A note on graph coloring extensions and list-colorings

Maria Axenovich

Department of Mathematics
Iowa State University, Ames, IA 50011, USA
axenovic@math.iastate.edu

Submitted: Oct 24, 2002; Accepted: Feb 10, 2003; Published: Mar 23, 2003
MR Subject Classifications: 05C15

Abstract

Let G be a graph with maximum degree $\Delta \geq 3$ not equal to $K_{\Delta+1}$ and let P be a subset of vertices with pairwise distance, $d(P)$, between them at least 8. Let each vertex x be assigned a list of colors of size Δ if $x \in V \setminus P$ and 1 if $x \in P$. We prove that it is possible to color $V(G)$ such that adjacent vertices receive different colors and each vertex has a color from its list. We show that $d(P)$ cannot be improved. This generalization of Brooks' theorem answers the following question of Albertson positively: If G and P are objects described above, can any coloring of P in at most Δ colors be extended to a proper coloring of G in at most Δ colors?

We say that a vertex-coloring of a graph $G = (V, E)$ is *proper* if the colors used on adjacent vertices are distinct. For an assignment of a color set (typically called a list) $l(x)$ to each vertex $x \in V$, we say that vertices are *colored from their lists* by a coloring c if $c(x) \in l(x)$ for each $x \in V$; c is called a *list-coloring* of G. A coloring c of $V(G)$ *extends* a coloring c' of vertices in P if it is a proper coloring with $c(x) = c'(x)$ for each $x \in P$. We denote by $d_G(x)$ the degree of x in a graph G and by $G[X]$ the subgraph of G induced by a set of vertices X.

The classic Brooks' theorem states that any simple connected graph G with maximum degree Δ can be colored properly in at most Δ colors unless $G = K_{\Delta+1}$ or G is an odd cycle. Recently, Albertson posed the following question. Take a graph described above, precolor a fixed set of vertices P in Δ colors arbitrarily. Under what condition on P can we extend that coloring to a proper coloring of G in at most Δ colors? He asks whether this condition is a large distance between the vertices in P. Albertson noticed though, that the maximum degree of a graph should be at least three. Indeed, it is easy to see that one cannot obtain a proper coloring of a path with an even number of vertices in two colors if the end-points are precolored in the same color. Here, we show that if the maximum degree is at least three, then there is a positive answer to Albertson's question when the pairwise distance, $d(P)$, between vertices of P is at least 8; moreover, this distance is optimal. The color extension problem is closely related to the concept of a list-coloring of graphs. Indeed, we can reformulate Albertson's question the following way. For set $S = \{1, \cdots, \Delta\}$, let the vertices of P be assigned lists of single colors from S and let every other vertex be assigned list S. Can G be properly list-colored from these lists if $d(P)$ is large enough? We answer this question by presenting a more general result. Our main tool is a corollary of the theorem about list-coloring of hypergraphs by Kostochka, Stiebitz and Wirth [4] which was also investigated independently by Borodin. The list-coloring version of Brooks' theorem was considered much earlier by Vizing [5]. We need a couple of definitions first. A *block* containing an edge e is a maximum 2-connected subgraph containing that edge or an edge e itself if such 2-connected subgraph does not exist. A *separating vertex* in a block is a vertex whose deletion disconnects the graph, i.e., a cutvertex of a graph. An *end-block* is a block with exactly one separating vertex. A *Gallai tree* is a graph all of whose blocks are either complete graphs, odd cycles, or single edges.

Theorem 1 (Kostochka, Stiebitz, Wirth). *Let $G = (V, E)$ be a connected graph. For each $x \in V$, let $l(x)$ be an assigned list of colors, $|l(x)| \geq d(x)$. If G is not list-colorable from these lists then it is a Gallai tree and $|l(x)| = d(x)$ for each $x \in V$.*

Figure 1 depicts graphs illustrating the exactness of our results. Next we give a formal description of graph G_1 from the figure.

A general construction Consider Δ copies of $K_{\Delta+1} \setminus e$, say $B_1, \cdots, B_\Delta$, where the deleted edge of B_i is $u_i v_i$ for each $i = 1, \cdots, \Delta$. Let B be a complete graph on vertices $w_1, \cdots, w_\Delta$. Then G_1 is formed from a disjoint union of $B, B_1, \cdots, B_\Delta$ and edges $u_1 w_1$, $u_2 w_2, \cdots, u_\Delta w_\Delta$. It is easy to see that the maximum degree of G_1 is Δ and G_1 is not equal to $K_{\Delta+1}$. Assign a list $\{1\}$ to each vertex in P and a list $\{1, \cdots, \Delta\}$ to every other vertex. Then, under any Δ-coloring c of B_is from the corresponding lists, $c(u_i) = c(v_i) = 1$. Thus $c(w_i) \neq 1$ for all $i = 1, \cdots, \Delta$. Since we need Δ colors for B, all different from 1, we need at least $\Delta + 1$ colors altogether to color G_1.

Theorem 2. *Let G be a graph with maximum degree $\Delta \geq 3$, not equal to $K_{\Delta+1}$. Let $P \subseteq V$, $d(P) \geq 8$. Let vertices in P and $V \setminus P$ be assigned arbitrary lists of sizes* 1 *and* Δ *respectively. Then G can be properly colored from these lists.*

Proof of Theorem 2. For each $x \in V$, let $l(x)$ be an assigned list of colors. The general idea of the proof is to list color all copies of $K_{\Delta+1} \setminus e$ in G which share a vertex of degree $\Delta - 1$ with P and then use Theorem 1 to list-color the rest. Let G have copies $B_1, \cdots, B_t$ of $K_{\Delta+1} \setminus e$ with $u_i v_i$ be the deleted edge, $u_i \in P$ for each $i = 1, \cdots, t$. Note that all B_is are vertex disjoint.

First we treat the case when $\Delta \geq 4$. When $\Delta = 3$ we need some more details to be considered separately. We shall color vertices of all B_is from their lists. For each $i = 1, \cdots, t$ we delete $l(u_i)$ from the lists of vertices in $B_i - \{u_i, v_i\}$ obtaining lists of size at least $\Delta - 1$. The degree of each vertex in $B_i - u_i$ is $\Delta - 1$; moreover, the new lists have size at least $\Delta - 1$ on $V(B_i) - \{u_i, v_i\}$ and Δ on v_i. Thus, by Theorem 1 we can properly

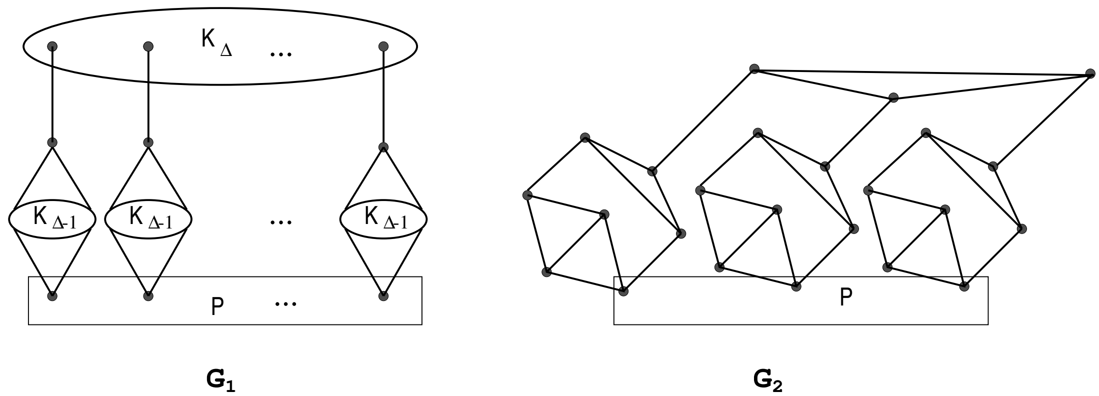

Figure 1: Two graphs with maximum degree Δ, which are not properly colorable from the list $\{1, \cdots, \Delta\}$ assigned to all vertices of $V \setminus P$ and the list $\{1\}$ assigned to all vertices of P.

color $B_i - u_i$ from the above lists, obtaining a proper coloring of B_i from the original lists. Let a_i be a color of v_i under some such coloring for each $i = 1, \cdots, t$.

Now, we consider a new graph G_1 obtained from G by deleting $V(B_i) - \{u_i, v_i\}$. Let $P_1 = P \cup \{v_1, \cdots, v_t\}$. Note that G_1 does not have copies of $K_{\Delta+1} \setminus e$ sharing a vertex of degree $\Delta - 1$ with P_1, and each vertex u_i or v_i for $i = 1, \cdots, t$ is adjacent to at most one vertex in G_1. Now, we need to color G_2 induced by $V(G_1) \setminus P_1$. We assign the new lists to $V(G_2)$ as follows.

$$l_2(x) = \begin{cases} l(x) \setminus l(u_i) & \text{if } xu_i \in E(G),\ xv_i \notin E(G), \\ l(x) \setminus \{a_i\} & \text{if } xv_i \in E(G),\ xu_i \notin E(G), \\ l(x) \setminus (\{a_i\} \cup l(u_i)) & \text{if } xu_i, xv_i \in E(G), \\ l(x) \setminus l(p) & \text{if } xp \in E(G),\ p \in P \setminus \{u_1, \cdots, u_t\}. \end{cases}$$

Note that if $x \in V(G_2)$ is adjacent to more than one vertex of P_1, these vertices must be u_i and v_i for some i, so only one of the above cases can hold. Assume that G_2 is not properly colorable from the lists l_2. Then, by Theorem 1 it is a Gallai tree with $d_{G_2}(x) = |l_2(x)|$ for each $x \in V(G_2)$. Thus, $d_{G_2}(x) = \Delta$, $\Delta - 1$ or $\Delta - 2$ when x is not adjacent to any vertex in P_1, when it is adjacent to one or two such vertices respectively. Thus each vertex in G_2 has degree at least 2.

We may assume that G_2 is connected since we can color the connected components separately. Let B be an end-block with a separating vertex x (if such exists) of G_2. B is a complete graph, or an odd cycle; moreover, $|V(B)| \geq 3$. If $B = G_2$ there must be an edge between $V(B)$ and P_1 since G is connected, if $B \neq G_2$ there is an edge between $V(B)$ and P_1 since $d_B(x) < d_{G_2}(x)$. Let uv be an edge of B. If $up, vq \in E(G)$ with $p, q \in P_1$, then either $p = q$ or $\{p, q\} = \{u_i, v_i\}$ for some i, otherwise the distance condition will be violated. Moreover, since $d_{G_1}(u_i) \leq 1$ and $d_{G_1}(v_i) \leq 1$ for each $i = 1, \cdots, t$, we have that all vertices of $B - x$ (or B if $G_2 = B$) are adjacent to the same vertex $p \in P$, and

$p \notin \{u_1, \cdots, u_t\} \cup \{v_1, \cdots, v_t\}$. Therefore $d_{G_2}(v) = \Delta - 1$ for each $v \in V(B - x)$, (or for each $v \in V(B)$ if $G_2 = B$), i.e., $B = K_\Delta$. But then $V(B) \cup \{p\}$ induces $K_{\Delta+1} \setminus e$ if $B \neq G_2$, a contradiction to the way we constructed G_1 or, if $B = G_2$, $V(B) \cup \{p\}$ induces $K_{\Delta+1}$ a contradiction to the condition of the theorem.

Now we treat the case when $\Delta = 3$. Assume, without loss of generality, that there are indices $1 \leq s' < s \leq t$, vertices w_i, $i = 1, \cdots, s$ and triangles $T_i = w_i w_i' w_i''$, $i = s'+1, \cdots, s$ such that w_i is adjacent to both u_i and v_i for $i = 1, \cdots, s'$, and $w_i' u_i, w_i'' v_i \in E(G)$ for $i = s' + 1, \cdots, s$. Note that all these w_i's are distinct. For each $i = 1, \cdots, s'$ let L_i be induced by $V(B_i)$ and w_i, for each $i = s'+1, \cdots, s$, let L_i be induced by $V(B_i)$ and $V(T_i)$, and, finally, for each $i = s + 1, \cdots, t$ let $L_i = B_i$. We properly color each L_i, $i = 1, \cdots, t$ from the original lists $l(x)$ and assume that w_i gets the color b_i for $i = 1, \cdots, s$ and v_i gets the color a_i for $i = s + 1, \cdots, t$.

We create G_1 from G by deleting vertices of $L_i - w_i$ for all $i = 1, \cdots, s$ and vertices of $B_i - \{u_i, v_i\}$ for $i = s + 1, \cdots, t$. Let $P_1 = (P \cap V(G_1)) \cup \{w_1, \cdots, w_s\} \cup \{v_{s+1}, \cdots, v_t\}$. Now, consider G_2, the subgraph of G_1 induced by $V(G_1) \setminus P_1$. Note that each vertex in G_2 has at most one neighbor in P_1, otherwise we violate the distance condition. Again, we create new lists for $l_2(x)$ for each vertex x of G_2 as follows.

$$l_2(x) = \begin{cases} l(x) \setminus l(u_i) & \text{if } xu_i \in E(G), \\ l(x) \setminus \{a_i\} & \text{if } xv_i \in E(G), \\ l(x) \setminus \{b_i\} & \text{if } xw_i \in E(G), \\ l(x) \setminus l(p) & \text{if } xp \in E(G),\ p \in P,\ p \neq u_i, v_i, \text{or } w_i \text{ for any } i \in \{1, \cdots, t\}. \end{cases}$$

Assume now that G_2 is not colored properly from the lists l_2. Then, by Theorem 1, we have $d_{G_2}(x) = |l_2(x)| = 3$ or 2. If G_2 is a block B, then it must be an odd cycle with all vertices adjacent to some vertices in P_1. It is easy to see that then all the vertices of G_2 must be adjacent to the same $p \in P_1$. In this case, we have $B \cup p$ induce K_4, a contradiction. If G_2 has a cut-vertex, let B be an end-block with a separating vertex x. B must be an odd cycle, either with all vertices in $B - x$ being adjacent to the same vertex in P and resulting in $K_4 \setminus e$, or with $V(B) - x = \{y, z\}$, where y and z are adjacent to u_i and v_i respectively for some i. In this case we get $B = K_3$ and $V(B_i) \cup V(B)$ induce a graph isomorphic to some L_j, a contradiction to the way we constructed G_2.

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Acknowledgments The author is indebted to T.I. Axenovich, the Institute of Cytology and Genetics of Russian Academy of Sciences for support and hospitality, and to D. Fon Der Flaass for useful comments.

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