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A NOTE ON FISCHER-MARSDEN'S CONJECTURE

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ABSTRACT. In this paper, we borrowed some ideas from general relativity and find a Robinson-type identity for the overdetermined system of partial differential equations in the Fischer-Marsden conjecture. We proved that if there is a nontrivial solution for such an overdetermined system on a 3-dimensional, closed manifold with positive scalar curvature, then the manifold contains a totally geodesic 2-sphere.

Let $\mathcal{M}$ denote the set of smooth Riemannian metrics on an n-dimensional closed manifold M whose derivatives are L_2-integrable. Then for any $g \in \mathcal{M}$, its scalar curvature R_g is an element in the space $\mathcal{W}$ of C^∞ functions. From the formula for R_g in local coordinates, we see that the scalar curvature map from $\mathcal{M}$ to $\mathcal{W}$ defines a quasi-linear differential operator of second order. The derivative R_g' at $g \in \mathcal{M}$ is given by

$$R_g'(h) = -\Delta_g(tr_g h) + \delta_g^*\delta_g(h) - g(Ric_g, h) \tag{1}$$

where δ is the divergence operator on the symmetric p-tensor on M, Ric_g is the Ricci curvature tensor of g, Δ is the Laplacian, and δ^* is the formal adjoint of δ. In the Riemannian case, if $\{e_i\}_{i=1}^n$ is a local orthonormal basis of vector fields, then

$$(\delta\alpha)(e_1, \cdots, e_p) = -\sum_{i=1}^n (D_{e_i}\alpha)(e_i, e_1, \cdots, e_p).$$

In particular, for any one-form α, we have

$$\delta^*\alpha = \frac{1}{2} L_{\hat{\alpha}} g \tag{2}$$

where $L_{\hat{\alpha}}$ is the Lie derivative of the vector field $\hat{\alpha}$ and $\hat{\alpha}$ is the dual of α.

It is easy to compute that the L_2-adjoint operator of R_g' is

$$R_g^{'*}(f) = -(\Delta f)g + D_g d(f) - fRic_g. \tag{3}$$

Since we can regard R_g' as a linear map from the space of symmetric tensors to the space of functions, so it is known that the following decompositions are true [Be-Eb]:

$$(Im(R_g'))^\perp = ker(R_g^{'*})$$

or

$$\mathcal{W} = Im(R_g') \oplus ker(R_g^{'*}). \tag{4}$$

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Definition 1. We say that the metric g is singular if R'_g is not a surjective map.

A natural question is: *What can we say about the singular metric g on a closed manifold M?*

From the decomposition (4), we know that g being singular is the same as the existence of a nontrivial solution to the following equation:

$$D_g d(v) = v\, Ric_g \; + \; (\Delta v) g. \tag{5}$$

Now we see that equation (5) is an overdetermined system, so the existence of a nontrivial solution should provide us with some information on the geometry of the underlying manifold M. Actually, Bourguignon [Bo] and Fischer-Marsden [F-M] proved the following:

Lemma 1. *Let (M, g) be a closed Riemannian manifold admitting a nontrivial solution to equation* (5). *Then either (M, g) is Ricci-flat and $Ker(R_g^{'*}) = \mathcal{W}$, or R_g is a positive constant.*

Based on this lemma, Fischer-Marsden made the following conjecture:

Conjecture 1. (M, g) must be an Einstein manifold if g is a singular metric.

We can easily rewrite (5) as:

$$D_g dv \; + \; \frac{R_g}{n(n-1)} vg \; = \; vT_g \tag{6}$$

or

$$v_{ij} + v\frac{R_g}{n(n-1)} g_{ij} = vT_{ij} \tag{7}$$

where T_g and T_{ij} are the traceless-Ricci tensors.

In order to prove the Conjecture, one only needs to show that (M, g) is Einstein if R_g is a positive constant. Now if (M, g) is an Einstein manifold with positive scalar curvature, then a theorem of Obata [Ob] tells us that (M, g) must be a round sphere. So the Fischer-Marsden Conjecture can be rephrased as the following:

Conjecture 2. If g is a singular metric on a Riemannian manifold M such that its scalar curvature is positive, then the existence of a nontrivial solution of the overdetermined system (5) implies that (M, g) is a round sphere.

The counterexample of the Fischer-Marsden Conjecture was provided by O. Kobayashi [Ko] and J. Lafontaine [Laf] independently. They proved that if we assume further that (M, g) is conformally flat, then (M, g) must be isometric to one of the following:

(a) Euclidean sphere S^n.
(b) Finite quotient of $(S^1, dt^2) \times (S^{n-1}, g_0)$, g_0 is the canonical metric.
(c) Finite quotient of a product torus $(S^1 \times S^{n-1}, dt^2 + h^2(t) g_0)$.

By looking at the examples listed above, one observes that (M, g) always contains a totally geodesic $(n-1)$-sphere if (M, g) is conformally flat. So it is interesting to know whether this is a general phenomenon without assuming that (M, g) is conformally flat. The present note gives a positive answer to the question if $dim\, M = 3$.

Theorem 1. *If g is a singular metric on a 3-dimensional closed Riemannian manifold M such that its scalar curvature R_g is positive, then $(M,\, g)$ contains a totally geodesic* 2*-sphere.*

The idea of the proof is borrowed from general relativity.

Recall that a geodesically complete spacetime $(N, \overset{4}{g})$ is called static if and only if there is an orientable, spacelike, 3-manifold (Σ, g) such that N is diffeomorphic to $\Sigma \times \mathbb{R}$, and

$$\overset{4}{g} = -U^2 dt^2 + g$$

where U is called the gravitational scalar potential of the static field.

The Einstein equations for a static perfect fluid with mass-energy density ρ and pressure p can be decomposed into:

$$Ric(g)_{ij} = U^{-1} U_{;ij} + 4\pi(\rho - p) g_{ij}, \tag{8}$$

$$\Delta U = 4\pi U(\rho + 3p). \tag{9}$$

Therefore, we know that our equation (5) is the equation for static perfect fluid. There has been active research among relativists for the proof of the spherical symmetries for steller models. We only provide the interested readers with a short list of papers: [Rob], [B-Mas], [I], [K], [Mas], [Lind]. The essential step in proving the spherical symmetry is to find some nice Robinson type identity that generally has the form that " divergence equals positive quantity".

Proof of the Theorem. We are going to construct a Robinson type identity first and then prove the theorem.

We may assume, without loss of generality, that $R_g = n(n-1)$. Hence equation (6) becomes

$$D_g dv + vg = vT_g \tag{10}$$

and

$$\Delta v + nv = 0. \tag{11}$$

It is easy to see that $\{x \in M | v(x) \neq 0\}$ is dense in M (see [Bo] or [F-M]). Let M_0 be a connected component of $\{x \in M | v(x) > 0\}$ such that $dv \neq 0$ on $\Sigma = \partial M_0$.

Define $W = |\nabla v|^2$ and $W_0 = 1 - v^2$. Since the scalar curvature is constant on ∂M_0, we can construct the Robinson type identity for the system (10):

$$\nabla^a [v^{-1} \nabla_a (W - W_0)] = 2v|T|^2. \tag{12}$$

Equation (12) can be easily verified by straightforward computation.

From equation (10), it is easy to check that

$$|W| = constant$$

on Σ. When restricted to Σ, we know that $v_{ij} = 0$. It is straightforward to check that Σ is totally geodesic and $-\frac{\nabla v}{W^{\frac{1}{2}}}$ is a unit outward normal vector field to Σ. We let e_1, e_2, e_3 be an orthonormal basis such that along Σ,

$$e_3 = \frac{\nabla v}{W^{\frac{1}{2}}}.$$

We remark that although $\{e_1, e_2, e_3\}$ is a local frame, e_3 is globally defined on Σ due to the embeddedness of Σ. The Gauss equation tells us that

$$R_{1212} = K \tag{13}$$

where R_{ijkl} is the full curvature tensor of (M, g) and K is the Gaussian curvature of Σ with the induced metric.

Integrating equation (12) and using Stoke's formula, we obtain

$$\int_{\Sigma} W^{-\frac{1}{2}} R^{ab} v_{;a} v_{;b} - 2W^{\frac{1}{2}} S_0 = -\int_{M_0} v|T|^2 \tag{14}$$

where S_0 is the area of Σ.

By using the facts

$$R_{11} = R_{1313} + K, \tag{15}$$

$$R_{22} = R_{2323} + K, \tag{16}$$

$$R_{33} = R_{1313} + R_{2323}, \tag{17}$$

$$R_{11} + R_{22} + R_{33} = 6, \tag{18}$$

we know that

$$\int_{\Sigma} W^{-\frac{1}{2}} R^{ab} v_{;a} v_{;b} = 3W^{\frac{1}{2}} S_0 - W^{\frac{1}{2}} \int_{\Sigma} K. \tag{19}$$

Therefore we get

$$\int_{\Sigma} K \geq S_0 > 0, \tag{20}$$

and the Gauss-Bonnet theorem immediately implies that Σ is a 2-sphere.

Remark. The area of the sphere Σ, S_0, is in general less than or equal to 4π. In case $S_0 = 4\pi$, then it is easy to see that (M_0, g) is isometric to the canonical 3-sphere since the manifold is Einstein.

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