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A Note on Generalized Solitons

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Abstract: In this paper, we initiate the study of a generalized soliton on a Riemannian manifold, we find a characterization for the Euclidean space, and in the compact case, we find a sufficient condition under which it reduces to a quasi-Einstein manifold. We also find sufficient conditions under which a compact generalized soliton reduces to an Einstein manifold. Note that Ricci solitons being self-similar solutions of the heat flow, this topic is related to the symmetry in the geometry of Riemannian manifolds. Moreover, generalized solitons being generalizations of Ricci solitons are naturally related to symmetry.

Keywords: generalized solitons; Euclidean space; quasi-Einstein manifolds

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1. Introduction

One of the most studied structures on a Riemannian manifold is a Ricci soliton and it is a stable solution of the Ricci flow introduced by Hamilton [1]. Moreover, Ricci solitons are natural generalizations of Einstein metrics [2]. There is another important generalization of Einstein manifolds namely quasi-Einstein manifolds [3], which are important in the Robertson–Walker spacetime. Moreover, an important generalization of Ricci solitons are Ricci almost solitons [4,5]. There are various types of solitons considered in [1,6–13]. It is a natural need to define a most general structure on a Riemannian manifold that should include existing structures such as Ricci solitons, Ricci almost solitons, quasi-Einstein manifolds, etc., as particular cases. A unitary approach of these soliton-types equations is given in the following. For an n-dimensional Riemannian manifold (M, g), a 1-form η and a vector field ξ on M, we consider the following equation

$$\frac{1}{2}\mathcal{L}_{\xi}g + \alpha Ric = \beta g + \gamma \eta \otimes \eta, \tag{1.1}$$

with α, β, and γ smooth functions on M, $\mathcal{L}_{\xi}g$ the Lie derivative of g in the direction of ξ, Ric the Ricci tensor of (M, g), which encompasses most of the concepts of solitons and quasi-Einstein manifold in the Riemannian setting, and we call $(M, g, \xi, \alpha, \beta, \gamma)$ a generalized soliton.

We shall denote by $\mathfrak{X}(M)$ the set of all smooth vector fields of M. If the vector field ξ in the generalized soliton is of the gradient type, i.e., $\xi := \nabla f$, for a smooth function f on M, and if we denote by $Hess(f)$ the Hessian of f, by ∇ the Levi-Civita connection of g, and by Q the Ricci operator, i.e.,

$$g(QX, Y) = Ric(X, Y),$$

for $X, Y \in \mathfrak{X}(M)$, then $\mathcal{L}_{\xi}g = 2Hess(f)$, and thus the definition of the generalized soliton takes the form

$$Hess(f) + \alpha Ric = \beta g + \gamma \eta \otimes \eta, \tag{1.2}$$

with α, β, and γ smooth functions on M, which is equivalent to

$$A_f + \alpha Q = \beta I + \gamma\eta \otimes t,$$

where A_f is the Hessian operator defined by $Hess(f)(X,Y) = g\left(A_f X, Y\right)$, t is the dual vector field to the 1-form η, and I is the identity endomorphism on $\mathfrak{X}(M)$. In this case, we say that $(M, g, \nabla f, \eta, \alpha, \beta, \gamma)$ defines a *gradient generalized soliton* on M.

Recall that for $\alpha = 1$, β a constant, and $\gamma = 0$, the generalized soliton is a Ricci soliton, and for $\alpha = 1$ and $\gamma = 0$, it is a Ricci almost soliton. Moreover, for ξ a Killing vector field and $\alpha = 1$, the generalized soliton is a quasi-Einstein manifold. It is easy to show that the Euclidean space $\mathbf{E}^n$ is a gradient generalized soliton $(\mathbf{E}^n, \langle,\rangle, \nabla f, \alpha, \beta, 0)$, where $\langle,\rangle$ is the Euclidean inner product and

$$f(x) = \frac{\beta}{2}\langle x, x\rangle, \quad x \in \mathbf{E}^n.$$

This raises a question: can we characterize the Euclidean space $\mathbf{E}^n$ using an n-dimensional complete and connected gradient generalized soliton $(M, g, \nabla f, \alpha, \beta, \gamma)$? We answer this question in Section 3, where we find a characterization of the Euclidean space $\mathbf{E}^n$.

Note that, if we allow in the definition of a generalized soliton $(M, g, \xi, \alpha, \beta, \gamma)$ the potential field ξ to be a Killing vector field, and the function $\alpha = 1$, then we get

$$Ric = \beta g + \gamma\eta \otimes \eta,$$

that is, the generalized soliton $(M, g, \xi, \alpha, \beta, \gamma)$ is a quasi-Einstein manifold (see [3]). However, the requirements that ξ be a Killing vector field and the function $\alpha = 1$ are quite strong conditions and naturally one would like to see whether some weaker conditions could be found that would render a generalized soliton $(M, g, \xi, \alpha, \beta, \gamma)$ into a quasi-Einstein manifold. In Section 4, we consider this question and find conditions on a compact generalized soliton $(M, g, \xi, \alpha, \beta, \gamma)$ under which it reduces to a quasi-Einstein manifold. In that section, we also find conditions under which a compact generalized soliton $(M, g, \xi, \alpha, \beta, \gamma)$ becomes an Einstein manifold.

It is worth noting that the generalized Ricci soliton $(M, g, \xi, \alpha, \beta, \gamma)$ generalizes structures such as Ricci solitons, Ricci almost solitons, Einstein manifolds, quasi-Einstein manifolds, therefore the study of generalized solitons has a modest scope. A future study could include questions of finding necessary and sufficient conditions on a generalized Ricci soliton $(M, g, \xi, \alpha, \beta, \gamma)$ to reduce it to a Ricci soliton or a Ricci almost soliton. Note that if the potential field ξ of a generalized Ricci soliton $(M, g, \xi, \alpha, \beta, \gamma)$ is a Killing vector field, and the function $\alpha = 1$, then the generalized Ricci soliton becomes a quasi-Einstein manifold. It will be an interesting question to analyze the impact of the restriction on the potential field ξ of a generalized Ricci soliton $(M, g, \xi, \alpha, \beta, \gamma)$ as a conformal vector field.

2. Preliminaries

Let $(M, g, \xi, \alpha, \beta, \gamma)$ be an n-dimensional generalized soliton and ∇ be the Riemannian connection with respect to metric g. We denote by ω the smooth 1-form dual to the potential field ξ of the generalized soliton $(M, g, \xi, \alpha, \beta, \gamma)$ and denote by φ the $(1,1)$ skew-symmetric tensor field defined by

$$\frac{1}{2}d\omega(X,Y) = g(\varphi X, Y), \quad X, Y \in \mathfrak{X}(M). \tag{2.1}$$

Note that

$$2g(\nabla_X\xi, Y) = g(\nabla_X\xi, Y) - g(\nabla_Y\xi, X) + g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X),$$

that is,

$$2g(\nabla_X \xi, Y) = d\omega(X,Y) + \left(\pounds_\xi g\right)(X,Y), \quad X, Y \in \mathfrak{X}(M),$$

where

$$d\omega(X,Y) = g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X)$$

and

$$\left(\pounds_\xi g\right)(X,Y) g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X).$$

Then, using Equations (1.1) and (2.1), we have

$$\nabla_X \xi = \beta X + \gamma \eta(X) t - \alpha Q X + \varphi X, \quad X \in \mathfrak{X}(M). \tag{2.2}$$

Using the expression for the scalar curvature $\tau = trQ$ and a local frame $\{v_1, \ldots, v_n\}$ with the skew-symmetry of φ, and the definition of the divergence

$$div\xi = \sum_{i=1}^{n} g(\nabla_{v_i} \xi, v_i),$$

we have

$$div\xi = \sum_{i=1}^{n} g(\beta v_i + \gamma \eta(v_i) t - \alpha Q v_i + \varphi v_i, v_i),$$

that is,

$$div\xi = n\beta + \gamma \|t\|^2 - \alpha\tau. \tag{2.3}$$

Now, we are in position to state the following:

Lemma 1. *Let $(M, g, \xi, \alpha, \beta, \gamma)$ be an n-dimensional compact generalized soliton. Then,*

$$\int_M \xi(\beta) = \int_M \left(\alpha\beta\tau - n\beta^2 - \gamma\beta\|t\|^2\right), \quad \int_M \xi(\gamma) = \int_M \left(\alpha\gamma\tau - n\beta\gamma - \gamma^2\|t\|^2\right).$$

Proof. Using Equation (2.3), we have $div(\beta\xi) = \xi(\beta) + n\beta^2 + \beta\gamma\|t\|^2 - \alpha\beta\tau$ and $div(\gamma\xi) = \xi(\gamma) + n\beta\gamma + \gamma^2\|t\|^2 - \alpha\gamma\tau$. Integrating these equations and using Stokes's theorem

$$\int_M divX = 0,$$

for a smooth vector field X, we get the results in the Lemma. □

Lemma 2. *Let $(M, g, \xi, \alpha, \beta, \gamma)$ be an n-dimensional generalized soliton. Then,*

$$\alpha^2 \left\| Q - \frac{\tau}{n} I \right\|^2 = 2\alpha\gamma Ric(t,t) + \|\nabla\xi\|^2 - n\left(\beta - \frac{\alpha}{n}\tau\right)^2 - \gamma^2\|t\|^4 - 2\beta\gamma\|t\|^2 - \|\varphi\|^2.$$

Proof. Using Equation (2.2), we have

$$\alpha\left(Q - \frac{\tau}{n} I\right)(X) = \left(\beta - \frac{\tau}{n}\right) X + \gamma\eta(X) t + \varphi X - \nabla_X \xi, \quad X \in \mathfrak{X}(M). \tag{2.4}$$

Taking a local orthonormal frame $\{v_1, \ldots, v_n\}$ on M, we have

$$\alpha^2 \left\| Q - \frac{\tau}{n} I \right\|^2 = \sum_{i=1}^{n} g\left(\alpha\left(Q - \frac{\tau}{n} I\right)(v_i), \alpha\left(Q - \frac{\tau}{n} I\right)(v_i)\right),$$

which, in view of Equation (2.4), implies

$$\alpha^2\left\|Q - \tfrac{\tau}{n}I\right\|^2 = n\left(\beta - \tfrac{\tau}{n}\right)^2 + \gamma^2\|t\|^4 + \|\varphi\|^2 + \|\nabla\xi\|^2 + 2\gamma\left(\beta - \tfrac{\tau}{n}\right)\|t\|^2 - 2\left(\beta - \tfrac{\tau}{n}\right)div\xi - 2\gamma g(\nabla_t\xi, t) - 2\sum_{i=1}^{n} g(\nabla_{v_i}\xi, \varphi v_i). \tag{2.5}$$

Using Equation (2.2), we get

$$g(\nabla_t\xi, t) = \beta\|t\|^2 + \gamma\|t\|^4 - \alpha Ric(t,t) \tag{2.6}$$

and

$$\sum_{i=1}^{n} g(\nabla_{v_i}\xi, \varphi v_i) = \sum_{i=1}^{n} g(\varphi v_i, \varphi v_i) = \|\varphi\|^2, \tag{2.7}$$

where we have used the skew-symmetry of φ and the symmetry of Q for concluding $tr(Q \circ \varphi) = 0$. Inserting Equations (2.3), (2.6), and (2.7) in Equation (2.5), we get the result. □

3. A Characterization of a Euclidean Space

Let $(M, g, \xi, \alpha, \beta, \gamma)$ be an n-dimensional gradient generalized soliton with $\xi = \nabla f$ for some smooth function f. In this section, we prove the following result that gives a characterization for a Euclidean space.

Theorem 1. *Let $(M, g, \nabla f, \alpha, \beta, 0)$ be an n-dimensional complete and connected gradient generalized soliton $n \geq 3$ with $\alpha \neq 0$, β constants. Then, the scalar curvature τ is constant with*

$$\alpha\tau(n\beta - \alpha\tau) \leq 0$$

and $n\beta \neq \alpha\tau$, if and only if $(M, g, \nabla f, \alpha, \beta, 0)$ is isometric to the Euclidean space.

Proof. Let $(M, g, \nabla f, \alpha, \beta, 0)$ be an n-dimensional complete and connected gradient generalized soliton. Then, Equation (1.2) with $\gamma = 0$ yields

$$A_f X = \beta X - \alpha Q X, \quad X \in \mathfrak{X}(M). \tag{3.1}$$

Since α and β are constants, we have

$$\left(\nabla A_f\right)(X, Y) = \nabla_X A_f Y - A_f(\nabla_X Y) = -\alpha(\nabla Q)(X, Y), \quad X, Y \in \mathfrak{X}(M). \tag{3.2}$$

Now, using the expression for the curvature tensor field

$$R(X, Y)\nabla f = \left(\nabla A_f\right)(X, Y) - \left(\nabla A_f\right)(Y, X)$$

and Equation (3.2), we conclude that

$$R(X, Y)\nabla f = \alpha\{(\nabla Q)(Y, X) - (\nabla Q)(X, Y)\}. \tag{3.3}$$

Note that the scalar curvature τ is a constant and therefore for a local orthonormal frame $\{v_1, \ldots, v_n\}$ on M, we have

$$\sum_{i=1}^{n} (\nabla Q)(v_i, v_i) = \frac{1}{2}\nabla\tau = 0. \tag{3.4}$$

Thus, Equation (3.3) on contraction yields

$$Ric(Y, \nabla f) = \alpha\left\{Y(\tau) - \frac{1}{2}Y(\tau)\right\} = 0,$$

that is,

$$Q(\nabla f) = 0. \tag{3.5}$$

We proceed to compute $divQ(\nabla f)$ as follows:

$$\begin{aligned} divQ(\nabla f) &= \sum_{i=1}^{n} g(\nabla_{v_i} Q(\nabla f), v_i) = \sum_{i=1}^{n} g\Big((\nabla Q)(v_i, \nabla f) + Q\Big(A_f v_i\Big), v_i\Big) \\ &= \sum_{i=1}^{n} g(\nabla f, (\nabla Q)(v_i, v_i)) + \sum_{i=1}^{n} g\Big(A_f v_i, Q v_i\Big). \end{aligned}$$

Using Equations (3.1), (3.4), and (3.5) in the above equation, we conclude

$$0 = \sum_{i=1}^{n} g(\beta v_i - \alpha Q v_i, Q v_i),$$

which implies $\beta\tau - \alpha\|Q\|^2 = 0$, that is,

$$\alpha^2\|Q\|^2 = \alpha\beta\tau.$$

Thus, we have

$$\alpha^2\left(\|Q\|^2 - \frac{1}{n}\tau^2\right) = \alpha\beta\tau - \frac{1}{n}\alpha^2\tau^2 = \frac{1}{n}\alpha\tau(n\beta - \alpha\tau) \leq 0$$

by virtue of the premise. Using Schwarz's inequality $\|Q\|^2 \geq \frac{1}{n}\tau^2$ in the above inequality, we conclude

$$\alpha^2\left(\|Q\|^2 - \frac{1}{n}\tau^2\right) = 0.$$

However, as the constant $\alpha \neq 0$, we conclude

$$\|Q\|^2 = \frac{1}{n}\tau^2;$$

this is the equality in Schwarz's inequality, and it holds if and only if $Q = \frac{\tau}{n}I$. Moreover, Equation (3.1) gives $A_f X = \beta X - \frac{1}{n}\alpha\tau X$, that is,

$$Hess(f) = \frac{1}{n}(n\beta - \alpha\tau)g = cg, \tag{3.6}$$

where $c = \frac{1}{n}(n\beta - \alpha\tau) \neq 0$ by virtue of a condition in the statement. Using the result in [11], we conclude that M is isometric to the n-dimensional Euclidean space $\mathbf{E}^n$.

Conversely, on the Euclidean space $\mathbf{E}^n$ for a nonzero constant β, we define $f : \mathbf{E}^n \to R$ by

$$f(x) = \frac{\beta}{2}\|x\|^2.$$

Then, we have $\nabla f = \beta I$ and $Hess(f) = \beta\langle , \rangle$, where $\langle , \rangle$ is the Euclidean inner product. Thus, on the Euclidean space $\mathbf{E}^n$, we have

$$Hess(f) + \alpha Ric = \beta\langle , \rangle$$

for a nonzero constant α. Hence, $(\mathbf{E}^n, \langle , \rangle, \nabla f, \alpha, \beta, 0)$ is a gradient generalized soliton. $\square$

4. Quasi-Einstein Manifolds

Recall that a Riemannian manifold (M, g) is said to be a quasi-Einstein manifold (see [3]) if its Ricci tensor has the form

$$Ric = \lambda g + \mu\varsigma \otimes \varsigma,$$

where λ, μ are smooth functions and ς is a smooth 1-form on M. In this section, we are interested in finding conditions under which an n-dimensional generalized soliton $(M, g, \xi, \alpha, \beta, \gamma)$ is a quasi-Einstein manifold. Note that from the definition of a generalized soliton, it follows that if the potential field ξ is a Killing vector field and the function $\alpha = 1$, then $(M, g, \xi, \alpha, \beta, \gamma)$ is a quasi-Einstein manifold.

Recall that a smooth vector field X on a Riemannian manifold (N, h) is said to be Killing if $\pounds_X h = 0$, and this condition is equivalent to the fact that the local flow of X consists of local isometries. It is worth pointing out that the presence of Killing vector fields on a Riemannian manifold (N, h) severely restricts its geometry as well as the topology of M; for instance, the Ricci curvature of a compact Riemannian manifold (N, h) possessing a nonparallel Killing vector field must have a positive Ricci curvature. Thus, asking the potential field ξ of a generalized Ricci soliton $(M, g, \xi, \alpha, \beta, \gamma)$ to be Killing is quite a strong condition.

Theorem 2. *Let $(M, g, \xi, \alpha, \beta, \gamma)$ be an n-dimensional compact generalized soliton with the function α nowhere zero. If the Ricci curvature $Ric(\xi, \xi)$ satisfies*

$$Ric(\xi, \xi) \geq (div\xi)^2 + \|\varphi\|^2,$$

then $(M, g, \xi, \alpha, \beta, \gamma)$ is a quasi-Einstein manifold.

Proof. Suppose $(M, g, \xi, \alpha, \beta, \gamma)$ is an n-dimensional compact generalized soliton. Then, using a local orthonormal frame $\{v_1, \ldots, v_n\}$ on M and Equation (1.1), we have

$$\frac{1}{4}|\mathcal{L}_\xi g|^2 = \sum_{i,j=1}^{n}\left[\frac{1}{2}(\mathcal{L}_\xi g)(v_i, v_j)\right]^2 = \sum_{i,j=1}^{n}\left(\beta g(v_i, v_j) + \gamma\eta(v_i)\eta(v_j) - \alpha Ric(v_i, v_j)\right)^2,$$

that is,

$$\begin{aligned} \frac{1}{4}|\mathcal{L}_\xi g|^2 &= n\beta^2 + \gamma^2\|t\|^4 + \alpha^2\|Q\|^2 + 2\beta\gamma\|t\|^2 - 2\alpha\beta\tau - 2\alpha\gamma Ric(t, t) \\ &= \alpha^2\left(\|Q\|^2 - \frac{1}{n}\tau^2\right) + n\left(\beta - \frac{\alpha}{n}\tau\right)^2 + \gamma^2\|t\|^4 + 2\beta\gamma\|t\|^2 - 2\alpha\gamma Ric(t, t). \end{aligned}$$

Note that

$$\alpha^2\left\|Q - \frac{1}{n}\tau I\right\|^2 = \alpha^2\left(\|Q\|^2 - \frac{1}{n}\tau^2\right)$$

and the above equation gives

$$\frac{1}{4}|\mathcal{L}_\xi g|^2 = \alpha^2\left\|Q - \frac{1}{n}\tau I\right\|^2 + n\left(\beta - \frac{\alpha}{n}\tau\right)^2 + \gamma^2\|t\|^4 + 2\beta\gamma\|t\|^2 - 2\alpha\gamma Ric(t, t).$$

Thus, we have

$$2\alpha\gamma Ric(t, t) - n\left(\beta - \frac{\alpha}{n}\tau\right)^2 - 2\beta\gamma\|t\|^2 - \gamma^2\|t\|^4 = \alpha^2\left\|Q - \frac{1}{n}\tau I\right\|^2 - \frac{1}{4}|\mathcal{L}_\xi g|^2. \tag{4.1}$$

Using Lemma 2 with the above equation, we conclude

$$-\frac{1}{4}|\mathcal{L}_\xi g|^2 + \|\nabla\xi\|^2 - \|\varphi\|^2 = 0. \tag{4.2}$$

Now, using Yano's integral formula (see [14])

$$\int_M \left(Ric(\xi, \xi) + \frac{1}{2}|\mathcal{L}_\xi g|^2 - \|\nabla\xi\|^2 - (div\xi)^2\right) = 0$$

and Equation (4.1), we have

$$\frac{1}{4}\int_M \left|\mathcal{L}_\xi g\right|^2 = \int_M \left(\|\varphi\|^2 + (div\xi)^2 - Ric(\xi,\xi)\right). \tag{4.3}$$

Now, using the condition in the hypothesis, we conclude

$$\frac{1}{4}\int_M \left|\mathcal{L}_\xi g\right|^2 \leq 0,$$

that is, $\mathcal{L}_\xi g = 0$. Consequently, as α is nowhere zero, Equation (1.1) implies

$$Ric = \frac{\beta}{\alpha}g + \frac{\gamma}{\alpha}\eta \otimes \eta,$$

that is, $(M, g, \xi, \alpha, \beta, \gamma)$ is a quasi-Einstein manifold. □

Theorem 3. *Let $(M, g, \xi, \alpha, \beta, \gamma)$ be an n-dimensional ($n > 2$), compact and connected generalized soliton with the function α nowhere zero. If γ is nonzero and constant along the integral curves of ξ and the following conditions hold*

$$\alpha\gamma\tau \leq n\beta\gamma, \quad Ric(\xi,\xi) \geq \|\varphi\|^2,$$

then $(M, g, \xi, \alpha, \beta, \gamma)$ is an Einstein manifold.

Proof. Suppose $(M, g, \xi, \alpha, \beta, \gamma)$ is an n-dimensional compact generalized soliton satisfying the conditions in the hypothesis. Then, using $\xi(\gamma) = 0$ in Lemma 1, we get

$$\int_M \gamma^2\|t\|^2 = \int_M (\alpha\gamma\tau - n\beta\gamma), \tag{4.4}$$

which, in view of $\alpha\gamma\tau \leq n\beta\gamma$, implies $\gamma^2\|t\|^2 = 0$. However, as $\gamma \neq 0$ and M is connected, we get $t = 0$. Using $t = 0$ in Equation (4.4) and $\alpha\gamma\tau \leq n\beta\gamma$, we conclude $\gamma(\alpha\tau - n\beta) = 0$. Furthermore, owing to the fact that $\gamma \neq 0$, we get

$$\alpha\tau - n\beta = 0. \tag{4.5}$$

Now, using $t = 0$ and Equation (4.5), we conclude through Equation (2.3) that $div\xi = 0$. Thus, Equation (4.3) takes the form

$$\frac{1}{4}\int_M \left|\mathcal{L}_\xi g\right|^2 = \int_M \left(\|\varphi\|^2 - Ric(\xi,\xi)\right).$$

Using the lower bound on $Ric(\xi,\xi)$ in the statement, we get $\mathcal{L}_\xi g = 0$. Thus, Equation (1.1) with $t = 0$ implies

$$\alpha Ric = \beta g$$

and as α is nowhere zero, we conclude $Ric = fg$, where $f = \beta\alpha^{-1} = \frac{1}{n}\tau$ (see Equation (4.5)), that is, $Q = fI$ and

$$(\nabla Q)(X,Y) = X(f)Y, \quad X, Y \in \mathfrak{X}(M).$$

Using a local orthonormal frame $\{v_1, \ldots, v_n\}$ on M in the above equation, we have

$$\sum_{i=1}^{n}(\nabla Q)(v_i, v_i) = \nabla f = \frac{1}{n}\nabla\tau,$$

that is

$$\frac{1}{2}\nabla\tau = \frac{1}{n}\nabla\tau$$

and as $n > 2$, we conclude τ is a constant and $(M, g, \xi, \alpha, \beta, \gamma)$ is an Einstein manifold. □

Along similar lines as in the above theorem, using the first part of Lemma 1, we prove the following:

Theorem 4. *Let $(M, g, \xi, \alpha, \beta, \gamma)$ be an n-dimensional $n > 2$, compact and connected generalized soliton with the function α nowhere zero. If $\beta\gamma > 0$ and β is a constant along the integral curves of ξ and the following conditions hold*

$$\alpha\beta\tau \le n\beta^2, \quad Ric(\xi, \xi) \ge \|\varphi\|^2,$$

then $(M, g, \xi, \alpha, \beta, \gamma)$ is an Einstein manifold.

5. Examples and Conclusions

First, we discuss a few examples of generalized solitons.

(i) Consider the open subset $M = \mathbf{E}^n - \{0\}$. Then, with the Euclidean metric g, (M, g) is a flat Riemannian manifold. Let

$$\Psi = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i}$$

be the position vector field, which is a nonzero vector field on M. Define

$$\xi = f\Psi, \quad f = \frac{1}{\|\Psi\|}.$$

Note that

$$X(f) = -\frac{1}{\|\Psi\|^2} X(\|\Psi\|) = -\frac{1}{2\|\Psi\|^3} X(g(\Psi, \Psi)) = -f^2 g(\xi, X), \quad X \in \mathfrak{X}(M).$$

Thus, we get

$$\nabla_X \xi = f(X - \eta(X)\xi), \quad X \in \mathfrak{X}(M),$$

where $\eta(X) = g(\xi, X)$; consequently, we have

$$(\mathcal{L}_\xi g)(X, Y) = 2fg(X, Y) - 2f\eta(X)\eta(Y), \quad X, Y \in \mathfrak{X}(M).$$

Choosing a nonzero function α on M, we get

$$\frac{1}{2}\mathcal{L}_\xi g + \alpha Ric = \beta g + \gamma\eta \otimes \eta,$$

where $\beta = f$ and $\gamma = -f$. Hence, $(M, g, \xi, \alpha, \beta, \gamma)$ is an n-dimensional generalized soliton.

(ii) Consider the unit sphere $\mathbf{S}^n$ with canonical metric g of constant curvature 1. Then, it is well-known that $\mathbf{S}^n$ possesses a conformal vector field ξ that satisfies (see [2])

$$\mathcal{L}_\xi g = 2\sigma g,$$

where σ is a smooth function on $\mathbf{S}^n$. Then, it follows for a smooth function α on $\mathbf{S}^n$, we have

$$\frac{1}{2}\mathcal{L}_\xi g + \alpha Ric = \beta g,$$

where $\beta = \sigma + (n-1)\alpha$. Hence, $(\mathbf{S}^n, g, \xi, \alpha, \beta, 0)$ is an n-dimensional generalized soliton.

(iii) Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional Einstein β-Kenmotsu manifold (see [8]). Then, the unit vector field ξ satisfies

$$\nabla_X \xi = \beta(X - \eta(X)\xi), \quad X \in \mathfrak{X}(M),$$

where β is a smooth function on M. Since M is an Einstein manifold, we have $Ric = \lambda g$ for a constant λ. Hence, for a smooth function α on M, we have

$$\frac{1}{2}\mathcal{L}_\xi g + \alpha Ric = \beta g + \gamma \eta \otimes \eta,$$

where $\gamma = -\beta$. Hence, $(M, g, \xi, \alpha, \beta, \gamma)$ is a $(2n+1)$-dimensional generalized soliton.

We saw through Theorem 1 that an n-dimensional complete and connected gradient generalized soliton $(M, g, \nabla f, \alpha, \beta, 0)$ was used to find a characterization of the Euclidean space $\mathbf{E}^n$, and example (ii) also showed that $(\mathbf{S}^n, g, \xi, \alpha, \beta, 0)$ was an n-dimensional generalized soliton. This naturally raises the question of whether we can find a characterization of the unit sphere $\mathbf{S}^n$ using an appropriate n-dimensional compact generalized soliton $(M, g, \xi, \alpha, \beta, 0)$. This could be an interesting question for future studies on this topic. Moreover, in the results of Section 4, we found conditions under which an n-dimensional generalized soliton $(M, g, \xi, \alpha, \beta, \gamma)$ was a quasi-Einstein and Einstein manifold, respectively. It will be interesting study to find conditions under which an n-dimensional generalized soliton $(M, g, \xi, \alpha, \beta, \gamma)$ is a Ricci almost soliton.

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