Mathematical Sciences And Applications E-Notes
Volume 2 No. 2 pp. 51–66 (2014) ©MSAEN

A NOTE ON FLUID APPROXIMATION OF RETRIAL QUEUEING SYSTEM WITH TWO ORBITS, ABANDONMENT AND FEEDBACK

AMINA ANGELIKA BOUCHENTOUF, ABBES RABHI, LAHCENE YAHIAOUI

(Communicated by Xiao-Jun YANG)

Abstract. This paper deals with the asymptotic analysis of a queueing system model consisting of two orbits, c_t servers, $t \geq 0$, abandoned and feedback customers. Two independent Poisson streams of customers arrive to the system, an arriving one of type i, $i = 1, 2$ is handled by an available server, if there is any; otherwise, he waits in an infinite buffer queue. A waiting customer of type i who did not get connected to a server will lose his patience and abandon after an exponentially distributed amount of time, the abandoned one may leave the system (loss customer) or move to the orbit depending of its type, from which he makes a new attempts to reach the primary queue, this latter may lose his patience and leave the system definitively (from the orbit) after an exponentially distributed amount of time. When a customer finishes his conversation with a server, he may comeback to the system for another service.

1. Introduction

During the past few decades, there has been increasing interest in studying retrial queueing systems because they are widely used in performance analysis of many practical systems, retrial queues have been investigated extensively because of their applications in telephone switching systems, telecommunication networks and computer systems for competing to gain service from a central processing unit and so on. Moreover, retrial queues are also used as mathematical models for several computer systems: packet switching networks, shared bus local area networks operating under the carrier-sense multiple access protocol and collision avoidance star local area networks, etc. Retrial queueing systems are characterized by the

Date: Received: May, 17, 2014; Accepted: August, 20, 2014.
2000 *Mathematics Subject Classification.* Primary 60K25; Secondary 68M20; Thirdly 90B22.
Key words and phrases. Queueing system, call center, retrial queue, fluid approximation, abandonment, feedback .

feature that a blocked customer (a customer who finds the server unavailable) may leave the service area temporarily and join a retrial group in order to retry his request after some random time. For excellent bibliography on retrial queues, the readers are referred to [15, 19, 16, 12, 29, 42, 8] and references therein.

Behavioral psychology concerning the use of service offered by mobile cellular networks includes repeated attempts and abandonments. Both phenomena reflect the impatience of subscribers when all channels are occupied. Following the arrival of a call, if all the available channels are occupied, a call is not be admitted into a network. Later, a subscriber initiates a repeated attempt for the admission of a call. An abandonment happens when a subscriber's call becomes rejected and the subscriber gets impatient and gives up after a certain time without getting service.

In feedback queueing model, if the service of the job is unsuccessful, it may try again and again until a successful service is completed. Takacs [40] was the first to study feedback queueing model. Studies on queue length, the total sojourn time and the waiting time for an $M/G/1$ queue with Bernoulli feedback were provided by Vanden Berg and Boxma [41]. Choudhury and Paul [9] derived the queue size distribution at random epoch and at a service completion epoch for $M/G/1$ queue with two phases of heterogeneous services and Bernoulli feedback system, Krishna Kumar *et al.*[25] considered a generalized $M/G/1$ feedback queue in which customers are either "positive" or "negative". In [17] Fayolle treated a simple telephone exchange with delayed feedback, Choi [8] considered an $M/M/c$ retrial queues with geometric loss and feedback when $c = 1, 2$.

A queueing system with two orbits and two exogenous streams of different type serves as a model for two competing job streams in a carrier sensing multiple access system, where the jobs, after a failed attempt to network access, wait in an orbit queue [34, 39]. An example of carrier sensing multiple access system is a local area computer network with bus architecture. The two types of customers can be interpreted as customers with different priority requirements.

A two-class retrial system with a single- server, no waiting room, batch arrivals and classical retrial scheme was introduced and analyzed in [26]. Then, in [14] author extended the analysis of the model in [26] to the multi-class setting with arbitrary number of classes. In [20] author has established equivalence between the multi-class batch arrival retrial queues with classical retrial policy and branching processes with immigration. In [33] a non-preemptive priority mechanism was added to the model of [14, 26]. In [28] authors have considered a multi-class retrial system where retrial classes are associated with different phases of service. Retrial queueing model $MMAP/M_2/1$ with two orbits was studied in [5], authors considered a retrial single-server queueing model with two types of customers. In case of the server occupancy at the arrival epoch, the customer moves to the orbit depending on the type of the customer, one orbit is infinite while the second one is a finite. Joint distribution of the number of customers in the orbits and some performance measures are computed. In [7] authors considered two retrial queueing system with balking and feedback, the joint generating function of the number of busy server and the queue length was found by solving Kummer differential equation, and by the method of series solution.

Call centers have become the central focus of many companies, as these centers stay in direct contact with the form's customers and form an integral part of their customer relationship management. So, at the present time, call centers are becoming an important means of communication with the customer. Therefore, the response-time performance of call centers is essential for the customer satisfaction. For call center managers, making the right staffing decisions is essential to the costs and the performances of call centers. Various models have been developed in order to decide on the right number of agents, see [18, 21], and the references therein. Thus, considering customer retrial behaviors in call centers is quite significant [18, 2, 38, 11] and reference therein.

Fluid models for call centers have been extensively studied, for instance see [43, 32]. In [31] the fluid and the diffusion approximation for time varying multi-server queue with abandonment and retrials as studied, it was shown that the fluid and the diffusion approximation can both be obtained by solving sets of non-linear differential equations. In [30] more general theoretical results for the fluid and diffusion approximation for Markovian service networks was given. In [1] authors extended the model by allowing customer balking behavior. Fluid models have also been applied in delay announcement of customers in call centers [22, 23].
And recently, in [10] authors study call centers with one redial and one orbit, using fluid limit they calculate the expected total arrival rate, which is then given as an input to the Erlang A model for the purpose of calculating service levels and abandonment rates. The performance of such a procedure is validated in the case of single intervals as well as multiple intervals with changing parameters.

In the present paper, an analysis of $M_t/M_t/c_t$ retrial queueing model with abandonment and feedback; a system with two orbits and two exogenous streams of different types is carried out.

The layout of the paper is given as follows. After the introduction, in section 2, we describe the mathematical model in more details and give the notations, assumptions and some results that will be used and useful throughput this paper. In section 3, our main result is given; an asymptotic analysis of the considered model is presented.

2. The mathematical model

Consider retrial queueing network with time dependent parameters, state dependent routing, abandonment and feedback (figure 1). The $M_t/M_t/c_t$ queue has a (time in homogeneous) Poisson arrival process with rate λ_{i_t}, a service rate (per server) with mean $\frac{1}{\mu_{i_t}}$, $i = 1, 2$ and c_t servers, for all $t > 0$.

Two independent Poisson streams of customers flow into c servers. An arriving customer of type i, $i = 1, 2$ is handled by an available server in FIFO manner, if there is any; otherwise, he waits in an infinite buffer queue. The customers are handled in the order of arrival. A waiting customer of type i who did not get connected to a server will lose his patience and abandon after an exponentially distributed amount of time at rate δ_{i_t}, the abandoned one may leave the entire network (loss customer) with probability ϕ_t or move into one of the orbits with

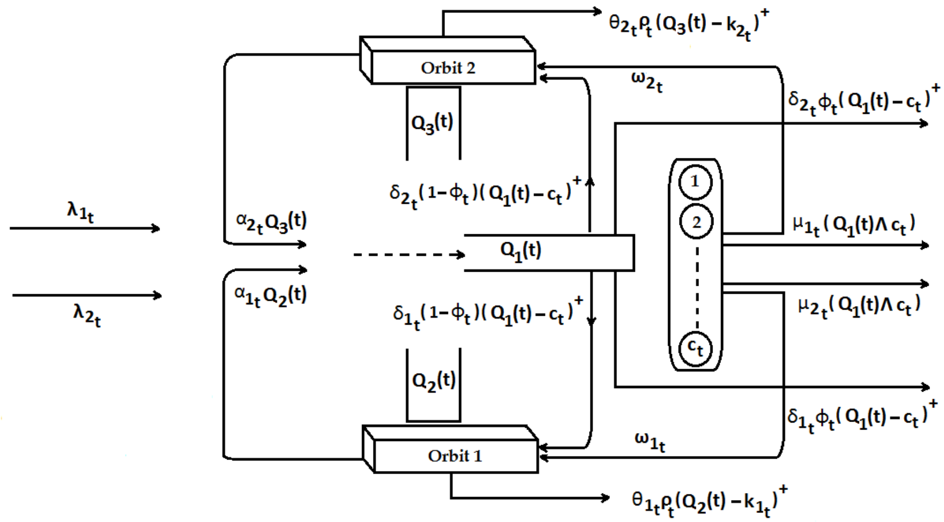

FIGURE 1. A retrial queueing model with two orbits, abandonment and feedback.

probability $1-\phi_t$, from which he makes a new attempts to reach the primary queue at rate α_{i_t}. Each customer waiting in the retrial pool may leave his patience and thus abandon the whole system at rate θ_{i_t} if at some moment he beholds that the queue length i is greater than k_{i_t} with $0 < k_{i_t} < Q_1(t)$, so after an exponentially distributed amount of time he have to decide either he still waiting for a new attempts or give up. An abandoning customer leave the system from the orbit with some probability ρ_{i_t}. When a customer finishes his conversation with a server or if the service of the job is unsuccessful, the customer may comeback to the system to the retrial pools depending on its type for another service or try again and again for a successful service at rate ω_{i_t}. Let's note that all the arrival and service processes are constructed from mutually independent Poisson processes.

After the description of the considered model let us introduce some notations and results helpful in our study.

Let $\{\Pi_i(\cdot)\}_{i\in I}$ a sequence of mutually independent, standard (rate 1) Poisson processes, indexed by a set I which is at most countably infinite; a separable Banach space $\mathbb{V}$, with norm $|\cdot|$; a sequence of jump vectors $\{v_i \in \mathbb{V} | i \in I\}$ with

$$\sum_{i\in I} v_i < \infty \tag{2.1}$$

a random initial state vector $Q(0)$ in $\mathbb{V}$ that is assumed to be independent of the sequence of Poisson processes $\{\Pi(\cdot)\}_{i\in I}$; and a collection of real-valued, non-negative Lipschitz rate functions on $\mathbb{V}$,

$$\{\nu_t(\cdot, i) | t \geq 0, i \in I\}, \tag{2.2}$$

that together satisfy

$$\|\nu_t(\cdot,i)\| \leq \xi_t \vartheta^{(i)}, \tag{2.3}$$

with ξ_t, a locally integrable function, and $\{\vartheta^{(i)}|i \in I\}$, a sequence of real numbers; with $\|\cdot\|$ a Lipschitz norm for real-valued functions on $\mathbb{V}$. In all what follows the number of elements in I is finite, $\mathbb{V} = \mathbb{R}^N$, $1 \leq N < \infty$ and $|\cdot|$ the standard Euclidean norm on $\mathbb{R}^N$.

Let the Markovian service network $\{Q(t)|t \geq 0\}$, be the $\mathbb{V}$-valued stochastic process whose sample paths are uniquely determined by $Q(0)$ and the functional equations

$$Q(t) = Q(0) + \sum_{i\in I} \Pi_i \left(\int_0^t \nu_s(Q(s),i)ds \right) v_i, \text{ for all } t \geq 0.$$

Let $\{Q^\eta|\eta > 0\}$ be the rescaled procees such that

$$Q^\eta(t) = Q^\eta(0) + \sum_{i\in I} \Pi_i \left(\eta \int_0^t \nu_s \left(\frac{Q^\eta(s)}{\eta}, i \right) ds \right) v_i, \tag{2.4}$$

The asymptotic analysis described above was carried out in [27] for the special case of rate functions having no explicit time dependence and state dependence that is continuously differentiable. The analysis was extend to the following general class of processes [30].

$$Q^\eta(t) = Q^\eta(0) + \sum_{i\in \mathbf{I}} \Pi_i \left(\int_0^t \nu_s^\eta \left(\frac{Q^\eta(s)}{\eta}, i \right) ds \right) v_i, \tag{2.5}$$

with

$$\|\nu_t^\eta(\cdot,i)\| \leq \eta \xi_t \vartheta^{(i)}. \tag{2.6}$$

In this extension, we permit the following hypotheses:

(H1) The rate functions $\nu_t^\eta(\cdot,i)$ are functions of time as well as state.

(H2) The rate functions, indexed by the parameter η, are such that for each $i \in I$, $\nu_t^\eta(\cdot,i)$ has the following asymptotic expansion as $\eta \to \infty$;

$$\nu_t^\eta(\cdot,i) = \eta\nu_t^{(0)}(\cdot,i) + \sqrt{\eta}\nu_t^{(1)}(\cdot,i) + 0(\sqrt{\eta}). \tag{2.7}$$

(H3) The rate functions, as a function of the state space $\mathbb{V}$, have a more general type of differentiability that include functions on the real line that are everywhere left and right differentiable.

These conditions allow to apply the limit theorems to a wider class of Markov processes that arise in the study of queueing networks with large numbers of servers. Now, let's introduce the first result where the sample path representation (2.5) of $\{Q^\eta|\eta > 0\}$ is strongly presented;

Theorem 2.1. [30] *Assume that (2.1) and (2.6) hold. Moreover, assume that*

$$\lim_{n\to\infty} \sum_{i\in I} \left(\int_0^t \left\| \frac{\nu_t^\eta(\cdot,i)}{\eta} - \nu_t^{(0)}(\cdot,i) \right\| \right) ds = 0, \tag{2.8}$$

for all $t \geq 0$. *If* $\{Q^{\eta}(0)|\eta > 0\}$ *is any family of random initial state vectors in* $\mathbb{V}$, *then*

$$\frac{Q^{\eta}(0)}{\eta} = Q^{(0)}(0), \;\; a.s \;\; implies \;\; \frac{Q^{\eta}(t)}{\eta} = Q^{(0)}(t) \;\; a.s \tag{2.9}$$

where the convergence is uniform on compact sets in t, *and* $Q^{(0)}$ *is the unique deterministic process* $\{Q^{(0)}(t)|t \geq 0\}$ *that solves the integral equation*

$$Q^{(0)}(t) = Q^{(0)}(0) + \int_0^t \nu_s^{(0)}(Q^{(0)}(s))ds, \;\; t \geq 0. \tag{2.10}$$

Here $\nu_t^{(0)}$, *given by*

$$\nu_t^{(0)}(x) = \sum_{i \in I} \nu_t^{(0)}(x,i)v_i, \quad x \in \mathbb{V}, \tag{2.11}$$

is a Lipschitz mapping of $\mathbb{V}$ *into itself and its Lipschitz norm* $\|\nu_t^{(0)}\|$, *is a locally integrable function of* t.
We call $Q^{(0)}$ *the fluid approximation associated with the family* $\{Q^{\eta}(t)|t \geq 0\}$. *It gives rise to first-order macroscopic fluid approximations of the form*

$$Q^{\eta}(t,\omega) = \eta Q^{(0)}(t) + o(\eta) \;\; a.s., \;\; t \geq 0. \tag{2.12}$$

We can now state the functional central limit theorem

Theorem 2.2. [30] *Assume that (2.1) and (2.6) hold. Moreover, assume that*

$$\sum_{i \in I} \overline{\lim}_{\eta \to \infty} \int_0^t \left\| \sqrt{\eta} \left(\frac{\nu_t^{\eta}(\cdot,i)}{\eta} - \nu_t^{(0)}(\cdot,i) \right) \right\| ds < \infty, \tag{2.13}$$

and

$$\lim_{\eta \to \infty} \sum_{i \in I} \int_0^t \left\| \sqrt{\eta} \left(\frac{\nu_t^{\eta}(\cdot,i)}{\eta} - \nu_t^{(0)}(\cdot,i) \right) - \nu_t^{(1)}(\cdot,i) \right\| ds = 0. \tag{2.14}$$

It follows that $\nu_t^{(0)}$, *given by (2.11), and* $\nu_t^{(1)}$, *given by*

$$\nu_t^{(1)}(x) = \sum_{i \in I} \nu_t^{(1)}(x,i)v_i, \;\; x \in \mathbb{V}, \tag{2.15}$$

are both Lipschitz mappings of $\mathbb{V}$ *into itself, and their Lipschitz norms are locally integrable functions of* t.
Moreover, if we assume that $\nu_t^{(0)}(\cdot)$ *has a scalable Lipschitz derivative* $\wedge\nu_t^{(0)}(Q^{(0)}(t);\cdot)$ *and we have a family of random initial state vectors* $\{Q^{\eta}(0)|\eta > 0\}$ *in* $\mathbb{V}$, *then for all random vectors* $Q^{(0)}(0)$ *and* $Q^{(1)}(0)$ *in* $\mathbb{V}$, *it follows that*

$$\lim_{\eta \to \infty} \sqrt{\eta} \left(\frac{Q^{\eta}(0)}{\eta} - Q^{(0)}(0) \right) =^d Q^{(1)}(0), \tag{2.16}$$

implies

$$\lim_{\eta \to \infty} \sqrt{\eta} \left(\frac{Q^{\eta}(t)}{\eta} - Q^{(0)}(t) \right) =^d Q^{(1)}(t), \tag{2.17}$$

the convergence being weak-convergence in $D_{\mathbb{V}}[0,\infty)$, *the space of* $\mathbb{V}$*-valued functions that are right-continuous with left-limits, equipped with the Skorohod* J_1 *topology.*

Finally, the limit $Q^{(1)} \equiv \{Q^{(1)}(t)|t \geq 0\}$ *is the unique stochastic process that solves the stochastic integral equation*

$$\begin{aligned} Q^{(1)}(t) = Q^{(1)}(0) + \int_0^t \left(\left(\wedge\nu_s^{(0)}(Q^{(0)}(s), Q^{(1)}(s)) \right) + \nu_s^{(1)}(Q^{(0)}(s)) \right) ds \\ + \sum_{i \in I} \Omega_i \left(\int_0^t \nu_s^{(0)}(Q^{(0)}(s), i) ds \right) v_i, \quad t \geq 0, \end{aligned} \tag{2.18}$$

where the $\{\Omega_i | i \in I\}$ *are a family of mutually independent, standard Brownian motions.*
We call $Q^{(1)}$ *the diffusion approximation associated with the family* $\{Q^\eta(t)|t \geq 0\}$. *It quantifies deviations from the fluid approximations, and it gives rise to second-order mesoscopic diffusion approximations of the form*

$$Q^\eta(t) =^d \eta Q^{(0)}(t) + \sqrt{\eta} Q^{(1)}(t) + o(\sqrt{\eta}), \tag{2.19}$$

as $\eta \to \infty$ *for all* $t \geq 0$, *with the approximation being in distribution*

Now consider the case of $\mathbb{V}$ being either a finite dimensional vector space or a Banach space that can be embedded into its own dual space (like a Hilbert space), so that the notion of a transpose can be defined, denoted by a superscript "$\top$" (for $\mathbb{V} = \mathbb{R}^N$, this corresponds to the standard transpose of a matrix). One consequence of the diffusion limit is an associated set of differential equations that become useful in the computation of its mean and covariance matrix.

Theorem 2.3. [30] *If conditions (2.1), (2.6), (2.13), and (2.14) all hold, then the mean vector and covariance matrix for* $Q^{(1)}(t)$ *solve the following set of differential equations:*

$$\frac{d}{dt}\mathbb{E}(Q^{(1)}(t)) = \mathbb{E}(\wedge\nu_t^{(0)}(Q^{(0)}(t), Q^{(1)}(t))) + \nu_t^{(1)}(Q^{(0)}(t)). \tag{2.20}$$

$$\begin{aligned} \tfrac{d}{dt} Cov(Q^{(1)}(t), Q^{(1)}(t)) = \left(Cov(Q^{(1)}(t), \wedge\nu_t^{(0)}(Q^{(0)}(t), Q^{(1)}(t))) \right) \\ + \sum_{i \in I} \nu_t^{(0)}(Q^{(0)}(t), i) v_i^\top \cdot v_i. \end{aligned} \tag{2.21}$$

for almost all t, *where*

$$Cov(Q^{(1)}(t), Q^{(1)}(t)) \equiv \mathbb{E}\left((Q^{(1)}(t))^T \cdot Q^{(1)}(t) \right) - \mathbb{E}(Q^{(1)}(t))^\top \cdot \mathbb{E}(Q^{(1)}(t)), \tag{2.22}$$

and for all operators A *on* $\mathbb{V}$,

$$\{A\} \equiv A + A^\top. \tag{2.23}$$

Moreover, if $\wedge\nu_t^{(0)}(Q^{(0)}(t), \cdot)$ is a linear operator for almost all t, then $\mathbb{E}[Q^{(1)}(t)]$ is the unique solution for (2.20) and $Cov[Q^{(1)}(t), Q^{(1)}(t)]$ is the unique solution for (2.21). Finally, for all $s < t$, $Cov[Q^{(1)}(s), Q^{(1)}(t)]$ solves the same set of differential equations in t as does $\mathbb{E}[Q^{(1)}(t)]$, but with a different set of initial conditions. Now, and after having stated all these results, we are able to give our main result.

3. Main result

Consider our queueing model presented in figure 1. The $M_t/M_t/c_t$ queue has a (time inhomogeneous) Poisson arrival process with external arrival rates λ_{i_t}, a service rates (per server) of μ_{i_t}, feedback rates ω_{i_t}, abandonment rates from the primary queue δ_{i_t}, abandonment rates from retrial pool i ω_{i_t}, $i = 1, 2$ and c_t servers, for all $t > 0$, $c_t = 1, 2, 3, \ldots$. With ϕ_t, $0 \leq \phi_t \leq 1$, the probability of no retrial at time t, ρ_t $0 \leq \rho_t \leq 1$ the probability of leaving the network from the orbit at time t.

Let $\mathbb{V} = \mathbb{R}^3$ and $Q(t) = \{Q_1(t), Q_2(t), Q_3(t)\}$. We can construct the sample paths for the $M_t/M_t/c_t$ queue length process as the unique set of solutions to the functional equation

(3.1)

$$\begin{aligned}
Q_1(t) = Q_1(0) &+ \Pi_1\left(\int_0^t \lambda_{1_s} ds\right) + \Pi_2\left(\int_0^t \lambda_{2_s} ds\right) + \Pi_3\left(\int_0^t \alpha_{1_s} Q_2(s) ds\right) \\
&+\Pi_4\left(\int_0^t \alpha_{2_s} Q_3(s) ds\right) - \Pi_5\left(\int_0^t (Q_1(s) - c_s)^+ \delta_{1_s} \phi_s ds\right) \\
&-\Pi_6\left(\int_0^t (Q_1(s) - c_s)^+ \delta_{2_s} \phi_s ds\right) - \Pi_7\left(\int_0^t (Q_1(s) - c_s)^+ \delta_{1_s} (1 - \phi_s) ds\right) \\
&-\Pi_8\left(\int_0^t (Q_1(s) - c_s)^+ \delta_{2_s} (1 - \phi_s) ds\right) - \Pi_9\left(\int_0^t \mu_{1_s} (Q_1(s) \wedge c_s) ds\right) \\
&-\Pi_{10}\left(\int_0^t \mu_{2_s} (Q_1(s) \wedge c_s) ds\right).
\end{aligned}$$

$$\begin{aligned}
Q_2(t) = Q_2(0) &+ \Pi_1^1\left(\int_0^t (Q_1(s) - c_s)^+ \delta_{1_s} (1 - \phi_s) ds\right) + \Pi_2^1\left(\int_0^t \omega_{1_s} ds\right) \\
&-\Pi_3^1\left(\int_0^t \alpha_{1_s} Q_2(s) ds\right) - \Pi_4^1\left(\int_0^t \rho_s \theta_{1_s} (Q_2(s) - k_{1_s})^+ ds\right).
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
Q_3(t) = Q_3(0) &+ \Pi_1^2\left(\int_0^t (Q_1(s) - c_s)^+ \delta_{2_s} (1 - \phi_s) ds\right) + \Pi_2^2\left(\int_0^t \omega_{2_s} ds\right) \\
&-\Pi_3^2\left(\int_0^t \alpha_{2_s} Q_3(s) ds\right) - \Pi_4^2\left(\int_0^t \rho_s \theta_{2_s} (Q_3(s) - k_{2_s})^+ ds\right),
\end{aligned} \tag{3.3}$$

where $\Pi_i(\cdot)$, $\Pi_i^1(\cdot)$, and $\Pi_i^2(\cdot)$, are given independent, standard (rate 1) Poisson processes, and for all real x and y, $x \wedge y \equiv \min(x, y)$.

For the $M_t/M_t/c_t$ queue, we create a family of associated processes. The $M_t/M_t/c_t$ queue is indexed by η, we want to have both the arrival rate and number of servers grow large, i.e., scaled up by η, but leave the service rate unscaled. We are then

interested in the asymptotic behavior of the process $Q^\eta(t) = (Q_1^\eta(t), Q_2^\eta(t), Q_3^\eta(t))$
(3.4)

$$\begin{aligned}
Q_1^\eta(t) &= Q_1^\eta(0) + \Pi_1\left(\int_0^t \eta\lambda_{1_s} ds\right) + \Pi_2\left(\int_0^t \eta\lambda_{2_s} ds\right) + \Pi_3\left(\int_0^t \alpha_{1_s} Q_2^\eta(s) ds\right)\\
&+\Pi_4\left(\int_0^t \alpha_{2_s} Q_3^\eta(s) ds\right) - \Pi_5\left(\int_0^t (Q_1^\eta(s) - \eta c_s)^+ \delta_{1_s}\phi_s ds\right)\\
&-\Pi_6\left(\int_0^t (Q_1^\eta(s) - \eta c_s)^+ \delta_{2_s}\phi_s ds\right) - \Pi_7\left(\int_0^t (Q_1^\eta(s) - \eta c_s)^+ \delta_{1_s}(1-\phi_s) ds\right)\\
&-\Pi_8\left(\int_0^t (Q_1^\eta(s) - \eta c_s)^+ \delta_{2_s}(1-\phi_s) ds\right) - \Pi_9\left(\int_0^t \mu_{1_s}(Q_1^\eta(s) \wedge \eta c_s) ds\right)\\
&-\Pi_{10}\left(\int_0^t \mu_{2_s}(Q_1^\eta(s) \wedge \eta c_s) ds\right).
\end{aligned}$$

(3.5)

$$\begin{aligned}
Q_2^\eta(t) &= Q_2^\eta(0) + \Pi_1^1\left(\int_0^t (Q_1^\eta(s) - \eta c_s)^+ \delta_{1_s}(1-\phi_s) ds\right) + \Pi_2^1\left(\int_0^t \eta\omega_{1_s} ds\right)\\
&-\Pi_3^1\left(\int_0^t \alpha_{1_s} Q_2^\eta(s) ds\right) - \Pi_4^1\left(\int_0^t \rho_s\theta_{1_s}((Q_2^\eta(s) - \eta k_{1_s})^+ ds\right).
\end{aligned}$$

(3.6)

$$\begin{aligned}
Q_3^\eta(t) &= Q_3^\eta(0) + \Pi_1^2\left(\int_0^t (Q_1^\eta(s) - \eta c_s)^+ \delta_{2_s}(1-\phi_s) ds\right) + \Pi_2^2\left(\int_0^t \eta\omega_{2_s} ds\right)\\
&-\Pi_3^2\left(\int_0^t \alpha_{2_s} Q_3^\eta(s) ds\right) - \Pi_4^2\left(\int_0^t \rho_s\theta_{2_s}(Q_3^\eta(s) - \eta k_{2_s})^+ ds\right)
\end{aligned}$$

as $\eta \to \infty$.

Let us note that servers and time-dependent parameters do not need to be scaled; The primary motivating models are call centers, where service involves an interaction between the customer and the server, because a customer is involved, it does not seem reasonable to scale the service rates with η. Thus, in order to accommodate the arrivals, whose rate is proportional to η, the number of servers must be scaled with η. Time dependent arrival rates should need no justification, since phenomena such as rush hours are quite common. Time dependent service rates can be used to model phenomena such as server fatigue or changes in the nature of services over the day. Finally, a time dependent number of servers arises with shift changes and in systems where the number of servers is varied to accommodate changes in the arrival rate.

The first-order asymptotic result takes the form of a functional strong law of large numbers and yields a fluid approximation for the original process.

Theorem 3.1. *Let Q^η be the uniform acceleration as in (2.4), the fluid limit for the multiserver queue with retrials abandonment and feedback is the unique solution* *to the differential equations*

(3.7)
$$\frac{d}{dt}Q_1^{(0)}(t) = \lambda_{1_t} + \lambda_{2_t} + \alpha_{1_t}Q_2^{(0)}(t) + \alpha_{2_t}Q_3^{(0)}(t) - (\mu_{1_t} + \mu_{2_t})(Q_1^{(0)}(t) \wedge c_t)$$
$$-(\delta_{1_t} + \delta_{2_t})(Q_1^{(0)}(t) - c_t)^+.$$

(3.8)
$$\frac{d}{dt}Q_2^{(0)}(t) = \omega_{1_t} - \alpha_{1_t}Q_2^{(0)}(t) + \delta_{1_t}(1-\phi_t)(Q_1^{(0)}(t) - c_t)^+ - \theta_{1_t}\rho_t(Q_2^{(0)}(t) - k_{1_t})^+.$$

(3.9)
$$\frac{d}{dt}Q_3^{(0)}(t) = \omega_{2_t} - \alpha_{2_t}Q_3^{(0)}(t) + \delta_{2_t}(1-\phi_t)(Q_1^{(0)}(t) - c_t)^+ - \theta_{2_t}\rho_t(Q_3^{(0)}(t) - k_{2_t})^+.$$

Furthermore, the diffusion limit for the multiserver queue with abandonment, feedback and retrials is the unique solution to the integral equations

(3.10)
$$Q_1^{(1)}(t) = Q_1^{(1)}(0) + \Omega_1\left(\int_0^t \lambda_{1_s}ds\right) + \Omega_2\left(\int_0^t \lambda_{2_s}ds\right) + \Omega_3\left(\int_0^t \alpha_{1_s}Q_2^{(0)}(s)ds\right)$$
$$+\Omega_4\left(\int_0^t \alpha_{2_s}Q_3^{(0)}(s)ds\right) - \Omega_5\left(\int_0^t (Q_1^{(0)}(s) - c_s)^+\delta_{1_s}(1-\phi_s)ds\right)$$
$$+\int_0^t \Big[\Big((\mu_{1_s} + \mu_{2_s})1_{\{Q_1^{(0)}(s)\le c_s\}} + (\delta_{1_s} + \delta_{2_s})1_{\{Q_1^{(0)}(s)>c_s\}}\Big)Q_1^{(1)}(s)^-$$
$$-\Big((\mu_{1_s} + \mu_{2_s})1_{\{Q_1^{(0)}(s)<c_s\}} + (\delta_{1_s} + \delta_{2_s})1_{\{Q_1^{(0)}(s)\ge c_s\}}\Big)Q_1^{(1)}(s)^+$$
$$+\alpha_{2_s}Q_3^{(1)}(s) + \alpha_{1_s}Q_2^{(1)}(s)\Big]ds - \Omega_6\left(\int_0^t (Q_1^{(0)}(s) - c_s)^+\delta_{2_s}(1-\phi_s)ds\right)$$
$$-\Omega_7\left(\int_0^t \mu_{1_s}(Q_1^{(0)}(s) \wedge c_s)ds\right) - \Omega_8\left(\int_0^t \mu_{2_s}(Q_1^{(0)}(s) \wedge c_s)ds\right)$$
$$-\Omega_9\left(\int_0^t (Q_1^{(0)}(s) - c_s)^+\delta_{1_s}\phi_s ds\right) - \Omega_{10}\left(\int_0^t (Q_1^{(0)}(s) - c_s)^+\delta_{2_s}\phi_s ds\right).$$

(3.11)

$$Q_2^{(1)}(t) = Q_2^{(1)}(0) + \Omega_1^1\left(\int_0^t (Q_1^{(0)}(s) - c_s)^+ \delta_{1_s}(1-\phi_s)ds\right) + \Omega_2^1\left(\int_0^t \omega_{1_s} ds\right)$$

$$+\int_0^t \left[Q_1^{(1)}(s)^+ 1_{\{Q_1^{(0)}(s)\geq c_s\}} - Q_1^{(1)}(s)^- 1_{\{Q_1^{(0)}(s)>c_s\}}\right]\delta_{1_s}(1-\phi_s)ds$$

$$-\Omega_3^1\left(\int_0^t \alpha_{1_s} Q_2^{(0)}(s)ds\right) - \Omega_4^1\left(\int_0^t \rho_s\theta_{1_s}(Q_2^{(1)}(s) - k_{1_s})^+ ds\right)$$

$$-\int_0^t \theta_{1_s}\rho_s\left[(Q_2^{(1)}(s))^+ 1_{\{(Q_2^{(0)}(s)\geq k_{1_s}\}} - (Q_2^{(1)}(s))^- 1_{\{(Q_2^{(0)}(s)>k_{1_s}\}}\right]ds$$

$$-\int_0^t \alpha_{1_s} Q_2^{(1)}(s)ds.$$

(3.12)

$$Q_3^{(1)}(t) = Q_3^{(1)}(0) + \Omega_1^2\left(\int_0^t (Q_1^{(0)}(s) - c_s)^+ \delta_{2_s}(1-\phi_s)ds\right) + \Omega_2^2\left(\int_0^t \omega_{2_s} ds\right)$$

$$+\int_0^t \left[Q_1^{(1)}(s)^+ 1_{\{Q_1^{(0)}(s)\geq c_s\}} - Q_1^{(1)}(s)^- 1_{\{Q_1^{(0)}(s)>c_s\}}\right]\delta_{2_s}(1-\phi_s)ds$$

(3.13)

$$-\Omega_3^2\left(\int_0^t \alpha_{2_s} Q_3^{(0)}(s)ds\right) - \Omega_4^2\left(\int_0^t \rho_s\theta_{2_s}(Q_3^{(1)}(s) - k_{2_s})^+ ds\right)$$

$$-\int_0^t \theta_{2_s}\rho_s\left[Q_3^{(1)}(s)^+ 1_{\{(Q_3^{(0)}(s)\geq k_{2_s}\}} - Q_3^{(1)}(s)^- 1_{\{(Q_3^{(0)}(s)>k_{2_s}\}}\right]ds$$

$$-\int_0^t \alpha_{2_s} Q_3^{(1)}(s)ds.$$

Getting these equations is based essentially on the theorem 2.1 and 2.2.
The following result provides ordinary differential equations for the mean vector, variance and covariance matrices of $Q_i^{(1)}$.

Theorem 3.2. *The mean vector for the diffusion limit solves the set of differential equations*

(3.14)

$$\frac{d}{dt}\mathbb{E}(Q_1^{(1)}(t)) = \left((\mu_{1_t} + \mu_{2_t})1_{\{Q_1^{(0)}(t)\leq c_t\}} + (\delta_{1_t} + \delta_{2_t})1_{\{Q_1^{(0)}(t)>c_t\}}\right)\mathbb{E}(Q_1^{(1)}(t)^-)$$

$$-\left((\mu_{1_t} + \mu_{2_t})1_{\{Q_1^{(0)}(t)<c_t\}} + (\delta_{1_t} + \delta_{2_t})1_{\{Q_1^{(0)}(t)\geq c_t\}}\right)\mathbb{E}(Q_1^{(1)}(t)^+)$$

$$+\alpha_{1_t}\mathbb{E}(Q_1^{(2)}(t)) + \alpha_{2_t}\mathbb{E}(Q_1^{(3)}(t)).$$

$$
\begin{aligned}
\frac{d}{dt}\mathbb{E}(Q_2^{(1)}(t)) = {} & \delta_{1_t}(1-\phi_t)\left(\mathbb{E}(Q_1^{(1)}(t)^+)1_{\{Q_1^{(0)}(t)\geq c_t\}} - \mathbb{E}(Q_1^{(1)}(t)^-)1_{\{Q_1^{(0)}(t)>c_t\}}\right) \\
& - \left(\rho_t\theta_{1_t}1_{\{Q_2^{(0)}(t)\geq k_{1_t}\}}\right)\times\mathbb{E}\left(Q_2^{(1)}(t)\right) - \alpha_{1_t}\mathbb{E}(Q_2^{(1)}(t)).
\end{aligned} \tag{3.15}
$$

$$
\begin{aligned}
\frac{d}{dt}\mathbb{E}(Q_3^{(1)}(t)) = {} & \delta_{2_t}(1-\phi_t)\left(\mathbb{E}(Q_1^{(1)}(t)^+)1_{\{Q_1^{(0)}(t)\geq c_t\}} - \mathbb{E}(Q_1^{(1)}(t)^-)1_{\{Q_1^{(0)}(t)>c_t\}}\right) \\
& - \left(\rho_t\theta_{2_t}1_{\{Q_3^{(0)}(t)\geq k_{2_s}\}}\right)\times\mathbb{E}\left(Q_3^{(1)}(t)\right) - \alpha_{2_t}\mathbb{E}(Q_3^{(1)}(t)).
\end{aligned} \tag{3.16}
$$

The covariance matrix for the diffusion limit solves the differential equations

$$
\begin{aligned}
\frac{d}{dt}Var(Q_1^{(1)}(t)) = {} & 2\left((\delta_{1_t}+\delta_{2_t})1_{\{Q_1^{(0)}(t)>c_t\}} + (\mu_{1_t}+\mu_{2_t})1_{\{Q_1^{(0)}(t)\leq c_t\}}\right) \\
& \times Cov(Q_1^{(1)}(t), Q_1^{(1)}(t)^-) + \lambda_{1_t} + \lambda_{2_t} + (\delta_{1_t}+\delta_{2_t})(Q_1^{(0)}(t)-c_t)^+ \\
& -2\left((\delta_{1_t}+\delta_{2_t})1_{\{Q_1^{(0)}(t)\geq c_t\}} + (\mu_{1_t}+\mu_{2_t})1_{\{Q_1^{(0)}(t)<c_t\}}\right) \\
& \times Cov(Q_1^{(1)}(t), Q_1^{(1)}(t)^+) + (\mu_{1_t}+\mu_{2_t})(Q_1^{(0)}(t)\wedge c_t) + \alpha_{1_t}Q_2^{(0)}(t) \\
& +\alpha_{2_t}Q_3^{(0)}(t) + 2\left[\alpha_{1_t}cov(Q_1^{(1)}(t), Q_2^{(1)}(t)) + \alpha_{2_t}cov(Q_1^{(1)}(t), Q_3^{(1)}(t))\right].
\end{aligned} \tag{3.17}
$$

$$
\begin{aligned}
\frac{d}{dt}Var(Q_2^{(1)}(t)) = {} & 2\delta_{1_t}(1-\phi_t)Cov(Q_2^{(1)}(t), Q_1^{(1)}(t)^+)1_{\{Q_1^{(0)}(t)\geq c_t\}} - 2\delta_{1_t}(1-\phi_t) \\
& \times Cov(Q_2^{(1)}(t), Q_1^{(1)}(t)^-)1_{\{Q_1^{(0)}(t)>c_t\}} - 2\alpha_{1_t}Var(Q_2^{(1)}(t)) \\
& +\delta_{1_t}(1-\phi_t)(Q_1^{(0)}(t)-c_t)^+ + \alpha_{1_t}Q_2^{(0)}(t) + \rho_t\theta_{1_t}(Q_2^{(0)}(t)-k_{1_t})^+ \\
& +\omega_{1_t} - 2\rho_t\theta_{1_t}1_{\{Q_2^{(0)}(t)\geq k_{1_t}\}}Var(Q_2(t)).
\end{aligned} \tag{3.18}
$$

$$
\begin{aligned}
\frac{d}{dt}Var(Q_3^{(1)}(t)) = {} & 2\delta_{2_t}(1-\phi_t)Cov(Q_1^{(1)}(t)^+, Q_3^{(1)}(t))1_{\{Q_1^{(0)}(t)\geq c_t\}} - 2\delta_{2_t}(1-\phi_t) \\
& \times Cov(Q_1^{(1)}(t)^-, Q_3^{(1)}(t))1_{\{Q_1^{(0)}(t)>c_t\}} - 2\alpha_{2_t}Var(Q_3^{(1)}(t)) \\
& +\delta_{2_t}(1-\phi_t)(Q_1^{(0)}(t)-c_t)^+ + \alpha_{2_t}Q_3^{(0)}(t) + \rho_t\theta_{2_t}(Q_3^{(0)}(t)-k_{2_t})^+ \\
& +\omega_{2_t} - 2\rho_t\theta_{2_t}1_{\{Q_3^{(0)}(t)\geq k_{2_t}\}}Var(Q_3(t)).
\end{aligned} \tag{3.19}
$$

(3.20)

$$\begin{aligned}\frac{d}{dt}Cov(Q_1^{(1)}(t),Q_2^{(1)}(t)) = & \Big((\mu_{1_t}+\mu_{2_t})1_{\{Q_1^{(0)}(t)\le c_t\}}+(\delta_{1_t}+\delta_{2_t})1_{\{Q_1^{(0)}(t)>c_t\}}\Big) \\ & \times Cov((Q_1^{(1)}(t))^-,Q_2^{(1)}(t))-\alpha_{1_t}Cov(Q_1^{(1)}(t),Q_2^{(1)}(t)) \\ & -\Big((\mu_{1_t}+\mu_{2_t})1_{\{Q_1^{(0)}(t)<c_t\}}+(\delta_{1_t}+\delta_{2_t})1_{\{Q_1^{(0)}(t)\ge c_t\}}\Big) \\ & \times Cov((Q_1^{(1)}(t))^+,Q_2^{(1)}(t))+\delta_{1_t}(1-\phi_t)(Q_1^{(0)}(t)-c_t)^+ \\ & -\Big(\theta_{1_t}\rho_t 1_{\{Q_2^{(0)}(t)\ge k_{1_t}\}}\Big)Cov(Q_1^{(1)}(t),Q_2^{(1)}(t)) \\ & +\delta_{1_t}(1-\phi_t)1_{\{Q_1^{(0)}(t)\ge c_t\}}Var(Q_1^{(1)}(t))+\alpha_{1_t}Var(Q_2^{(1)}(t)) \\ & +\alpha_{2_t}Cov(Q_3^{(1)}(t),Q_2^{(1)}(t))+\alpha_{1_t}Q_2^{(0)}(t).\end{aligned}$$

$\frac{d}{dt}Cov(Q_1^{(1)}(t),Q_3^{(1)}(t))$ *will be given easily, in the same manner.*

(3.21)

$$\begin{aligned}\frac{d}{dt}Cov(Q_2^{(1)}(t),Q_3^{(1)}(t)) = & \ \delta_{2_t}(1-\phi_t)1_{\{Q_1^{(0)}(t)\ge c_t\}}Cov(Q_2^{(1)}(t),Q_1^{(1)}(t))+\delta_{1_t}(1-\phi_t) \\ & \times 1_{\{Q_1^{(0)}(t)\ge c_t\}}Cov(Q_3^{(1)}(t),Q_1^{(1)}(t))-\Big(\alpha_{1_t}+\alpha_{2_t}+\theta_{1_t}\rho_t \\ & \times 1_{\{Q_2^{(0)}(t)\ge k_{1_t}\}}+\theta_{2_t}\rho_t 1_{\{Q_3^{(0)}(t)\ge k_{2_t}\}}\Big)Cov(Q_2^{(1)}(t),Q_3^{(1)}(t)).\end{aligned}$$

The proof of this theorem is based on Theorems 2.2 and 2.3; Given the integral equations (3.10)-(3.13) that $Q_i^{(1)}(t)$ solves, we immediately have for i=1,2,3

(3.22)

$$\mathbb{E}(Q_i^{(1)}(t))=\mathbb{E}(Q_i^{(1)}(0))+\int_0^t\mathbb{E}(\wedge\nu_s^{(0)}(Q_i^{(0)}(s),Q_i^{(1)}(s))ds+\int_0^t\nu_s^{(1)}(Q^{(0)}(s)).$$

Differentiating this equation we get (3.14), (3.15) and (3.16).

Then The solution to the integral equations (3.10)-(3.13) also solves the stochastic differential equation

(3.23)

$$d(Q_i^{(1)})(t)=(\wedge\nu_t^{(0)}(Q_i^{(0)}(t),Q_i^{(1)}(t)))+\nu_s^{(1)}(Q^{(0)}(t))dt+\sum_{i\in I}\sqrt{\nu_t^{(0)}(Q_i^{(0)}(t),i)}vi\ d\Omega_i^*(t).$$

Using Ito's formula [24] (page 149) we get

$$d((Q_i^{(1)}(t))^\top, Q_i^{(1)}(t)) = (\wedge\nu_t^{(0)}(Q_i^{(0)}(t), Q_i^{(1)}(t))) + \nu_t^{(1)}(Q_i^{(0)}(t))^\top \cdot Q_i^{(1)}(t)dt$$
$$+\sum_{i\in I}\sqrt{\nu_t^{(0)}(Q_i^{(0)}(t), i)}v_i^\top \cdot Q_i^{(1)}(t)d\Omega_i^*(t) + (Q_i^{(1)}(t))^\top$$
$$\cdot(\wedge\nu_t^{(0)}(Q_i^{(0)}(t), Q_i^{(1)}(t))) + \nu_t^{(1)}(Q_i^{(0)}(t))^\top dt$$
$$+\sum_{i\in I}\sqrt{\nu_t^{(0)}(Q_i^{(0)}(t), i)}(Q_i^{(1)}(t))^\top \cdot v_i d\Omega_i^*(t)$$
$$+\sum_{i\in I}\nu_t^{(0)}(Q_i^{(0)}(t), i)v_i^\top \cdot v_i dt.$$

Taking the expectations, we get

(3.24)

$$\tfrac{d}{dt}\mathbb{E}(Q_i^{(1)}(t))^\top, Q_i^{(1)}(t)) = \mathbb{E}\left((\wedge\nu_t^{(0)}(Q_i^{(0)}(t), Q_i^{(1)}(t))) + \nu_t^{(1)}(Q_i^{(0)}(t))^\top \cdot Q_i^{(1)}(t)\right)$$
$$+\mathbb{E}\left(Q_i^{(1)}(t)\cdot(\wedge\nu_t^{(0)}(Q_i^{(0)}(t), Q_i^{(1)}(t))) + \nu_t^{(1)}(Q_i^{(0)}(t))^\top\right)$$
$$+\sum_{i\in I}\nu_t^{(0)}(Q_i^{(0)}(t), i)v_i^\top \cdot v_i.$$

for almost all t. Using the derivative of (3.22), we obtain

(3.25)

$$\mathbb{E}(Q_i^{(1)}(t))^\top)\mathbb{E}(Q_i^{(1)}(t)) = \mathbb{E}(\wedge\nu_t^{(0)}(Q_i^{(0)}(t), Q_i^{(1)}(t))) + \nu_t^{(1)}(Q_i^{(0)}(t))^\top \cdot \mathbb{E}(Q_i^{(1)}(t)$$
$$+\mathbb{E}(Q_i^{(0)}(t))^\top \cdot \mathbb{E}(\wedge\nu_t^{(0)}(Q_i^{(0)}(t), Q_i^{(1)}(t))) + \nu_t^{(1)}(Q_i^{(0)}(t)).$$

Subtracting (3.26) from (3.24) gives us (3.17)-(3.21).

Now, observe that (3.22) can be written as

$$\mathbb{E}(Q_i^{(1)}(t)) = \mathbb{E}(Q_i^{(1)}(0)) + \int_0^t \mathbb{E}(Q_i^{(1)}(s))A_s ds + \int_0^t \nu_s^{(1)}(Q^{(0)}(s)). \tag{3.26}$$

With A_t is the matrix that represents its action on $\mathbb{V}$;

$$\nu_t^{(0)}(Q_i^{(0)}(t), Q_i^{(1)}(t))) = Q_i^{(1)}(t))A_t,$$

and $|A_t| \leq \|\nu_t^{(0)}\|$ So, let

$$cov(Q_i^{(1)}(t), \wedge\nu_t^{(0)}(Q_i^{(0)}(t), Q_i^{(1)}(t)))) = cov(Q_i^{(1)}(t), Q_i^{(1)}(t))))A_t,$$

for almost all t, and so the integral equation for the covariance matrix is

$$cov(Q_i^{(1)}(t), Q_i^{(1)}(t)) = cov(Q_i^{(1)}(0), Q_i^{(1)}(0)) + \int_0^t cov(Q_i^{(1)}(s), Q_i^{(1)}(s))A_s ds$$
$$+\int_0^t \sum_{i\in I}\nu_t^{(0)}(Q_i^{(0)}(t), i)v_i^\top \cdot v_i ds.$$

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Mathematics Laboratory, Djillali LIABES University of Sidi Bel Abbes, B.P. 89, Sidi Bel Abbes 22000, Algeria.

E-mail address: bouchentouf_amina@yahoo.fr

Mathematics Laboratory, Djillali LIABES University of Sidi Bel Abbes, B.P. 89, Sidi Bel Abbes 22000, Algeria.

E-mail address: rabhi_abbes@yahoo.fr

Mathematics Laboratory, Moulay Tahar University, Saida, B.P. 138, En-Nasr Saida 20 000, Algeria.

E-mail address: lahceneya8@gmail.com