphi T \perp E_2$$

and γ is f-biharmonic with $f = c_1(1+t^2)^{3/2}$, where $c_1 > 0$ is a constant. It is easy to show that γ satisfies Theorem 3.3 $(1)(b)$.

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