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# A Note on Freedom from Detachment in the Logic of Paradox

Jc Beall, Thomas Forster, and Jeremy Seligman

**Abstract** We shed light on an old problem by showing that the logic LP cannot define a binary connective $\odot$ obeying detachment in the sense that every valuation satisfying $\varphi$ and $(\varphi \odot \psi)$ also satisfies $\psi$, except trivially. We derive this as a corollary of a more general result concerning variable sharing.

## 1 Introduction

One approach to resolving logico-cum-semantic paradoxes (see Beall [4], Goodship [8]) is to reject the existence of any *detachable conditional* or, more generally, any *detachable connective*—a binary connective $\odot$ for which "modus ponens" holds (i.e., $\varphi$ and $\varphi \odot \psi$ jointly imply $\psi$). There is a roundabout proof that LP, the logic of paradox (see Asenjo [2], Priest [12]), is "detachment-free," and so suitable for such an approach to paradox. The argument first shows, via a Kripke construction (see Dowden [6], Woodruff [15]), that target LP truth theories (or, similarly, "naive set" theories) are "nontrivial" (i.e., that while such theories are negation-inconsistent, not all sentences are true in them); in turn, one notes that if LP contained a detachable connective, the theories would be trivial (i.e., contain all sentences), and concludes that LP does not contain a detachable connective.

In this note, we offer a more direct, much simpler proof that LP is "detachment-free" (in a sense to be defined) by showing that LP has a surprisingly strong variable-sharing property. We review LP in Section 2, set up terminology in Section 3, and give the result in Section 4. We close in Section 5 with a few remarks on a related logic in the vicinity of LP.

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## 2 The Logic LP

LP has the unary connective $\neg$ and binary connectives $\wedge$ and $\vee$. A *valuation* for LP is a function $v$ from the primitive propositions to $\wp(\{0,1\}) \setminus \{\emptyset\}$ (see Dunn [7]). Intuitively, values $\{1\}$ and $\{0\}$ correspond to the standard classical values of "just true" and "just false," respectively, while $\{1,0\}$ is the nonclassical value "both" (or, if you like, "deviant"). For our purposes here, we pass quickly by further discussion of the philosophical interpretation of LP's semantic values, leaving this to other work on the topic (see Beall [3], Belnap [5], Lewis [10], Priest [13]).

An LP valuation $v$ is extended to a function $v^*$ on formulas with the same recursive clauses used in classical logic:[1]

$$\begin{aligned} v^*(p) &= v(p) \text{ when } p \text{ is a propositional letter;} \\ v^*(\neg\varphi) &= \big\{1-n : n \in v^*(\varphi)\big\}; \\ v^*(\psi \wedge \varphi) &= \big\{\min\{n,m\} : n \in v^*(\varphi) \text{ and } m \in v^*(\psi)\big\}; \\ v^*(\psi \vee \varphi) &= \big\{\max\{n,m\} : n \in v^*(\varphi) \text{ and } m \in v^*(\psi)\big\}. \end{aligned}$$

We say that a valuation $v$ *satisfies* a formula iff $1 \in v^*(\psi)$. As with its classical counterpart, satisfaction has the *agreement* property: if $v_1(p) = v_2(p)$ for all propositional variables in $\varphi$, then $v_1^*(\varphi) = v_2^*(\varphi)$. We say that a valuation $v$ *equivocates on $p$* if $v(p) = \{0,1\}$.[2]

**Remark 2.1** Any formula $\varphi$ is satisfied by any valuation that equivocates on all the propositional variables appearing in $\varphi$.

**Proof** By structural induction, if $v(p) = \{0,1\}$ for each propositional variable $p$, then also $v^*(\varphi) = \{0,1\}$. The result follows from agreement. □

Finally, where $\Gamma$ is any set of formulas, we define the (model-theoretic or "semantic") entailment relation $\models$ in familiar terms:

$$\Gamma \models \varphi \text{ iff any valuation that satisfies } \Gamma \text{ satisfies } \varphi.$$

We follow standard conventions of abbreviation for the set of premises, writing $\models \varphi$ for $\emptyset \models \varphi$ and $\Gamma_1, \Gamma_2 \models \varphi$ for $\Gamma_1 \cup \Gamma_2 \models \varphi$.

LP owes its usefulness for reasoning in the face of contradiction (e.g., paradox) to the fact that, unlike classical logic, it is *paraconsistent*, meaning that $(\varphi \wedge \neg\varphi) \not\models \psi$.[3] But LP is not an "anticlassical" logic: it is not only a sublogic of classical logic (anything LP-valid is classically valid), but it also enjoys the tautologies of classical logic: $\models \varphi$ for every (classical-logic) tautology $\varphi$ (see [12]).

## 3 Detachment-Free Logics

Let $\odot$ be a binary connective that is definable in LP in the sense that there is a formula $\varphi(p,q)$ of our language such that $(\psi_1 \odot \psi_2)$ is an abbreviation for $\varphi(\psi_1, \psi_2)$, namely, the result of replacing $p$ by $\psi_1$ and $q$ by $\psi_2$ in $\varphi$. We say that $\odot$ *obeys detachment* if and only if for all $\psi_1$ and $\psi_2$,

$$\psi_1, (\psi_1 \odot \psi_2) \models \psi_2.$$

It is standard in the literature that the usual candidate for such a connective, to wit, *the hook* or *horseshoe*, defined by

$$(\varphi \supset \psi) := (\neg\varphi \vee \psi),$$

fails to obey detachment because of any valuation $v$ such that $v(\varphi) = \{0, 1\}$ and $v(\psi) = \{0\}$. Such a valuation is the key to showing the results below.

Finally, a connective might obey detachment *trivially*, for example, conjunction ($\wedge$). Trivially? Yes: $\varphi, (\varphi \wedge \psi) \models \psi$ but only because $(\varphi \wedge \psi) \models \psi$. Such cases are of no interest to us: we say that they are *trivial*.[4]

## 4 LP is Detachment-Free

**Theorem 4.1** *No connective definable in LP nontrivially obeys detachment.*

We derive this as a corollary of a more general (and rather striking) result.[5]

**Theorem 4.2** *If* $\Gamma_1, \Gamma_2 \models \varphi$ *and none of the formulas in* $\Gamma_1$ *contain propositional variables that also appear in* $\varphi$*, then* $\Gamma_2 \models \varphi$.

**Proof** Let $v$ be a valuation that satisfies everything in $\Gamma_2$. Modify $v$ to $v'$ by making it equivocate on all the variables that do not appear in $\varphi$. We are assuming that each formula $\psi$ in $\Gamma_1$ contains only variables that do not appear in $\varphi$ and so is satisfied by $v'$ (by Remark 2.1). But then $v'$ satisfies everything in $\Gamma_1$ and in $\Gamma_2$ and so also satisfies $\varphi$. Finally, by the agreement property (see Section 2), $v$ also satisfies $\varphi$. $\square$

Although, strictly speaking, LP is not a "relevant(-ance) logic" because of examples such as $\varphi \models (\psi \vee \neg\psi)$, Theorem 4.2 shows why LP is "almost relevant" in the sense that if $\Gamma \models \varphi$, then $\Gamma$ and $\varphi$ must share a variable, unless $\models \varphi$.

Finally, observe that Theorem 4.1 is an easy corollary of Theorem 4.2: the formula $p$ fails to contain the variable $q$; so, if $p, (p \odot q) \models q$, then $(p \odot q) \models q$. The result can also be strengthened by a result of Arieli, Avron, and Zamansky [1], according to which LP is a "maximally paraconsistent logic" in the sense that there is no proper paraconsistent extension of the entailment relation $\models$ satisfying some fairly minimal conditions. In particular, there can be no paraconsistent way of adding a rule of detachment to any LP-definable connective, via some proof-theoretic presentation or alternative semantics.[6] Nonetheless, it is possible to add a *new* detachable connective to LP. Several examples of such connectives have been considered in the literature, notably the relevant logic RM3 and the logic $L^{\supset}$ from Middelburg [11] defined by the following tables:

| | $\{1\}$ | $\{0,1\}$ | $\{0\}$ |
|---|---|---|---|
| $\{1\}$ | $\{1\}$ | $\{0\}$ | $\{0\}$ |
| $\{0,1\}$ | $\{1\}$ | $\{0,1\}$ | $\{0\}$ |
| $\{0\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |

RM3

| | $\{1\}$ | $\{0,1\}$ | $\{0\}$ |
|---|---|---|---|
| $\{1\}$ | $\{1\}$ | $\{0,1\}$ | $\{0\}$ |
| $\{0,1\}$ | $\{1\}$ | $\{0,1\}$ | $\{0\}$ |
| $\{0\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |

$\mathrm{LP}^{\supset}$

From the results of Ariely, Avron, and Zemansky [1], these two logics are also maximally paraconsistent. $L^{\supset}$ but not RM3 also has the deduction theorem. Nonetheless, neither of these logics can claim to be a logic of *paradox*; both fall to Curry's paradox, in the form $(p \leftrightarrow (p \rightarrow q)) \models q$, which leads to triviality when the logic is applied to theories of truth, sets, or properties that allow self-reference, specifically, for each proposition $q$ a proposition $p$ that is arrow-equivalent to $(p \rightarrow q)$.

## 5 Detachment in Some Closely Related Logics

One might think that the definition of entailment in LP is a little biased toward truth. Let us say that a valuation $v$ *falsifies* $\varphi$ iff $0 \in v^*(\varphi)$. Then a reasonable requirement for $\varphi$ to be a consequence of $\Gamma$ is "backward falsity-preservation," namely, that there is no valuation that falsifies the conclusion $\varphi$ without also falsifying one of the premises in $\Gamma$. We will write this as $\Gamma \Dashv \varphi$. Since $(p \wedge \neg p)$ is falsified by every valuation, $(p \wedge \neg p) \Dashv q$ holds, and so $\Dashv$ is not itself paraconsistent. Moreover, $\Dashv$ lacks LP's property of sharing the tautologies of classical logic. In fact, $\Dashv$ is *paracomplete*,[7] meaning that $\not\Dashv (p \vee \neg p)$.[8] On a more positive note, it has a detachable connective:[9]

$$\varphi, (\neg\varphi \vee \psi) \Dashv \psi.$$

Despite this, $\Dashv$ may still play a role in detachment-free approaches to paradox if we take both it and $\models$ to be necessary and jointly sufficient conditions for entailment. In particular, define $|\!\models$ as follows:

$$\Gamma \,|\!\models \varphi \quad \text{iff } \Gamma \models \varphi \text{ and } \Gamma \Dashv \varphi.$$

The combined logic $|\!\models$ is both paraconsistent and paracomplete and is detachment-free for the obvious reason that if $\varphi, (\varphi \odot \psi) \,|\!\models \psi$, then $\varphi, (\varphi \odot \psi) \models \psi$, which we have seen in Theorem 4.1 not to be the case. Interestingly, the analogue of Theorem 4.2 does not apply, as the following fact implies: $(\varphi \wedge \neg\varphi) \,|\!\models (\psi \vee \neg\psi)$.[10]

## Notes

1. This is equivalent to the well-known "strong Kleene" (K3) valuations. In particular, the table for negation is given by

| $\varphi$ | $\neg\varphi$ |
|---|---|
| $\{0\}$ | $\{1\}$ |
| $\{0,1\}$ | $\{0,1\}$ |
| $\{1\}$ | $\{0\}$ |

K3 and LP differ not in the set of valuations but in their accounts of satisfaction—the latter "designating" both $\{1\}$ and $\{1,0\}$ while K3 designates only $\{1\}$. See below.

2. D. Lewis [10] suggests an interpretation of the third truth value of LP (and related logics) as representing an ambiguity between true and false readings of a sentence, and relates this to the fallacy of equivocation, but we do not intend any more direct connection to his work.

3. The terminology comes from the thought of having coherent theories that go "beyond consistency," so-called negation-inconsistent but nontrivial theories: theories $T$ such that, for some $\varphi$ and $\psi$, both $\varphi \in T$ and $\neg\varphi \in T$ but $\psi \notin T$.

4. To be precise, say that $\odot$ satisfies detachment *trivially* if for all $\psi_1$ and $\psi_2$, $(\psi_1 \odot \psi_2) \models \psi_2$. Other trivially detaching connectives are obtained by taking any formula $\varphi(\varphi, \psi)$ that entails $\psi$ as the definition of $\odot$, for example, $\psi$, $\neg\neg\psi$, $\neg(\psi \supset \varphi)$, and so on.

5. Our first reaction to this theorem was to describe it as "interpolation-like," but an anonymous referee convinced us that the matter is delicate. As the referee noted, Takano's result in [14] immediately delivers interpolation for LP; however, it is not clear that interpolation is really of relevance to detachment freedom.

6. Our thanks to the journal's anonymous referee for this insight. The possibility of adding a detachable connective, specifically $\mathrm{LP}^{\supset}$, was pointed out to us by Koji Tanaka and Patrick Girard.

7. The terminology comes from the thought of having coherent theories that go "beyond completeness," so-called negation-incomplete but nonempty theories: theories $T$ such that, for some $\varphi$ and $\psi$, both $\varphi \notin T$ and $\neg\varphi \notin T$ but $\psi \in T$.

8. $\mathrel{=\!\!|}$ is "dual" to $\models$ in that $\varphi \mathrel{=\!\!|} \neg\psi$ iff $\psi \models \neg\varphi$. In fact, this logic is none other than Kleene's (see [9]) strong three-valued logic K3 in disguise. Satisfaction for K3 is defined by taking $\{1\}$ to be the only designated value, so that a valuation K3 satisfies $\varphi$ iff it does not falsify $\varphi$ (see note 5).

9. Proof: Suppose for contradiction that $v$ falsifies $\psi$ but neither premise; $0 \notin v^*(\varphi)$ so $v^*(\varphi) = \{1\}$. But then $v^*(\neg\varphi) = \{0\}$, and so $0 \in v^*(\neg\varphi \vee \psi)$, contradicting the assumption that $v$ does not falsify $(\neg\varphi \vee \psi)$.

10. Proof: We have $(\varphi \wedge \neg\varphi) \mathrel{=\!\!|} (\psi \vee \neg\psi)$ since $\mathrel{=\!\!|}$ is not paraconsistent; $(\varphi \wedge \neg\varphi) \models (\psi \vee \neg\psi)$ because $\models$ is not paracomplete.

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Beall
Department of Philosophy and UConn Logic Group
University of Connecticut
Storrs, Connecticut 06268
USA
jc.beall@uconn.edu
http://entailments.net

Forster
Department of Philosophy
Auckland University
Arts 2 Building
18 Symonds Street
Auckland, New Zealand
T.Forster@dpmms.cam.ac.uk
http://www.dpmms.cam.ac.uk/~tf/

Seligman
Dept. of Philosophy
Auckland University
Arts 2 Building
18 Symonds Street
Auckland, New Zealand
j.seligman@auckland.ac.nz
http://auckland.academia.edu/JeremySeligman