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A NOTE ON GRADE

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All rings that occur in this note will be assumed to be commutative with unity and all modules will be finitely generated and unitary.

The grade of a module M over a noetherian local ring R is defined to be the length of a maximal R-sequence contained in the annihilator of M. If M has finite projective dimension it is well-known that grade $M \leq$ proj. dim M. We can say more when R is a regular local ring. We state the

THEOREM. *Let R be a regular local ring and M a given R-module. Let N be any other R-module such that* $\mathrm{Hom}(M,N) \neq (0)$. *Let p be the least integer such that* $\mathrm{Ext}_R^p(M,N) = (0)$. *Then* $\operatorname{grade} M \leq \inf(p-1, \text{proj. dim } N)$. *If q is the least integer such that* $\mathrm{Ext}_R^q(M,M) = (0)$, *then projective dimension of M equals $q-1$.*

Remark. Taking $N = k$, we get $\operatorname{grade} M \leq$ proj. dim M, the result mentioned in the introduction.

The proof of theorem depends on the following

LEMMA. *Let R be a regular local ring; let M, N be any two R-modules. If* $\mathrm{Ext}_R^p(M,N) = (0)$ *for some integer $p \geq 1$, then there exists a natural isomorphism* $\mathrm{Ext}_R^{p-1}(M,R) \otimes_R N \cong \mathrm{Ext}_R^{p-1}(M,N)$.

Proof. Define $\Omega^0 = M$ and for $p \geq 1$, define Ω^p to be the pth syzygy module of M taken with respect to a fixed minimal resolution of M,

$$\to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0 \qquad (1)$$

Taking the R-dual sequence of (1) and using $*$ to denote R-duals we define $D\Omega^p = \operatorname{cokernel}(F_p^* \to F_{p+1}^*)$ for $p \geq 0$. According to [3] for every integer $p \geq 0$, there exists an exact sequence

$$\mathrm{Tor}_2^R(D\Omega^p, N) \to \mathrm{Ext}_R^p(M,R) \otimes N \to \mathrm{Ext}_R^p(M,N) \to \mathrm{Tor}_1^R(D\Omega^p, N) \to 0 \qquad (2)$$

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where the maps are certain natural homomorphisms. Suppose $\mathrm{Ext}^p_R(M,N)$ is the zero module. Applying (2) we obtain $\mathrm{Tor}^R_1(D\Omega^p, N) = (0)$. Using [6] this implies $\mathrm{Tor}^R_j(D\Omega^p, N) = (0)$ for $j \geq 1$. The second application of (2) yields $\mathrm{Ext}^p_R(M,R) \otimes N = (0)$. Since $N \neq (0)$, we conclude that $\mathrm{Ext}^p_R(M,R) = (0)$, i.e. $\mathrm{Ext}^1_R(\Omega^{p-1}, R) = (0)$. Taking R-duals in the exact sequence $0 \to \Omega^p \to F_{p-1} \to \Omega^{p-1} \to 0$ and using the fact that $\mathrm{Ext}^1_R(\Omega^{p-1}, R) = (0)$, we obtain the following exact sequence

$$0 \to (\Omega^{p-1})^* \to F^*_{p-1} \to (\Omega^p)^* \to 0 . \tag{3}$$

Using the definition of $D\Omega^p$, we get an exact sequence

$$0 \to (\Omega^p)^* \to F^*_p \to F^*_{p+1} \to D\Omega^p \to 0 . \tag{4}$$

Putting (2) and (3) together and making use of the definition of $D\Omega^{p-1}$, we get the following exact sequence,

$$0 \to D\Omega^{p-1} \to F^*_{p+1} \to D\Omega^p \to 0 . \tag{5}$$

The exact sequence (5) gives $\mathrm{Tor}^R_j(D\Omega^{p-1}, N) = \mathrm{Tor}^R_{j+1}(D\Omega^p, N) = (0)$ for $j \geq 1$. The lemma follows after using this information in the exact sequence (2) with p replaced by $p-1$.

Proof of the theorem: If $\mathrm{grade}\, M > \mathrm{proj.\,dim}\, N$, then clearly $\mathrm{depth}\, N > \mathrm{Krull\ dim}\, M$ and so applying [4] we find that $\mathrm{Hom}(M, N) = (0)$, a contradiction. Hence $\mathrm{grade}\, M \leq \mathrm{proj.\,dim}\, N$. The lemma gives an isomorphism $\mathrm{Ext}^{p-1}_R(M,R) \otimes N \cong \mathrm{Ext}^{p-1}_R(M,N)$. Now if $\mathrm{grade}\, M \geq p$, it is well-known that $\mathrm{Ext}^i_R(M,R) = (0)$ for $0 \leq i \leq p-1$, so that $\mathrm{Ext}^{p-1}_R(M, N) = (0)$, a contradiction to the minimality of p. Hence $\mathrm{grade}\, M \leq p-1$. Combining with the inequality $\mathrm{grade}\, M \leq \mathrm{proj.\,dim}\, N$ established before we find that $\mathrm{grade}\, M \leq \inf(p-1, \mathrm{proj.\,dim}\, N)$. This proves the first part of the theorem. As for the second part we observe that $\mathrm{Ext}^q_R(M, M) = (0)$ implies, as in the lemma above that $\mathrm{Tor}^R_j(D\Omega^{q-1}, M) = (0)$ for $j \geq 1$. Using this in the exact sequence $0 \to \Omega^{q-1} \to F_{q-2} \to \cdots \to F_0 \to M \to 0$ we get $\mathrm{Tor}^R_j(D\Omega^{q-1}, \Omega^{q-1}) = (0)$ for $j \geq 1$. An application of (2) with $p = 0$ and M replaced by Ω^{q-1} shows that the natural map $(\Omega^{q-1})^* \otimes \Omega^{q-1} \to \mathrm{Hom}(\Omega^{q-1}, \Omega^{q-1})$ is an isomorphism. Hence Ω^{q-1} is projective, i.e. $\mathrm{proj.\,dim}\, M \leq q-1$. The minimality of q implies that $\mathrm{proj.\,dim}\, M = q-1$.

The theorem is proved.

We recall the following conjecture of M. Auslander

TOR CONJECTURE: If M is a module of finite projective dimension over a noetherian local ring R and N any other R-module such that

$$\mathrm{Tor}_1^R(M,N) = (0)\ , \quad \text{then} \quad \mathrm{Tor}_j^R(M,N) = (0) \quad \text{for} \quad j \geq 1\ .$$

It is well-known that this conjecture is true if R is regular local [6] and trivially so if proj. dim $M \leq 1$. We remark that if the above conjecture is true then the lemma is valid for any noetherian local ring provided N has finite projective dimension. Consequently the second part of the theorem is also valid for any noetherian local ring provided M has finite projective dimension.

M. Auslander and O. Goldman have proved that a reflexive module M over a regular local ring R is free if and only if $\mathrm{Hom}(M,M)$ is free [1]. In his article on the purity of the Branch locus [2] M. Auslander asks if this result is true for any noetherian local ring provided one assumes that M has finite projective dimension. We shall show that the answer is yes if the Tor conjecture mentioned above is true. In fact we prove the following

PROPOSITITION. *Let M, N be reflexive modules of finite projective dimensions over a noetherian local ring R such that* $\mathrm{Hom}(M,N)$ *is a nonzero free R-module. Then if the Tor conjecture is true M and N are both free modules.*

Proof. By induction on the Krull-dimension of R and [1, Lemma 4.8] we easily find that $\mathrm{Ext}_R^1(M,N) = (0)$. As in the proof of the lemma we get an isomorphism $M^* \otimes N \cong \mathrm{Hom}(M,N)$. Hence $M^* \otimes N$ is a nonzero free module. From this it is easy to conclude that both M^* and N are free. Since M is reflexive, M and N are both free modules.

REFERENCES

[1] M. Auslander and O. Goldman: Maximal order, Trans. Amer. Math. Soc. Vol. **97** (1960).

[2] M. Auslander: On the purity of the branch locus, American Journal of Mathematics Vol. **84** (1962).

[3] M. Auslander: Coherent functors, Proceedings of a conference on categorical algebra, La Jolla 1965, Springer-Verlag Berlin, Heidelberg, New York 1966.

[4] F. Ischebeck: Eine Dualitat zwischen den Funktoren Ext und Tor, J. Algebra **11** (1969), 510–531.

[5] D. Rees: The grade of an ideal or module, Proc. Camb. Phil. Soc. **53** (1957).

[6] S. Lichtenbaum: On the vanishing of Tor in regular local rings Illinois. J. of Math. Vol. **10** (1966).

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