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A note on Frobenius inner product and the m-distance matrices of a tree

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Abstract
The m-distance matrix D_m of a simple connected undirected graph has an important role in computing the distance matrix D of the graph from the powers of the adjacency matrix using Hadamard product. This paper shows that for an undirected tree T with diameter d, $\{D_0.D_1,...,D_d\}$ is an orthogonal basis for the space spanned by the binary equivalent matrices of the first $d+1$ powers of the adjacency matrix of T and it gives an invertible conversion matrix for finding the m-distance matrix of T using Frobenius inner product on matrices.

Keywords
Adjacency matrix, Distance matrix, Binary matrix , Diameter, Hadamard product, m-distance matrix , Frobenius inner product, Frobenius norm.

AMS Subject Classification
05C12, 05C40, 05C50, 05C62, 05B20, 46C05, 15A60.

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Contents

1. Introduction

Consider a simple, connected, undirected graph $G=(V,E)$ of order n with vertex set V and edge set E throughout this paper unless otherwise specified. Let A_G be the $n\times n$ adjacency matrix [1] of G. Then ij^{th} entry of A_G^m (m^{th} power of A_G), represent the number of walks of length m between the vertices v_i and v_j of G.

Definition 1.1. *([2]) The distance matrix, $D=(d_{ij})$ of G is defined as,*

$$d_{ij}=\begin{cases} d(v_i,v_j) & ,\text{if } i\neq j\\ 0 & ,\text{if } i=j\end{cases}$$

Where, $d(v_i,v_j)$ is the distance between the vertices v_i and v_j.

Definition 1.2. *([3]) The diameter of a graph G is the maximum distance between any two vertices of G and it is denoted by $Diam(G)$.*

Definition 1.3. *(Hadamard product [4]) Consider the vector space $\mathbb{R}^{m\times n}$ of all $m\times n$ real matrices over the real field $\mathbb{R}$. For P, $F\in\mathbb{R}^{m\times n}$, the Hadamard product $\circ$ is a binary operation on $\mathbb{R}^{m\times n}$ defined by,*

$$(P\circ F)_{ij}:=(P)_{ij}(F)_{ij},\quad \forall\, i,j.$$

Let $B=\{0,1\}$ and let $B_{m\times n}$ denote the set of all binary matrices in $\mathbb{R}^{m\times n}$.
Then for P, $F\in B_{m\times n}$,

$$(P\circ F)_{ij}=\begin{cases} 1, & \text{if } p_{ij}=1 \text{ and } f_{ij}=1\\ 0, & \text{otherwise}\end{cases}$$

Definition 1.4. *(Frobenius inner product [5]) Consider the vector space $\mathbb{R}^{n\times n}$ over the field $\mathbb{R}$. Then the Frobenius inner product $\langle,\rangle_F:\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}\to\mathbb{R}$ is defined by,*

$$\langle P,Q\rangle_F=\sum_{i=1}^{n}\sum_{j=1}^{n}p_{ij}q_{ij}=\sum_{i=1}^{n}\sum_{j=1}^{n}(P\circ Q)_{ij}=Tr(PQ^T),$$

where $P,Q\in\mathbb{R}^{n\times n}$. The Frobenius norm $\|\cdot\|$ induced by this inner product on $\mathbb{R}^{n\times n}$ is, $\|P\|_F=(\langle P,P\rangle_F)^{\frac{1}{2}}$, $P\in\mathbb{R}^{n\times n}$. Then *the metric d_F induced by this norm on $\mathbb{R}^{n\times n}$ is $d_F(P,Q) = \|P-Q\|_F$, $P,Q \in \mathbb{R}^{n\times n}$.*

Remark 1.5. *(i)* $\|P\|_F = (\langle P,P\rangle_F)^{\frac{1}{2}} = \left(Tr(PP^T)\right)^{\frac{1}{2}}$, $P \in \mathbb{R}^{n\times n}$

(ii) Let $X,Y \in \mathbb{R}^{1\times n}$. Then $\langle X,Y\rangle_F = \sum_{i=1}^{n} x_i y_i$ and

$$d_f(X,Y) = \|X-Y\|_F = \left(\sum_{i=1}^{n}(x_i-y_i)^2\right)^{\frac{1}{2}}$$

which is the Euclidean distance between the vectors X and Y of $\mathbb{R}^n$.

(iii) For $P \in B_{n\times n}$,

$$d_F(P,0) = \|P-0\|_F = \|P\|_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}^2\right)^{\frac{1}{2}}$$

Then $(d_F(P,0))^2$ represents the number of 1's in P. For $P,Q \in B_{n\times n}$,

$$d_F(P,Q) = \|P-Q\|_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{n}(p_{ij}-q_{ij})^2\right)^{\frac{1}{2}}$$

Then $(d_F(P,Q))^2$ represent the number of non identical entries in P and Q and the number of identical entries in P and Q is $n^2-(d_F(P,Q))^2$

2. Preliminaries

Definition 2.1. *([6]) The function $\delta : \mathbb{R} \to B = \{0,1\}$ is defined by*

$$\delta(a) := \begin{cases} 0, & a \le 0 \\ 1, & \text{otherwise} \end{cases}, a \in \mathbb{R}$$

Also, $\delta : \mathbb{R}^{m\times n} \to B_{m\times n}$ is defined by $\delta(D) = (\delta(D))_{ij} := (\delta(d_{ij}))$, $\forall\ i,j$ and $D \in \mathbb{R}^{m\times n}$. Then $\delta(A_G^m) \in B_{n\times n}$, $\forall\ m \in \{0,1,2,\dots\}$ and it is the equivalent binary matrix representation of A_G^m. Let $A_G^{(m)} = \delta(A_G^m)$. Then

$$(A_G^{(m)})_{ij} = \begin{cases} 1, & \text{if there exist a walk of length } m \\ & \text{from } v_i \text{ to } v_j \text{ in } G \\ 0, & \text{otherwise} \end{cases}$$

Also, for $C,F \in B_{m\times n}$, $\delta(C\circ F) = \delta(C)\circ\delta(F)$.

Definition 2.2. *(m-distance matrix [6]) Consider the graph $G=(V,E)$ with n vertices. Then the m- distance matrix D_m of G is an $n\times n$ symmetric binary matrix defined by*

$$(D_m)_{ij} := \begin{cases} 1, & \text{if } d(v_i,v_j) = m \\ 0, & \text{otherwise} \end{cases}$$

where, $d(v_i,v_j)$ is the distance between the vertices v_i and v_j.

Remark 2.3. *(i)* $D_m \in B_{n\times n}$, $\forall\ m = 0,1,2,\dots$

(ii) $D_0 = I_n$

(iii) $D_0 + D_1 + \dots + D_d =$

$$J_n = \begin{bmatrix} 1 & 1 & . & . & . & 1 \\ 1 & 1 & . & . & . & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 1 & . & . & . & 1 \end{bmatrix} = \text{The matrix of ones}$$

(iv) $$Diam(G) = \underset{m}{Max}\{m : D_m \ne 0\}$$

(v) $D_i \circ D_j = \begin{cases} 0, & \text{if } i \ne j \\ D_i, & \text{if } i = j \end{cases}$, for $0 \le i,j \le d$.
i.e., $\{D_0, D_1, \dots, D_d\}$ is an orthogonal subset $\mathbb{R}^{n\times n}$ with respect to the binary operation $\circ$.

Theorem 2.4. *([6]) Let A_G be the adjacency matrix of $G = (V,E)$ with diameter d. Then $\beta_2 = \{D_0, D_1, \dots, D_d\}$ is a Linearly independent subset of the vector space $\mathbb{R}^{n\times n}$.*

Theorem 2.5. *([6]) Let A_G be the adjacency matrix of $G = (V,E)$ with diameter d. Then for $1 \le m \le d$, $D_m = A_G^{(m)} - \delta\left(\sum_{s=0}^{m-1} A_G^{(m)} \circ A_G^{(s)}\right)$, where D_m is the m-distance matrix of G.*

3. Main Resuls

Theorem 3.1. *Let A_G be the adjacency matrix of $G = (V,E)$. Let $d = diam(G)$. Then for $0 \le K, m \le d$,*

$$D_k \circ A_G^{(m)} = \begin{cases} D_m, & \text{if } K = m \\ 0, & \text{if } m < K. \end{cases}$$

Proof. **Case(i):** $K = m$

$$\begin{aligned}\left(D_m \circ A_G^{(m)}\right)_{ij} &= (D_m)_{ij}(A_G^{(m)})_{ij} \\ &= \begin{cases} 1, & \text{if } (D_m)_{ij} = 1 \text{ and } (A_G^{(m)})_{ij} = 1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{if } d(v_i,v_j) = m \text{ and } (A_G^{(m)})_{ij} = 1 \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

But $d(v_i,v_j) = m \Rightarrow (A_G^{(m)})_{ij} = 1$

$$\begin{aligned}\therefore \left(D_m \circ A_G^{(m)}\right)_{ij} &= (D_m)_{ij}(A_G^{(m)})_{ij} \\ &= \begin{cases} 1, & \text{if } d(v_i,v_j) = m \\ 0, & \text{otherwise} \end{cases} = (D_m)_{ij}\end{aligned}$$

That is, $(D_m \circ A_G^{(m)})_{ij} = D_m$.
Case(ii): $K > m$
$(D_k \circ A_G^{(m)})_{ij} = (D_k)_{ij}(A_G^{(m)})_{ij}$

If $(D_k)_{ij} = 1$, then $d(v_i, v_j) = K$.
That is, if there exist a shortest path of length K between v_i and v_j. But then this path cannot be traversed by any of the walk between v_i and v_j of length$< K$. Thus $(A_G^{(m)})_{ij} = 0$. $(\because K > m)$
$\therefore (D_K)_{ij}(A_G^{(m)})_{ij} = 0$
If $(D_K)_{ij} = 0$, then also $(D_K)_{ij}(A_G^{(m)})_{ij} = 0$.
Therefore, $(D_K \circ A_G^{(m)})_{ij} = 0, \forall\, i, j, \forall\, K > m$.
ie., $D_K \circ A_G^{(m)} = 0$ if $K > m$. Hence

$$D_K \circ A_G^{(m)} = \begin{cases} D_m, & \text{if } K = m \\ 0, & \text{if } m < K \end{cases}, \text{ for } K \leq d.$$

□

Theorem 3.2. *Let A_G be the adjacency matrix of $G = (V, E)$. Let $d = diam(G)$. Then $\beta_1 = \{A_G^{(0)}, A_G^{(1)}, \ldots A_G^{(d)}\}$ is a linearly independent subset of the vector space $\mathbb{R}^{n\times n}$.*

Proof. Suppose

$$c_0 \cdot A_G^{(0)} + c_1 \cdot A_G^{(1)} + \ldots + c_d \cdot A_G^{(d)} = 0 \tag{3.1}$$

for some $c_0, c_1, \ldots, c_d \in \mathbb{R}$. By taking Hadamard product $\circ$ by D_d

$$(3.1) \Rightarrow D_d \circ (c_0 \cdot A_G^{(0)} + c_1 \cdot A_G^{(1)} + \ldots + c_d \cdot A_G^{(d)}) = 0$$
$$c_0 \cdot (D_d \circ A_G^{(0)}) + c_1 \cdot (D_d \circ A_G^{(1)}) + \ldots + c_d \cdot (D_d \circ A_G^{(d)}) = 0$$
$$\Rightarrow 0 + 0 + \ldots 0 + c_d \cdot (D_d \circ A_G^{(d)}) = 0 (\because Theorem\ 3.1)$$
$$\Rightarrow c_d \cdot D_d = 0 \Rightarrow c_d = 0, \text{ as } D_d \neq 0$$
$$(3.1) \Rightarrow c_0 \cdot A_G^{(0)} + c_1 \cdot A_G^{(1)} + \ldots + c_{d-1} \cdot A_G^{(d-1)} = 0 \tag{3.2}$$

By taking Hadamard product $\circ$ by D_{d-1} on both side of equation (3.2).

$$(3.2) \Rightarrow D_{d-1} \circ (c_0 \cdot A_G^{(0)} + c_1 \cdot A_G^{(1)} + \ldots + c_{d-1} \cdot A_G^{(d-1)}) = 0$$
$$c_0 \cdot (D_{d-1} \circ A_G^{(0)}) + c_1 \cdot (D_{d-1} \circ A_G^{(1)}) + \ldots$$
$$+ c_{d-1} \cdot (D_{d-1} \circ A_G^{(d-1)}) = 0$$
$$\Rightarrow 0 + 0 + \ldots + 0 + c_{d-1} \cdot (D_{d-1} \circ A_G^{(d-1)}) = 0$$
$$\Rightarrow c_{d-1} D_{d-1} = 0 \Rightarrow c_{d-1} = 0, \text{ as } D_{d-1} \neq 0$$
$$(3.2) \Rightarrow c_0 \cdot A_G^{(0)} + c_1 \cdot A_G^{(1)} + \ldots + c_{d-1} \cdot A_G^{(d-1)} = 0$$

Continue the above process further a finite number of times, we get $c_0 = 0$, $c_1 = 0, \ldots, c_d = 0$.
$\Rightarrow \beta_1 = \{A_G^{(0)}, A_G^{(1)}, \ldots, A_G^{(d)}\}$ is a linearly independent subset of $\mathbb{R}^{n\times n}$. □

Remark 3.3. $\beta_1 = \{A_G^{(0)}, A_G^{(1)}, \ldots, A_G^{(d)}\}$ *is a basis for span* $\{A_G^{(0)}, A_G^{(1)}, \ldots, A_G^{(d)}\}$.

Lemma 3.4. *Let $G = (V, E)$ be a connected undirected graph of order $n (n > 1)$. Let d_{ij} denote the distance between v_i and v_j. Then there always exist at least one walk between v_i and v_j of length $d_{ij} + 2r$, $\forall\, r = 0, 1, 2, \ldots$. That is,*

$$\left(A_G^{(d_{ij}+2r)}\right)_{ij} = 1, \forall r = 0, 1, 2, \ldots.$$

Proof. Since G is a connected undirected graph, there exist at least one path in between v_i and v_j. Let $P_{ij} : v_0 (= v_i) - v_1 - v_2 - \ldots - v_{d_{ij}-1} - v_{d_{ij}} (= v_j)$ denote the shortest path between v_i and v_j of length d_{ij}. If we traverse back and forth once along the last edge in P_{ij}, then we get a walk $v_0 (= v_i) - v_1 - v_2 - \ldots - v_{d_{ij}-1} - v_{d_{ij}} (= v_j) - v_{d_{ij}-1} - v_{d_{ij}} (= v_j)$ of length $d_{ij} + 2$ between v_i and v_j. Also If we traverse back and forth twice along the last edge in P_{ij}, then we get a walk $v_0 (= v_i) - v_1 - v_2 - \ldots - v_{d_{ij}-1} - v_{d_{ij}} - v_{d_{ij}-1} - v_{d_{ij}} - v_{d_{ij}-1} - v_{d_{ij}} (= v_j)$ of length $d_{ij} + 4$ between v_i and v_j. If we proceed like this, then we always get a walk of length $d_{ij} + 2r$, $\forall$ $r = 0, 1, 2, \ldots$ between v_i and v_j in G. This walk will reflect as 1 in the ij^{th} entry of $A_G^{(d_{ij}+2r)}$.

$$\therefore \left(A_G^{(d_{ij}+2r)}\right)_{ij} = 1, \forall\, r = 0, 1, 2, \ldots$$

□

Theorem 3.5. *Let $T = (V, E)$ be an undirected tree of order $n (n > 1)$. Let d_{ij} denote the distance between v_i and v_j. Then,*

$$\left(A_T^{(m)}\right)_{ij} = \begin{cases} 1, & \text{if } m = d_{ij} + 2r, \forall r = 0, 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

Proof. Let T_{ij} denote the unique path in T from v_i to v_j of distance d_{ij}. Consider a walk W_{ij} from v_i to v_j such that length of W_{ij}, $l(W_{ij}) > d_{ij}$. Now delete the edges of T_{ij} from W_{ij}. Let W_{ij}^1 be the remaining part of the walk W_{ij}. Then W_{ij}^1 should be disconnected. Otherwise W_{ij}^1 and some of the edges of T_{ij} together would form a cycle. But T has no cycle.
Let $W_{ij,1}$, $W_{ij,2}$, $\ldots$, $W_{ij,s}$ be the component walks of W_{ij}^1. Then $W_{ij,h} \cap T_{ij}$, $(1 \leq h \leq s)$ should be a single vertex $v_{w_{ij},h}$, otherwise some of the edges of $W_{ij,h}$ and T_{ij} together form a cycle. Since each vertex $v_{w_{ij},h}$ is a part of the walk W_{ij}, $W_{ij,h}$ should be either the single vertex $v_{w_{ij},h}$ or a closed walk from $v_{w_{ij},h}$ to $v_{w_{ij},h}$.
But length of a closed walk in a tree is always even. So $l(W_{ij}^1) = \sum_{h=1}^{s} l(W_{ij,h})$ must be even.
Let $l(W_{ij}^1) = \sum_{h=1}^{s} l(W_{ij,h}) = 2r$, for some $r \in \{0, 1, 2, \ldots\}$
$\Rightarrow l(W_{ij}) = l(T_{ij}) + l(W_{ij}^1) = d_{ij} + \sum_{h=1}^{s} l(W_{ij,h}) = d_{ij} + 2r$, for some $r \in \{0, 1, 2, \ldots\}$
$\Rightarrow$ length of every walk from v_i to v_j must be of the form $d_{ij} + 2r$, for $r \in \{0, 1, 2, \ldots\}$.
By Lemma 3.4, there always exist a walk between v_i and v_j of length $d_{ij} + 2r$, for all $r = 0, 1, 2, \ldots$. So

$$\left(A_T^{(m)}\right)_{ij} = \begin{cases} 1, & \text{if } m = d_{ij} + 2r, \forall r = 0, 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

□

Theorem 3.6. *Let A_T be the adjacency matrix of a tree $T=(V,E)$ of order $n(n>1)$. Let $d = diam(T)$. Then for $0 \le K \le d$,*

(i) $D_K \circ A_T^{(K+2r)} = D_K$.

(ii) $D_K \circ A_T^{(K+2r+1)} = 0$, $\forall\, r = 0,1,2,\ldots$.

Proof.

$$\left(D_K \circ A_T^{(m)}\right)_{ij} = (D_K)_{ij}(A_T^{(m)})_{ij} = \begin{cases} 1, & \text{if } (D_K)_{ij}=1 \text{ and } (A_T^{(m)})_{ij}=1 \\ 0, & \text{otherwise} \end{cases}$$

$(D_K)_{ij} = 1 \Rightarrow d_{ij} = K$.
But then $(A_T^{(K+2r)})_{ij} = 1$, $\forall\, r = 0,1,2,\ldots$ ($\because$ Theorem 3.5)
ie., $(D_K)_{ij} = 1 \Rightarrow (A_T^{(K+2r)})_{ij} = 1$, $\forall\, r = 0,1,2,\ldots$.

$$\therefore \left(D_K \circ A_T^{(K+2r)}\right)_{ij} = \begin{cases} 1, & \text{if } K = d_{ij} \\ 0, & \text{otherwise} \end{cases} = (D_K)_{ij}, \forall i,j \text{ and } r = 0,1,2,\ldots.$$

$\Rightarrow D_K \circ A_T^{(K+2r)} = D_K$, $\forall\, r = 0,1,2,\ldots$.
By Theorem 3.5 there does not exist a walk of length $d_{ij}+2r+1$ between v_i and v_j, $\forall\, r = 0,1,2,\ldots$.
$\Rightarrow \left(A_T^{(d_{ij}+2r+1)}\right)_{ij} = 0$, $\forall\, i,j$ and $r = 0,1,2,\ldots$.
$\Rightarrow D_K \circ A_T^{(K+2r+1)} = 0\ \forall\, r = 0,1,2,\ldots$. □

Remark 3.7. *Let A_T be the adjacency matrix of a tree $T=(V,E)$ of order $n(n>1)$. Let $d = diam(T)$. Then for $0 \le K \le d$. For $m < K$, $A_T^{(m)} \circ D_K = 0$, (by Theorem 3.1).*
For $m > K$,

(i) $D_K \circ A_T^{(K+2r)} = D_K$.

(ii) $D_K \circ A_T^{(K+2r+1)} = 0$. $\forall r = 0,1,2,\ldots$, *(by Theorem 3.6)*

$$\therefore A_T^{(m)} \circ D_K = \begin{cases} D_K, & m = K+2r, \forall r = 0,1,2,\ldots \\ 0, & \text{otherwise} \end{cases}$$

Theorem 3.8. *Let A_T be the adjacency matrix of $T=(V,E)$ with $d = diam(G)$. Then $\beta_2 = \{D_0, D_1, \ldots, D_d\}$ is an orthogonal basis for span $\{A_T^{(0)}, A_T^{(1)}, \ldots A_T^{(d)}\}$.*

Proof. By Theorem 2.4, $\beta_2 = \{D_0, D_1, \ldots, D_d\}$ is linearly independent. By Remark 2.3(v) β_2 is orthogonal also. So it is enough to prove that $A_T^{(m)}$ is a linear combination of $D_0, D_1, \ldots, D_d$, for $0 \le m \le d$. By Remark 3.7,

$$\therefore A_T^{(m)} \circ D_K = \begin{cases} D_K, & m = K+2r, \forall r = 0,1,2,\ldots \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow \sum_{K=0}^{d} A_T^{(m)} \circ D_K$ is a linear combination of $D_0, D_1, \ldots, D_d$.
i.e., $\sum_{K=0}^{d} A_T^{(m)} \circ D_K = \sum_{K=0}^{d} c_{m,K} D_K$, for $c_{m,0}, c_{m,1}, \ldots, c_{m,d} \in B = \{0,1\}$. Now

$$A_T^{(m)} = A_T^{(m)} \circ J_n = A_T^{(m)} \circ (D_0 + D_1 + \cdots + D_d), (\because Remark 2.3(iii)) = \sum_{K=0}^{d} A_T^{(m)} \circ D_K = \sum_{K=0}^{d} c_{m,K} D_K$$

$\Rightarrow A_T^{(m)}$ is a linear combination of $D_0, D_1, \ldots, D_d$.
For finding the scalars $c_{m,k}$, consider

$$\begin{aligned}\langle A_T^{(m)}, D_j\rangle_F &= \langle c_{m,0}\cdot D_0 + c_{m,1}\cdot D_1 + \cdots + c_{m,d}\cdot D_d, D_j\rangle_F \\ &= c_{m,0}\cdot\langle D_0, D_j\rangle_F + c_{m,1}\cdot\langle D_1, D_j\rangle_F + \cdots \\ &\quad + c_{m,d}\cdot\langle D_d, D_j\rangle_F \\ &= 0 + 0 + \cdots + c_{m,j}\langle D_j, D_j\rangle_F + 0 + \cdots + 0 \\ &= c_{m,j}\langle D_j, D_j\rangle_F (\because Remark 2.3(v))\end{aligned}$$

$$\Rightarrow c_{m,j} = \frac{\langle A_T^{(m)}, D_j\rangle_F}{\langle D_j, D_j\rangle_F}, j = 0,1,\ldots,d$$

the scalars, $c_{m,j} = \dfrac{\langle A_T^{(m)}, D_j\rangle_F}{\langle D_j, D_j\rangle_F} \in B = \{0,1\}, \forall 0 \le m,j \le d$,

So,

$$\begin{aligned}A_T^{(m)} &= c_{m,0}\cdot D_0 + c_{m,1}\cdot D_1 + \cdots + c_{m,d}\cdot D_d \\ &= \frac{\langle A_T^{(m)}, D_0\rangle_F}{\langle D_0, D_0\rangle_F}\cdot D_0 + \frac{\langle A_T^{(m)}, D_1\rangle_F}{\langle D_1, D_1\rangle_F}\cdot D_1 + \cdots \\ &\quad + \frac{\langle A_T^{(m)}, D_d\rangle_F}{\langle D_d, D_d\rangle_F}\cdot D_d \\ A_T^{(m)} &= \sum_{j=0}^{d} \frac{\langle A_T^{(m)}, D_j\rangle_F}{\langle D_j, D_j\rangle_F} D_j\end{aligned}$$

$\Rightarrow \beta_2 = \{D_0, D_1, \ldots, D_d\}$ is an orthogonal basis for span $\{A_T^{(0)}, A_T^{(1)}, \ldots, A_T^{(d)}\}$. □

Remark 3.9. *(i) We have*

$$A_T^{(m)} = \sum_{j=0}^{d} \frac{\langle A_T^{(m)}, D_j\rangle_F}{\langle D_j, D_j\rangle_F} D_j (\because Theorem 3.8)$$

$$A_T^{(m)} = \frac{\langle A_T^{(m)}, D_d\rangle_F}{\langle D_d, D_d\rangle_F} D_d + \sum_{j=0}^{d-1} \frac{\langle A_T^{(m)}, D_j\rangle_F}{\langle D_j, D_j\rangle_F} D_j$$

$$\frac{\langle A_T^{(m)}, D_d\rangle_F}{\langle D_d, D_d\rangle_F} D_d = A_T^{(m)} - \sum_{j=0}^{d-1} \frac{\langle A_T^{(m)}, D_j\rangle_F}{\langle D_j, D_j\rangle_F} D_j$$

$\Rightarrow$ The orthogonal basis for span $\{A_T^{(0)}, A_T^{(1)}, \ldots, A_T^{(d)}\}$ obtained from the basis β_1 by Gram-Schmidt process is nothing but β_2 itself.

(ii) $A_T^{(m)} = \sum_{j=0}^{d} c_{m,j} D_j$, *by Theorem 3.8, where*

$$c_{m,j} = \frac{\langle A_T^{(m)}, D_j\rangle_F}{\langle D_j, D_j\rangle_F}, m,j = 0,1,\ldots,d.$$

This can be written in the matrix form as given below,

$$\begin{bmatrix} A_T^{(0)} \\ A_T^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ A_T^{(d)} \end{bmatrix} = \begin{bmatrix} c_{00} & c_{01} & \cdot & \cdot & \cdot & c_{0d} \\ c_{10} & c_{11} & \cdot & \cdot & \cdot & c_{1d} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{d0} & c_{d1} & \cdot & \cdot & \cdot & c_{dd} \end{bmatrix} \begin{bmatrix} D_0 \\ D_1 \\ \cdot \\ \cdot \\ \cdot \\ D_d \end{bmatrix} \tag{3.3}$$

Let

$$Y = \begin{bmatrix} A_T^{(0)} \\ A_T^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ A_T^{(d)} \end{bmatrix}, C = \begin{bmatrix} c_{00} & c_{01} & \cdot & \cdot & \cdot & c_{0d} \\ c_{10} & c_{11} & \cdot & \cdot & \cdot & c_{1d} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{d0} & c_{d1} & \cdot & \cdot & \cdot & c_{dd} \end{bmatrix}, X = \begin{bmatrix} D_0 \\ D_1 \\ \cdot \\ \cdot \\ \cdot \\ D_d \end{bmatrix}$$

Then (3.3) $\Rightarrow$

$$Y = CX \tag{3.4}$$

where, X, Y are $(d+1)\times 1$ column matrices whose elements are $n\times n$ binary matrices and C is a $(d+1)\times(d+1)$ real matrix. But by Theorem 3.1, $c_{ii}=1$ and $c_{ij}=0$, when $i<j, (i,j=0,1,\dots,d)$. Which implies that C is a unit lower triangular matrix.
$\therefore C$ is invertible and C^{-1} is a lower triangular unit matrix with $|C|=1$.
So, by multiplying C^{-1} on both sides od equation (3.4)*. Then* (3.4) $\Rightarrow$

$$X = C^{-1}Y \tag{3.5}$$

Then (3.3) $\Rightarrow$

$$\begin{bmatrix} D_0 \\ D_1 \\ \cdot \\ \cdot \\ \cdot \\ D_d \end{bmatrix} = \begin{bmatrix} c_{00} & c_{01} & \cdot & \cdot & \cdot & c_{0d} \\ c_{10} & c_{11} & \cdot & \cdot & \cdot & c_{1d} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{d0} & c_{d1} & \cdot & \cdot & \cdot & c_{dd} \end{bmatrix}^{-1} \begin{bmatrix} A_T^{(0)} \\ A_T^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ A_T^{(d)} \end{bmatrix}$$

$\Rightarrow C^{-1}$ and C are the conversion matrices for getting the orthogonal basis β_2 from β_1 and vice versa.

4. Illustration

Consider the following tree $T=(V,E)$. Then the adjacency matrix A_T of T is

$$A_T = \begin{array}{c|cccccc|} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 4 & 1 & 1 & 1 & 0 & 1 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 & 1 \\ 6 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

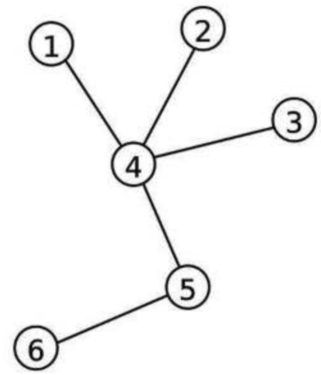

Figure 1

Here the maximum distance $d=3, A_T^{(0)}=I, A_T^{(1)}=\delta(A_T)=A_T$

$$A_T^2 = \begin{bmatrix} 1&1&1&0&1&0 \\ 1&1&1&0&1&0 \\ 1&1&1&0&1&0 \\ 0&0&0&4&0&1 \\ 1&1&1&0&2&0 \\ 0&0&0&1&0&1 \end{bmatrix},$$

$$A_T^{(2)} = \delta(A_T^{(2)}) = \begin{bmatrix} 1&1&1&0&1&0 \\ 1&1&1&0&1&0 \\ 1&1&1&0&1&0 \\ 0&0&0&1&0&1 \\ 1&1&1&0&1&0 \\ 0&0&0&1&0&1 \end{bmatrix}$$

$$A_T^{(2)} * A_T^{(1)} = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}$$

$$A_T^3 = \begin{bmatrix} 0&0&0&4&0&1 \\ 0&0&0&4&0&1 \\ 0&0&0&4&0&1 \\ 4&4&4&0&5&0 \\ 0&0&0&5&0&2 \\ 1&1&1&0&2&0 \end{bmatrix}$$

$$A_T^{(3)} = \delta(A_T^{(3)}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$A_T^{(3)} * A_T^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$A_G^{(3)} * A_T^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} A_T^{(1)} * I =& 0 \\ D_0 =& A_T^{(0)} = I \\ D_1 =& A_T^{(1)} - \delta(A_T^{(1)} * I) = A_T^{(1)} - 0 = A_T^{(1)} \end{aligned}$$

$$\left(A_T^{(2)} * I + A_T^{(2)} * A_T^{(1)}\right)$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_2 = A_T^{(2)} - \delta(A_T^{(2)} * I + A_T^{(2)} * A_T^{(1)})$$
$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$(A_T^{(3)} * I + A_T^{(3)} * A + A_T^{(3)} * A^2) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$D_3 = A_T^{(3)} - \delta(A_T^{(3)} * I + A_T^{(3)} * A_T^{(1)} + A_T^{(3)} * A_T^{(2)})$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Let

$$Y = \begin{bmatrix} A_T^{(0)} \\ A_T^{(1)} \\ A_T^{(2)} \\ A_T^{(3)} \end{bmatrix}, X = \begin{bmatrix} D_0 \\ D_1 \\ D_2 \\ D_3 \end{bmatrix}$$

Then $Y = CX$, where

$$c_{m,j} = \frac{\langle A_T^{(m)}, D_j \rangle_F}{\langle D_j, D_j \rangle_F}, m, j = 0, 1, 2, 3$$

$$\begin{aligned}
c_{00} &= \frac{\langle A_T^{(0)}, D_0 \rangle_F}{\langle D_0, D_0 \rangle_F} = \frac{6}{6} = 1, c_{01} = c_{02} = c_{03} = 0 \\
c_{10} &= \frac{\langle A_T^{(1)}, D_0 \rangle_F}{\langle D_0, D_0 \rangle_F} = \frac{0}{6} = 0, \\
c_{11} &= \frac{\langle A_T^{(1)}, D_1 \rangle_F}{\langle D_1, D_1 \rangle_F} = \frac{10}{10} = 1, c_{12} = c_{13} = 0 \\
c_{20} &= \frac{\langle A_T^{(2)}, D_0 \rangle_F}{\langle D_0, D_0 \rangle_F} = \frac{6}{6} = 1, c_{21} = \frac{\langle A_T^{(2)}, D_1 \rangle_F}{\langle D_1, D_1 \rangle_F} = \frac{0}{10} = 0, \\
c_{22} &= \frac{\langle A_T^{(2)}, D_2 \rangle_F}{\langle D_2, D_2 \rangle_F} = \frac{14}{14} = 1, c_{23} = 0, \\
c_{30} &= \frac{\langle A_T^{(3)}, D_0 \rangle_F}{\langle D_0, D_0 \rangle_F} = \frac{0}{6} = 0, c_{31} = \frac{\langle A_T^{(3)}, D_1 \rangle_F}{\langle D_1, D_1 \rangle_F} = \frac{10}{10} = 1, \\
c_{32} &= \frac{\langle A_T^{(3)}, D_2 \rangle_F}{\langle D_2, D_2 \rangle_F} = \frac{0}{14} = 0, c_{33} = \frac{\langle A_T^{(3)}, D_3 \rangle_F}{\langle D_3, D_3 \rangle_F} = \frac{6}{6} = 1.
\end{aligned}$$

So,

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, C^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Then $Y = CX$, also $X = C^{-1}Y$

$$\begin{aligned}
Y = CX &\Rightarrow & X = C^{-1}Y &\Rightarrow \\
A_T^{(0)} &= D_0 & D_0 &= A_T^{(0)} \\
A_T^{(1)} &= D_1 & D_1 &= A_T^{(1)} \\
A_T^{(2)} &= D_0 + D_2 & D_2 &= (-1)A_T^{(0)} + A_T^{(2)} \\
A_T^{(3)} &= D_1 + D_3 & D_3 &= (-1)A_T^{(1)} + A_T^{(3)}
\end{aligned}$$

5. Conclusion

Generally, $\beta_2 = \{D_0, D_1, \ldots, D_d\}$ is not an orthogonal basis for span $\{A_G^{(0)}, A_G^{(1)}, \ldots, A_G^{(d)}\}$, for a simple connected undirected graph G with diameter d. But we proved that this is true for an undirected tree T with diameter d and also derived an invertible conversion matrix for computing one basis β_1 from the other basis β_2 and vice versa. Further study may be done on exploring some other connected undirected graphs having this property other than trees.

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