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A Note on Finite Groups Having Perfect Order Subsets[1]

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Abstract

A group G is said to be a POS-group if for each x in G the cardinality of the set $\{y \in G | o(y) = o(x)\}$ is a divisor of the order of G. In this paper we study the structure of POS-groups of order $2m$ with $(2, m) = 1$, and confirm a conjecture of Das's.

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Throughout this paper G denotes a finite group, $o(x)$ the order of a group element x, and $|X|$ the cardinality of a set X. Denote by $\pi(G) = \{p | p \text{ is a prime divisor of } |G|\}$. As in [2], the order subset (or, order class) of G determined by an element $x \in G$ is defined to be the set $OS(x) = \{y \in G | o(y) = o(x)\}$. Clearly, for every $x \in G$, $OS(x)$ is a disjoint union of some of the conjugacy classes in G. The group G is said to have perfect order subsets (in short, G is called a POS-group) if $|OS(x)|$ is a divisor of $|G|$ for all $x \in G$. In the paper [2], Finch and Jones first classified the Abelian POS-groups. Afterwards they continued the study of nonabelian POS-groups (see [3],[4]). Recently, Das gave some properties of POS-groups and listed some examples of nonablian POS-groups in his paper [1]. In this note we study the structure of POS-groups of order $2m$ with $(2, m) = 1$. Firstly we cite some lemmas.

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Lemma 1. (Theorem 1, [6]) If every order of element of finite group G is a power of prime, then $|\pi(G)| \leq 3$. Moreover, if G is solvable, then $|\pi(G)| \leq 2$.

Recall that a 2-Frobenius group G is ABC, where A and AB are normal subgroups of G, AB and BC are Frobenius group with kernel A, B and complements B, C respectively. The following lemma is due to Gruenberg and Kegel (see [7]).

Lemma 2. If $\pi(G) = \{p, q\}$ with p, q both odd primes, and G has no element of order pq, then G is a Frobenius group or 2-Frobenius.

Lemma 3. (Theorem 3, [8]) Let G be a finite group. Then the number of elements whose orders are multiples of n is either zero, or a multiple of the greatest divisor of $|G|$ that is prims to n.

Now we give the main result of the structure of POS-groups of order $2m$ with $(2, m) = 1$. The following integers n and l are positive.

Theorem 4. If G is a POS-group of order $2m$ with $(2, m) = 1$, then G is one of the following groups:

(a) the cyclic group Z_2 of order 2.

(b) dihedral groups $D_{2\cdot 3^n}$.

(c) $Z_2 \times Z_{3^n}$.

(d) POS-groups $F : Z_2$, where F is a Frobenius group which kernel is a p-group and complement a cyclic 3-group, and $p = 2 \cdot 3^k + 1$ a prime.

Proof. Since $|G| = 2m$ with $(2, m) = 1$, there exists a normal subgroup M of order m. Clearly, G is solvable. We can assume that $M > 1$. Let s_m be the number of elements of order m in G. Suppose that $p, q \in \pi(M)$ and $p \neq q$. Then M has no element of order pq. In fact, otherwise $(p-1)(q-1) = \phi(pq) \mid s_{pq} \mid |G|$, where ϕ is Euler totient function, we have $4 \mid |G|$, a contradiction. Thus every order of element of M is a power of prime. Then we have $|\pi(M)| \leq 2$ by Lemma 1. We divide into two cases:

Case 1. $|\pi(M)| = 1$. Suppose that $\pi(G) = \{2, p\}$ and $|G| = 2 \cdot p^n$. Since $p-1 = \phi(p) \mid |G| = 2 \cdot p^n$, we have $p-1 | 2$, and hence $p = 3$. So $|G| = 2 \cdot 3^n$. By Proposition 2.8 of the article [1], the Sylow 3-subgroup M of G is cyclic. Since $M \lhd G$, every element u of order 2 of G is an automorphism of M, and thus u maps the generated element c of M to c or c^{-1}. So we have G is $Z_2 \times Z_{3^n}$ or dihedral groups $D_{2\cdot 3^n}$.

Case 2. $|\pi(M)| = 2$. Similarly, suppose that $\pi(G) = \{2, q, p\}$ and $|G| = 2 \cdot q^n \cdot p^l$. Without loss of generalities, we assume that $q < p$. If $q > 3$, then $q - 1 \mid |G|$, and thus $q - 1 \mid 2$, i.e., $q = 3$, a contradiction. Therefore, we have $q = 3$. Similarly, since $p - 1 \mid 2 \cdot 3^n$, we have $p = 2 \cdot 3^k + 1$, where $1 \leq k \leq n$.

Since M has no element of order pq, M is a Frobenius group or 2-Frobenius group by Lemma 2.

(I.) M is a Frobenius group. Let $M = K : H$, where K and H are kernel and complement, respectively. If $|K| = 3^n$ and $|H| = p^l$, then H is cyclic and the intersection of every two subgroups of order p^l is trivial. So the number of cyclic subgroups of order p^l in M is $|M : N_M(H)| = 3^n$. Since $(|M|, |G/M|) = 1$, the number of ones in G is also 3^n. Then the number of elements of order p^l of G is $2 \cdot 3^k \cdot p^{l-1} \cdot 3^n$, which is a divisor of $|G| = 2 \cdot 3^n \cdot p^l$. So $k = 0$, which contradicts the fact that $k \geq 1$. If $|K| = p^l$ and $|H| = 3^n$, obviously, H is cyclic. Suppose that $\Omega = \{x \in G \mid o(x) = 2\}$ and K acts on the set Ω. Then $|\Omega| \equiv |C_\Omega(K)| (\text{mod } p)$. Now since $s_2 \mid 3^n p^l = 3^n(2 \cdot 3^k + 1)^l$, we have $|C_\Omega(K)| \geq 1$

(II.) M is a 2-Frobenius group. Let $M = KHK_0$, where KH and HK_0 are Frobenius group with kernel K, H and complements H, H_0 respectively. Suppose that $exp(KK_0) = p^e$. If $|K| = p^{l_1}, |H| = 3^n$ and $|K_0| = p^{l_2}$, where $l_1 + l_2 = l$, then

$$3^n \mid s_{p^e},\ s_{p^{e-1}} + s_{p^e},\ \cdots,\ s_p + s_{p^2} + \cdots + s_{p^e}$$

by Lemma 3, and hence $3^n \mid s_p$. While $s_p = 2 \cdot 3^k \cdot c_p$ with c_p the number of cyclic subgroups of order p, and $s_p \mid 2 \cdot 3^n \cdot p^l$, we have $s_p = 2 \cdot 3^n$ since $c_p \equiv 1(\text{mod } p)$. Since there exists just $(p-1) \cdot 3^n$ elements of order p in HK_0, $s_p > (p-1) \cdot 3^n$, a contradiction. Similarly, if $|K| = 3^{n_1}, |H| = p^l$ and $|K_0| = 3^{n_2}$, where $n_1 + n_2 = n$, then $s_3 = 2 \cdot p^l$. But HK_0 has just $2 \cdot p^l$ elements of order 3, so $s_3 > 2 \cdot p^l$, it is impossible. □

Unfortunately, the item (d) above is not classified completely. By use of the GAP software [5], it seems that the kernel of F is cyclic. We have the following conjecture.

Conjecture 5. The kernel of F of the item (d) of above Theorem 4 is cyclic.

In Das's paper [1], he posed a conjecture (see 5.2). Using Theorem 4, we can give a positive answer directly as follows.

Corollary 6. If G is a POS-group and $|G| = 42 \cdot m$ with $(42, m) = 1$, then $|G| = 42$.

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