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A NOTE ON GORENSTEIN RINGS OF EMBEDDING CODIMENSION THREE

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1. Let $A = R/\mathfrak{a}$, where R is a regular local ring of arbitrary dimension and $\mathfrak{a}$ is an ideal of R. If A is a Gorenstein ring and if height $\mathfrak{a} = 2$, it is easily proved that A is a complete intersection, i.e., $\mathfrak{a}$ is generated by two elements (Serre [5], Proposition 3). Hence Gorenstein rings which are not complete intersections are of embedding codimension at least three. An example of these rings is found in Bass' paper [1] (p. 29). This is obtained as a quotient of a three dimensional regular local ring by an ideal which is generated by five elements, i.e., generated by a regular sequence plus two more elements. In this paper, suggested by this example, we prove that if A is a Gorenstein ring and if height $\mathfrak{a} = 3$, then $\mathfrak{a}$ is minimally generated by an odd number of elements. If A has a greater codimension, presumably there is no such restriction on the minimal number of generators for $\mathfrak{a}$, as will be conceived from the proof.

In the following the basic results of the two famous papers Bass [2] and Matlis [4] are taken for granted.

2. In this paper we shall consider only Noetherian local rings. If R is a local ring with the maximal ideal $\mathfrak{m}$, we sometimes say that the pair $(R, \mathfrak{m})$ is a local ring. Let R be a ring. If $x, y, \cdots, z$ are elements of R, $(x, y, \cdots, z)$ denotes the ideal they generate. For an R-module M, $\operatorname{hd} M$ denotes the homological dimension of M over R. If R is a regular local ring, $\operatorname{hd} M < \infty$ for any finite R-module M and it holds that $\operatorname{hd} M + \operatorname{depth} M = \dim R$.

LEMMA 1. *Let R be a regular local ring and let $\mathfrak{q}$ be a primary ideal belonging to the maximal ideal of R. Suppose that $\mathfrak{q} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ is an irredundant decomposition of $\mathfrak{q}$ by n irreducible ideals $\mathfrak{q}_i$. Let*

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$0 \to F_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to 0$ *be a minimal free resolution of* $R/\mathfrak{q}$. *Then the rank of* F_d *is equal to* n.

Proof. Since depth $R/\mathfrak{q} = 0$, $\dim R = d$. Therefore we have an isomorphism $\mathrm{Ext}^d_R(R/\mathfrak{q}, R) \cong \mathrm{Hom}_R(R/\mathfrak{q}, E)$, where E denotes the injective envelope of the residue class field. (See [2] Theorem 4.1) Thus the rank of F_d is equal to the minimal number of generators for $\mathrm{Hom}_R(R/\mathfrak{q}, E)$. On the other hand, the injective envelope of the module $\mathrm{Hom}_R(\mathrm{Hom}_R(R/\mathfrak{q}, E), E) \cong R/\mathfrak{q}$ is an n copies of E, and in general these two numbers are identical, because a minimal surjection $F \to \mathrm{Hom}_R(R/\mathfrak{q}, E) \to 0$ with F free gives an essential injection $0 \to \mathrm{Hom}_R(\mathrm{Hom}_R(R/\mathfrak{q}, E), E) \to \mathrm{Hom}_R(F, E)$. (cf. [4] Theorem 2.3 and Theorem 4.2)

COROLLARY. *Let* R *be a Gorenstein ring and* $\mathfrak{q}$ *a perfect ideal of grade* d. *Let* $0 \to F_d \to F_{d-1} \to \cdots \to F_0 \to 0$ *be as in the Lemma. Then the rank of* F_d *is the "type" of the Cohen-Macaulay ring* $R/\mathfrak{q}$.

Proof. Let $x_1, x_2, \cdots, x_r$ be a maximal regular sequence for both R and $R/\mathfrak{q}$. Then it is well known that the complex:

$$0 \longrightarrow F_d \otimes R/\mathfrak{x} \longrightarrow F_{d-1} \otimes R/\mathfrak{x} \longrightarrow \cdots \longrightarrow F_0 \otimes R/\mathfrak{x} \longrightarrow 0$$

is a minimal free resolution of $R/\mathfrak{q} + \mathfrak{x}$, over $R/\mathfrak{x}$, where $\mathfrak{x} = (x_1, \cdots, x_r)$. (To prove this we only have to show the acyclicity, and this can be done by induction on r.) Since the isomorphisms used in the proof of Lemma 1 hold for a Gorenstein ring $R/\mathfrak{x}$, the assertion follows.

LEMMA 2. *Let* A *be an Artin Gorenstein local ring and* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two ideals of* A. *If* $0 : \mathfrak{a} = 0 : \mathfrak{b}$, *then* $\mathfrak{a} = \mathfrak{b}$.

Proof. Since for any ideal $\mathfrak{a}$ of A, we have $0 : [0 : \mathfrak{a}] = \mathfrak{a}$, the assertion is clear. (cf. [3] Satz 1.44)

LEMMA 3. *Let* $(R, \mathfrak{m})$ *be a local ring and* $\mathfrak{q}$ *an* $\mathfrak{m}$*-primary irreducible ideal, and let* y *be an element of* R *which is not in* $\mathfrak{q}$. *Assume that* $\mathfrak{q} : y = \mathfrak{q} + (f_1, f_2, \cdots, f_n)$. *Then we have* $\bigcap_{i=1}^n [\mathfrak{q} : f_i] = \mathfrak{q} : (f_1, \cdots, f_n) = \mathfrak{q} + (y)$. *Moreover the following two conditions are equivalent to each other:*

i) *the intersection of ideals* $\bigcap_{i=1}^n [\mathfrak{q} : f_i]$ *is irredundant.*

ii) $\{f_1, f_2, \cdots, f_n\}$ *is (a set of representatives of) a minimal generators for the ideal* $\mathfrak{q} + (f_1, f_2, \cdots, f_n)$ *modulo* $\mathfrak{q}$.

Proof. The equality $\bigcap_{i=1}^{n}[\mathfrak{q}:f_i]=\mathfrak{q}:(f_1,\cdots,f_n)$ is easily verified (without the assumption that $\mathfrak{q}$ is irreducible and $\mathfrak{m}$-primary). We prove the second equality. It is obvious that $\mathfrak{q}+(y)\subset\mathfrak{q}:(f_1,\cdots,f_n)$. Assume $z\in\mathfrak{q}:(f_1,\cdots,f_n)$. Then $zf_i\in\mathfrak{q}$ for each i, which implies that $\mathfrak{q}:z\supset\mathfrak{q}:y$. Therefore $\mathfrak{q}:(y)=\mathfrak{q}:(y,z)$, and considering everything modulo $\mathfrak{q}$, we conclude by Lemma 2 that $\mathfrak{q}+(y)=\mathfrak{q}+(y,z)$, which proves $\mathfrak{q}:(f_1,\cdots,f_n)\subset\mathfrak{q}+(y)$. (Recall that $R/\mathfrak{q}$ is an Artin Gorenstein ring if and only if $\mathfrak{q}$ is $\mathfrak{m}$-primary and irreducible.)

To prove the second assertion assume $\bigcap_{i=1}^{n}[\mathfrak{q}:f_i]=\bigcap_{i=2}^{n}[\mathfrak{q}:f_i]$ for instance. Then $\mathfrak{q}:(f_1,f_2,\cdots,f_n)=\mathfrak{q}:(f_2,\cdots,f_n)$, and again by Lemma 2, $\mathfrak{q}+(f_1,\cdots,f_n)=\mathfrak{q}+(f_2,\cdots,f_n)$. This shows that ii) implies i). The other implication is immediate.

LEMMA 4. *Let $\mathfrak{a}$ be an irreducible $\mathfrak{m}$-primary ideal of a local ring $(R,\mathfrak{m})$. If $\mathfrak{b}$ is another irreducible ideal which contains $\mathfrak{a}$, then there is an element y such that $\mathfrak{b}=\mathfrak{a}:y$. Conversely for any element y of R which is not in $\mathfrak{a}$, $\mathfrak{a}:y$ is irreducible.*

Proof. Let E be the injective envelope of $R/\mathfrak{m}$. From the canonical epimorphism $R/\mathfrak{a}\to R/\mathfrak{b}\to 0$ we obtain a monomorphism $0\to\operatorname{Hom}_R(R/\mathfrak{b},E)\to\operatorname{Hom}_R(R/\mathfrak{a},E)$. Since $R/\mathfrak{a}$ and $R/\mathfrak{b}$ are both self-injective, $\operatorname{Hom}_R(R/\mathfrak{a},E)\cong R/\mathfrak{a}$ and $\operatorname{Hom}_R(R/\mathfrak{b},E)\cong R/\mathfrak{b}$. Therefore the above monomorphism shows the existence of y satisfying $\mathfrak{b}=\mathfrak{a}:y$. An $\mathfrak{m}$-primary ideal $\mathfrak{q}$ is irreducible if and only if $\dim_k\operatorname{Hom}_R(k,R/\mathfrak{q})=1$, where $k=R/\mathfrak{m}$. Consequently the irreducibility of $\mathfrak{a}:y$ follows immediately from the fact that we can define a monomorphism $R/[\mathfrak{a}:y]\to R/\mathfrak{a}$ by $1 \bmod [\mathfrak{a}:y]\mapsto y \bmod \mathfrak{a}$.

THEOREM. *Let $(R,\mathfrak{m})$ be a regular local ring and $\mathfrak{a}$ be an ideal of height three, such that $R/\mathfrak{a}$ is a Gorenstein ring. Then $\mathfrak{a}$ is minimally generated by an odd number of elements.*

Proof. We denote by $\mu(I)$ the number of minimal generators of an ideal I of a local ring. With this notation it is easy to see that if $x\in\mathfrak{m}$ is a regular element on $R/\mathfrak{a}$, then $\mu(\mathfrak{a})=\mu(\mathfrak{a}+(x)/(x))$, where $\mathfrak{a}+(x)/(x)$ is an ideal of $R/(x)$. Note also that, in this case, height $\mathfrak{a}=$ height $\mathfrak{a}+(x)/(x)$ and $R/\mathfrak{a}+(x)=R/(x)/\mathfrak{a}+(x)/(x)$ is a Gorenstein ring. Thus we may assume depth $R/\mathfrak{a}=0$, because whenever depth $R/\mathfrak{a}>0$, there is a regular element on $R/\mathfrak{a}$ in $\mathfrak{m}-\mathfrak{m}^2$. This amounts to assuming that

$\mathfrak{a}$ is $\mathfrak{m}$-primary and dimension $R = 3$, since $R/\mathfrak{a}$ is a Cohen-Macaulay ring.

Let $\mu(\mathfrak{a}) = N = n + 3$. If $n = 0$, there is nothing to prove. Let $n > 0$, and let $\mathfrak{a} = (x_1, x_2, x_3, f_1, f_2, \cdots, f_n)$, where we may assume that $(x_1, x_2, x_3) = \mathfrak{x}$ is already $\mathfrak{m}$-primary. Since both $\mathfrak{a}$ and $\mathfrak{x}$ are irreducible, by Lemma 4, there is y such that $\mathfrak{a} = \mathfrak{x} : y$.

This y can be chosen in such a way that x_2, x_3, y is a regular sequence. For suppose that $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_t$ are the associated primes of (x_2, x_3) and that

$$\begin{aligned} y \notin \mathfrak{p}_i \quad i &= 1, 2, \cdots, s \\ y \in \mathfrak{p}_i \quad i &= s + 1, \cdots, t \,. \end{aligned}$$

Since x_2, x_3 is a regular sequence, height $\mathfrak{p}_i = 2$ for every i. If $s = t$, the sequence x_2, x_3, y is a regular sequence. Let $0 \leqq s < t$ and $D = \mathfrak{x} \cap \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$. Then $\mathfrak{p}_{s+1} \cup \cdots \cup \mathfrak{p}_t \not\supset D$. For any element $z \in D$ such that $z \notin \mathfrak{p}_{s+1} \cup \cdots \cup \mathfrak{p}_t$, $x_2, x_3, y + z$ is a regular sequence and obviously $\mathfrak{x} : y + z = \mathfrak{a}$. From now on y is assumed to be chosen in this way.

We are interested in the ideal $\mathfrak{q}$ generated by x_1, x_2, x_3 and y. We assert first that these four elements are a minimal generating set for $\mathfrak{q}$. For if $x_1 \in (x_2, x_3, y)$, $x_1 = a_2x_2 + a_3x_3 + by$ with suitable elements a_2, a_3, b. This b is an element of $\mathfrak{x} : y$, so that b is a linear combination of x_1 and f_i. But this contradicts the fact that x_i and f_i form a minimal basis for $\mathfrak{a}$. The same is true with x_2 and x_3. Since it is clear that y cannot be omitted, the above assertion is proved. On the other hand, by Lemma 3, $\mathfrak{q} = \bigcap_{i=1}^{n} [\mathfrak{x} : f_i]$, and therefore, by Lemma 1 and Lemma 4, $\dim_k \operatorname{Tor}_3^R (k, R/\mathfrak{q}) = n$, where $k = R/\mathfrak{m}$.

Let $0 \to F_3 \to F_2 \to F_1 \to F_0 \to R/\mathfrak{q} \to 0$ be a minimal free resolution of $R/\mathfrak{q}$. So far we have proved that $\operatorname{rank} F_1 = 4$ and $\operatorname{rank} F_3 = n$. Since $\operatorname{rank} F_0 = 1$, $\operatorname{rank} F_2 = N$. We may assume that the homomorphism $F_1 \to F_0$ is defined by the column vector φ such that ${}^t\varphi = [x_1\, x_2\, x_3\, y]$ (where ${}^t\varphi$ denotes the transposed matrix of φ); if elements of F_1 are represented by row vectors, their images by φ are obtained by the usual matrix product. Let M be a matrix that defines $F_2 \to F_1$, and let I_i be the ideal generated by those elements that appear in the i-th column of M $(i = 1, 2, 3, 4)$. It is easy to see that these I_i depend only on the vector φ and not on the choice of M. In fact I_1 is nothing but $(x_2, x_3, y) : x_1$, for example. Note $I_4 = \mathfrak{a}$. By Lemma 4 and by the choice of y, I_1 is irreducible. We are going to prove that $\mu(I_1) = N - 2$, which completes the proof of the theorem by induction on μ of irreducible $\mathfrak{m}$-primary ideals, because the least μ is three.

Consider the following $N \times 4$ matrix M_1:

$$M_1 = \begin{bmatrix} -y & 0 & 0 & x_1 \\ 0 & -y & 0 & x_2 \\ 0 & 0 & -y & x_3 \\ a_{11} & a_{12} & a_{13} & f_1 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & f_n \end{bmatrix}$$

where a_{ij} are elements satisfying $a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + f_i y = 0$, their existence being a consequence of the assumption that $f_i \in \mathfrak{x} : y$. Since each row of M_1 is in $\operatorname{Ker}\varphi$, there is an $N \times N$ matrix $\boldsymbol{T}$ such that $\boldsymbol{TM} = \boldsymbol{M}_1$. This $\boldsymbol{T}$ can be regarded as an $\boldsymbol{R}$-endomorphism of each I_i. Then since $\boldsymbol{T}$ must be an R-automorphism of I_4, T is invertible and it follows that $I_1 = (y, a_{11}, a_{21}, \cdots, a_{n1})$. We want to show that these elements are precisely a minimal basis for I_1. Let $\boldsymbol{v}$ denote the first and $\boldsymbol{v}_j$ the $(3 + j)$-th row of $\boldsymbol{M}_1$, where $j = 1, 2, \cdots, n$. Assume for instance $a_{11} \in (y, a_{21}, \cdots, a_{n1})$. Then there are elements b and c_j such that the first component of $\boldsymbol{u} = b\boldsymbol{v} + \sum_{j=2}^{n} c_j \boldsymbol{v}_j$ is a_{11}. Set $\boldsymbol{u}' = \boldsymbol{v}_1 - \boldsymbol{u}$. Then the first component of $\boldsymbol{u}'$ is 0, and the fourth component of $\boldsymbol{u}'$ has the form $f_1 - \alpha$, where α is an element of $(x_1, f_2, \cdots, f_n)$. Let d_2 and d_3 be the 2nd and the 3rd component of $\boldsymbol{u}'$ respectively. Since $\boldsymbol{u}' \in \operatorname{Ker}\varphi$, d_2, d_3 and $f_1 - \alpha$ give a relation of x_2, x_3, y, i.e., $d_2x_2 + d_3x_3 + (f_1 - \alpha)y = 0$. Since x_2, x_3, y is a regular sequence, it follows that $f_1 - \alpha \in (x_2, x_3)$, whence $f_1 \in (x_1, x_2, x_3, f_2, \cdots, f_n)$, which is impossible. That y is not superfluous is similarly proved. Q.E.D.

COROLLARY. *Let R be a Gorenstein ring and $\mathfrak{a}$ be an ideal of homological dimension two. If $R/\mathfrak{a}$ is a Gorenstein ring, then $\mu(\mathfrak{a})$ is odd.*

Proof. By the first part of the proof of Corollary to Lemma 1, we may assume that $R/\mathfrak{a}$ is Artinian.

Let $\mathfrak{x}, y, \mathfrak{q}$ etc. be as in the proof of the theorem. In order to repeat the same argument as before we only have to show that $\operatorname{hd} R/\mathfrak{q}$ is finite. But we have an exact sequence: $0 \to R/\mathfrak{a} \xrightarrow{\varphi} R/\mathfrak{x} \to R/\mathfrak{q} \to 0$, where φ is defined by $\varphi(1 \bmod \mathfrak{a}) = y \bmod \mathfrak{x}$. Since $\operatorname{hd} R/\mathfrak{x} = \operatorname{hd} R/\mathfrak{a} = 3$, it follows that $\operatorname{hd} R/\mathfrak{q} \leqq 3$. (In fact $\operatorname{hd} R/\mathfrak{q} = 3$, since $R/\mathfrak{q}$ is Artinian.) Q.E.D.

Remark. It can be proved that over a local ring R the existence of an ideal $\mathfrak{a}$ of finite homological dimension such that $R/\mathfrak{a}$ is a Gorenstein ring implies that R itself is a Gorenstein ring. Therefore in the above corollary the condition that R is a Gorenstein ring is unnecessary.

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