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# A Note on Height Pairings, Tamagawa Numbers, and the Birch and Swinnerton-Dyer Conjecture

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## Introduction

Let $G$ be an algebraic group defined over a number field $k$. By choosing a lifting of $G$ to a group scheme over $\mathcal{O}_S \subset k$, the ring of $S$-integers for some finite set of places $S$ of $k$, we may define $G(\mathcal{O}_v)$, where $\mathcal{O}_v \subset k_v$ is the ring of integers in the $v$-adic completion of $k$ for all non-archimedean places $v \notin S$. In this way, we can define the adelic points $G(\mathbf{A}_k)$. Since different choices of lifting will change $G(\mathcal{O}_v)$ for only a finite number of $v$, $G(\mathbf{A}_k)$ is intrinsically defined independent of the choice of $\mathcal{O}_S$-scheme structure.

It may happen that $G(k) \subset G(\mathbf{A}_k)$ is discrete. This will be the case, for example, if $G$ is affine. If so, we may try to compute the volume of $G(\mathbf{A}_k)/G(k)$. Writing $\mathbb{F}_v$ = residue field at $v$, $q_v = \#\mathbb{F}_v$, $N_v = \#G(\mathbb{F}_v)$, the natural volume form gives $\mathrm{Vol}(G(\mathcal{O}_v)) = N_v q_v^{-1}$ for all $v \notin S$. It can happen that $\prod_v N_v q_v^{-1}$ does not converge (example: $G = G_m$), but in many cases there is an $L$-function $L(G,s)$ available such that $L(G,s) = \prod_{v \notin S} L_v(G,s)$ where the product converges absolutely for $\mathrm{Re}\, S \gg 0$ and extends meromorphically to the whole plane with $L_v(G,1) = \frac{q_v}{N_v}$. Suppose $\lim_{s \to 1} L(G,s)(s-1)^{-r} \neq 0, \infty$. The Tamagawa number $\tau(G)$ is defined by modifying the measure on $G(\mathbf{A}_k)$ so $\mathrm{Vol}(G(\mathcal{O}_v)) = 1$, all $v \notin S$, computing the measure of $G(\mathbf{A}_k)/G(k)$, and then multiplying by $\lim_{s \to 1} L(G,s)(s-1)^{-r}$. For more details, the reader should see [10].

The Tamagawa number has been computed for all except a few particularly stubborn affine algebraic groups, and takes the value (see [10, 4–6])

$$\tau(G) = \frac{\#\,\mathrm{Pic}(G)}{\#\,Ш(G)},$$

where $\mathrm{Pic}(G)$ = Picard group, and $Ш(G) = \mathrm{Ker}(H^1(\bar{k}/k, G(\bar{k}))) \to \prod_v H^1(\bar{k}_v/k_v, G(\bar{k}))$. Moreover, $r \leqq 0$, and $r = 0$ if $G(A_k)/G(k)$ is compact.

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Suppose now that $G$ is not necessarily affine, but that $G(k)$ is discrete in $G(\mathbf{A}_k)$. One conjectures that $Ш(G)$ is finite. (This is not known for a single abelian variety $G$!) $\mathrm{Pic}(X)$ may be infinite but $\mathrm{Pic}(X)_{\text{torsion}}$ is finite and one may

(0.1) *Conjecture.* $\tau(G)=\dfrac{\#\,\mathrm{Pic}(G)_{\text{tors}}}{\#\,Ш(G)}$. Moreover, $r\leqq 0$ and $r=0$ if and only if $G(\mathbf{A}_k)/G(k)$ has finite volume.

We refer to this in the sequel as the Tamagawa number conjecture.

Consider now the case of an abelian variety $A$. Conjecture (0.1) makes sense only if $A(k)$ is finite. The Hasse-Weil $L$-function $L(A,s)=\prod_{v\notin S} L_v(A,s)$, where $S$ = set of bad reduction places, and

$$L_v(A,s)=\frac{1}{\det(1-q_v^{-s}F_v\,|\,H^1_{\text{et}}(A_{\overline{\mathbb{F}}_v},\mathbb{Q}_l))}\quad (F_v=\text{geometric frobenius}).$$

Birch and Swinnerton-Dyer conjecture that $L(A,s)$ has a zero of order $r=rk\,A(k)$ at $s=1$ (so $r\geqq 0$) and that

$$\lim_{s\to 1} L(A,s)(s-1)^{-r}=\frac{\#\,Ш(A)\cdot\det\langle\ \rangle\cdot V_\infty\cdot V_{\text{bad}}}{\#\,A(k)_{\text{tors}}\cdot\#\,\mathrm{Pic}(A)_{\text{tors}}},\tag{0.2}$$

where $V_\infty=\text{Volume }A(k\otimes_{\mathbb{Q}}\mathbb{R})$ and $V_{\text{bad}}=\text{Volume }\prod_{v\in S}A(k_v)$. Finally, $\langle\ \rangle$ denotes the height pairing [1, 3]

$$\langle\ \rangle\colon A(k)\times A'(k)\to\mathbb{R}$$

with $A'(k)=\mathrm{Pic}^0(A)$.

The purpose of this note is to deduce (0.2) from (0.1), and thus to give a purely volume-theoretic interpretation of Birch and Swinnerton-Dyer. An element $\alpha\in\mathrm{Pic}(A)$ corresponds to a $\mathbb{G}_m$-torseur $X_\alpha\to A$. If $\alpha\in\mathrm{Pic}^0(A)=A'(k)$, $X_\alpha$ is a group extension of $A$ by $G_m$. We construct in this way an extension

$$0\to T\to X\to A\to 0\tag{0.3}$$

where $T$ is the split torus with character group $\cong A'(k)/\text{torsion}$. An important point is that the "logarithmic modulus" map factors

$$\begin{array}{ccccc} 0\to & T(\mathbf{A}_k) & \to & X(\mathbf{A}_k) & \\ & {\scriptstyle\text{log. mod.}}\downarrow & \swarrow & & \\ & \mathrm{Hom}(A'(k),\mathbb{R}) & & & \end{array}$$

The product formula shows $\text{log.mod.}(T(k))=(0)$, so by restriction to global points, we obtain

$$A(k)\cong X(k)/T(k)\dashrightarrow\mathrm{Hom}(A'(k),\mathbb{R}),$$

or again

$$A(k)\times A'(k)\to\mathbb{R}.\tag{0.4}$$

Using the axiomatic characterization of Neron's local pairings [1, 3], we show that (0.4) is the height pairing. From this it follows without difficulty that $X(k)$ is discrete and cocompact in $X(\mathbf{A}_k)$, and that (0.1) for $X$ implies (0.2) for $A$.

It seems likely that this technique will lead to height pairings in many new situations, e.g., for algebraic cycles other than zero cycles and divisors. I hope to return to this question in the future. I am indebted to W. Messing for several helpful discussions regarding the Neron model.

## 1. The Global Construction

Let $A$ be an abelian variety over a number field $k$. Let $N$ be the Neron model of $A$ over the ring of integers $\mathcal{O}_k$, $N^0 \subset N$ the largest open subgroup scheme whose fibres are connected. Let $A'$ be the dual abelian variety, $N'$ = Neron model of $A'$. It is known (cf. [11], p. 53) that

$$N' \cong \mathbf{Ext}^1_{\mathcal{O}\text{-group scheme}}(N^0, \mathbb{G}_m). \tag{1.1}$$

In particular, if we fix once for all a splitting

$$A'(k) = B \oplus A'(k)_{\text{tors}} \tag{1.2}$$

and use $A'(k) = N'(\mathcal{O}_k)$, we can build an extension over $\mathcal{O}_k$

$$0 \to T \to X \to N^0 \to 0, \tag{1.3}$$

where $T$ is the $k$-split torus with character group $B$. Let $A_k$ denote the adeles of $k$. Since $H^1(\operatorname{Sp} R, \mathbb{G}_m) = (0)$ for $R$ local, we get exact sequences

$$\begin{array}{ccccccccc} 0 \to & T(k) & \to & X(k) & \to & A(k) & \to 0 \\ 0 \to & T(\mathbf{A}_k) & \to & X(\mathbf{A}_k) & \to & N^0(\mathbf{A}_k) & \to 0. \end{array} \tag{1.4}$$

Define $T^1 \subset T(\mathbf{A}_k)$ by the exact sequence

$$0 \to T^1 \to T(\mathbf{A}_k) \xrightarrow{l} \operatorname{Hom}(B, \mathbb{R}) \to 0 \tag{1.5}$$

where $l$ is induced from the usual logarithmic modulus map from the ideles to $\mathbb{R}$. Define further, for $v$ a place of $k$

$$X_v^1 = \begin{cases} X(\mathcal{O}_v) & v \text{ non-archimedean} \\ X(k_v)_{\text{max. compact}} & v \text{ archimedean} \end{cases} \tag{1.6}$$

$$\tilde{X}^1 = T^1 \cdot \prod_v X_v^1 \subset X(\mathbf{A}_k).$$

Finally, let $X^1$ be the rational saturation of $\tilde{X}^1$, i.e.,

$$X^1 = \{a \in X(\mathbf{A}_k) \mid \exists n \geqq 1,\ n \in \mathbb{Z},\ na \in \tilde{X}^1\}. \tag{1.7}$$

(1.8) **Lemma.** *There exists a diagram with exact rows and columns*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & T^1 & \longrightarrow & X^1 & \longrightarrow & N^0(\mathbf{A}_k) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & T(\mathbf{A}_k) & \longrightarrow & X(\mathbf{A}_k) & \longrightarrow & N^0(\mathbf{A}_k) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle l} & & \downarrow & & & & \\
 & & \operatorname{Hom}(B,\mathbb{R}) & = & \operatorname{Hom}(B,\mathbb{R}) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

*Proof.* It suffices to show $X^1 \cap T(\mathbf{A}_k) = T^1$, and $X^1 \twoheadrightarrow N^0(\mathbf{A}_k)$. The first point is straightforward, using that $T(\mathbf{A}_k)/T^1$ is torsion-free, and $l$ is trivial on $T(\mathcal{O}_v)$ and $T(k_v)_{\text{max. compact}}$. For the second, note that the image of $\tilde{X}^1$ in $N^0(\mathbf{A}_k)$ contains $A(k_v) = N^0(\mathcal{O}_v)$ for almost all $v$ and the cokernel

$$W \underset{\text{def}}{=} N^0(\mathbf{A}_k)/\operatorname{Im}(\tilde{X}^1 \to N^0(\mathbf{A}_k))$$

is finite. It follows easily that $X(\mathbf{A}_k)/\tilde{X}^1 \cong \operatorname{Hom}(B,\mathbb{R}) \oplus W$. Replacing $\tilde{X}^1$ by $X^1$ eliminates torsion in the quotient, and the lemma follows by diagram chasing. Q.E.D.

Combining (1.4) and (1.8), and using the fact that $T(k) \subset T^1$ (product formula) we get

$$A(k) \cong X(k)/T(k) \to X(\mathbf{A}_k)/T(k) \to \operatorname{Hom}(B,\mathbb{R})$$

and hence a pairing

$$\langle\ \rangle : A(k) \times A'(k) \to \mathbb{R}.$$

(1.9) **Theorem.** *The above pairing coincides with the height pairing.*

We postpone the proof until the next section.

(1.10) **Theorem.** $X(k) \subset X(\mathbf{A}_k)$ *is discrete and cocompact.*

*Proof.* Let $U = X(k) \cap X^1 \subset X(\mathbf{A}_k)$. Since the height pairing is perfect, we get $0 \to T(k) \to U \to A(k)_{\text{tors}} \to 0$, and hence exact sequences

$$(1.11)\qquad \begin{array}{l}
0 \to T^1/T(k) \to \quad X^1/U \quad \to N^0(\mathbf{A}_k)/A(k)_{\text{tors}} \to 0 \\[1em]
0 \to \quad X^1/U \to X(\mathbf{A}_k)/X(k) \to \dfrac{\operatorname{Hom}(B,\mathbb{R})}{\operatorname{Image} A(k)} \to 0
\end{array}$$

The image of $A(k)$ in $\operatorname{Hom}(B,\mathbb{R})$ is known to be discrete and cocompact (perfectness of height pairings), and compactness is known for $T^1/T(k)$ (classical theorem about ideles) and $N^0(\mathbf{A}_k)$. The assertions of the theorem follow. Q.E.D.

What about the Tamagawa number of $X$? With notation as in the introduction, let $r = rk\,A'(k)$. We choose convergence factors in the sense of [10] for the measure on $X(\mathbf{A}_k)$:

$$(1.12)\qquad \begin{array}{ll} (1-q_v^{-1})^r L_v(A,1)^{-1} & v \text{ non-archimedean, } A \text{ has good reduction at } v, \\ (1-q_v^{-1})^r & v \text{ non-archimedean, } A \text{ does not have good reduction at } v, \\ 1 & v \text{ archimedean.} \end{array}$$

These correspond to convergence factors $(1-q_v^{-1})^r$ on $T(\mathbf{A}_k)$ and $L_v(A,1)^{-1}$ on $N^0(\mathbf{A}_k)$ ($v$ good reduction place). Writing $\zeta_k(s)$ for the zeta function of $k$ we get from (1.11)

$$\operatorname{Volume}(T^1/T(k)) = \lim_{s\to 1}(\zeta_k(s)(s-1))^r$$

$$(1.13)\qquad \operatorname{Vol.} N^0(\mathbf{A}_k) = \underbrace{\operatorname{Vol.}(A(K\otimes_{\mathbb{Q}}\mathbb{R}))}_{V_\infty}\cdot\underbrace{\prod_{v\ \mathrm{bad}}\operatorname{Vol.} A(k_v)}_{V_{\mathrm{bad}}}$$

$$\operatorname{Vol.}(X^1/U) = \frac{1}{\# A(k)_{\mathrm{tors}}}\lim_{s\to 1}(\zeta_k(s)(s-1))^r V_\infty V_{\mathrm{bad}}$$

$$\operatorname{Vol.}(X(\mathbf{A}_k)/X(k)) = \frac{1}{\# A(k)_{\mathrm{tors}}}\lim_{s\to 1}(\zeta_k(s)(s-1))^r V_\infty V_{\mathrm{bad}} R$$

where $R$ is the absolute value of the discriminant of the height pairing.

We assume now that $\lim_{s\to 1}\zeta_k(s)^r L(A,s) \neq 0, \infty$, i.e., that the $L$-function of $X$ has no zero or pole at $s=1$ as predicted by the Tamagawa number conjecture, or equivalently that the $L$-function of $A$ has a zero of order $r = rk\,A(k)$ as predicted by Birch and Swimmerton-Dyer. To define the Tamagawa number $\tau(X)$ we eliminate the (non-canonical) choice of convergence factors by dividing the volume computed above by $\lim_{s\to 1}\zeta_k(s)^r L(A,s)$, getting

$$(1.14)\qquad \tau(X) = \frac{1}{\# A(k)_{\mathrm{tors}}}\lim_{s\to 1} L(A,s)^{-1}(s-1)^r V_\infty V_{\mathrm{bad}} R.$$

Conjecture (0.2) is thus equivalent to

$$(1.15)\qquad \tau(X) \stackrel{?}{=} \frac{\# A'(k)_{\mathrm{tors}}}{\# Ш(A)}.$$

(1.16) **Lemma.** *$Ш(A) \cong Ш(X)$ and $A'(k) \cong \operatorname{Pic}(X)_{\mathrm{tors}}$.*

*Proof.* The first isomorphism follows from chasing the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & Ш(X) & \longrightarrow & Ш(A) & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^1(\bar{k}/k, X) & \longrightarrow & H^1(\bar{k}/k, A) & \longrightarrow & H^2(\bar{k}/k, T) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod_v H^1(\bar{k}_v/k_v, X) & \longrightarrow & \prod_v H^1(\bar{k}_v/k_v, A) & \longrightarrow & \prod_v H^2(\bar{k}_v/k_v, T)
\end{array}$$

For the second isomorphism, note that if $T$ is a split torus over a ring $R$ with character group $\hat{T}$, then taking units in the ring of regular functions on $T$ yields an exact sequence

$$0 \to R^* \to R[T]^* \to \hat{T} \to 0.$$

Let $\pi: X \to A$ be the projection. The above sequence globalizes

$$0 \to \mathbb{G}_{m,A} \to \pi_* \mathbb{G}_{m,X} \to B_A \to 0$$

where $B_A$ is the constant Zariski sheaf on $A$ with stalk $B$. The boundary map

$$B = \Gamma(A, B_A) \to H^1(A, \mathbb{G}_m) = \operatorname{Pic} A$$

is the natural inclusion, so we obtain

$$H^1(A, \pi_* \mathbb{G}_{m,X}) \cong (\operatorname{Pic} A)/B.$$

Locally over $A$, $Z \cong \mathbb{G}_m^r \times A$, so $R^1 \pi_* \mathbb{G}_{m,X} = (0)$ and we find

$$\operatorname{Pic} X \cong (\operatorname{Pic} A)/B$$

and a similar result holds for torsion. Q.E.D.

Combining (1.15) and (1.16) yields

(1.17) **Theorem.** *The Birch and Swinnerton-Dyer conjecture holds for $A$ if and only if*

$$\tau(X) = \frac{\# \operatorname{Pic}(X)_{\text{tors}}}{\# Ш(X)}.$$

## 2. The Local Neron Pairing

The purpose of this section is to prove (1.9). Let $k$ be a local field, $A$ an abelian variety over $k$, $N$ = Neron model of $A$, $N^0 \subset N$ the subgroup scheme with connected fibres. The Néron model of the dual variety $A' = \mathbf{Ext}^1(A, \mathbb{G}_m)$ is then $N' = \mathbf{Ext}^1_{\mathcal{O}}(N^0, \mathbb{G}_m)$ ([11], p. 53). Thus given a divisor $\Delta$ on $A$ defined over $k$ and algebraically equivalent to 0, we get a corresponding extension

(2.1) $$0 \to \mathbb{G}_m \to X_\Delta \to N^0 \to 0.$$

If $\mathscr{L}_\Delta$ is the line bundle associated to $\Delta$,

$$X_\Delta \cong V(\mathscr{L}_\Delta) - (0\text{-section})$$

as a $G_m$-torseur. The extension (2.1) depends only on the linear equivalence class of $\Delta$.

Restricting to $\operatorname{Sp} k$, the extension (2.1) is split as a torseur over $A-|\Delta|$ ($|\Delta| = \operatorname{Supp} \Delta$)

$$\text{(2.2)} \qquad \begin{array}{ccccccccc} 0 & \to & \mathbb{G}_{m,k} & \to & X_{\Delta,k} & \to & A & \to & 0 \\ & & & & & {}^{\sigma_\Delta}\nwarrow & \cup & & \\ & & & & & & A-|\Delta| & & \end{array}$$

where $\sigma_\Delta$ is canonical up to translation by $\mathbb{G}_{m,k}(k) = k^*$ (choosing $\sigma_\Delta$ is tantamount to choosing a rational section of $\mathscr{L}_\Delta$ corresponding to the divisor $\Delta$). Let $Z_{\Delta,k}$ = group of zero cycles $\mathfrak{A} = \sum n_i(p_i)$ on $A$ defined over $k$ such that $\sum n_i \deg p_i = 0$ and $\operatorname{Supp} \mathfrak{A} \subset A - |\Delta|$. We get a homomorphism

$$\text{(2.3)} \qquad \sigma_\Delta \colon Z_{\Delta,k} \to X_\Delta(k).$$

Define

$$X_\Delta^1 = \begin{cases} X_\Delta(Sp\,\mathcal{O}_k) & k \text{ non-archimedean} \\ X_\Delta(k)_{\text{max. compact}} & k \text{ archimedean,} \end{cases}$$

$$\mathbb{G}_m^1 = \begin{cases} \mathcal{O}_k^* & k \text{ non-archimedean} \\ (k^*)_{\text{max compact}} & k \text{ archimedean,} \end{cases}$$

$$F = \begin{cases} \mathbb{Z} & k \text{ non-archimedean, } N = N^0 \\ \mathbb{Q} & k \text{ non-archimedean, } N \neq N^0 \\ \mathbb{R} & k \text{ archimedean.} \end{cases}$$

(2.4) **Lemma.** *Assume either $v$ archimedean or $N = N^0$. Then there is a diagram with exact rows and columns*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \mathbb{G}_m^1 & \longrightarrow & X_\Delta^1 & \longrightarrow & A(k) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & k^* & \longrightarrow & X_\Delta(k) & \longrightarrow & A(k) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota} & & \downarrow & & & & \\ & & F & = & F & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

*Proof.* The map labeled $l$ is either the logarithm or the valuation map. As in the proof of (1.8), the only thing we need to show is $X^1_\Delta \twoheadrightarrow A(k)$. In the non-archimedean case we have $A(k) \cong N(\mathcal{O}) = N^0(\mathcal{O})$. Surjectivity $X^1_\Delta = X(\mathcal{O}) \twoheadrightarrow N^0(\mathcal{O})$ follows from $H^1(Sp\,\mathcal{O}, \mathbb{G}_m) = (0)$. In the archimedean case, the existence of an exponential implies the connected component of 0 in $A(k)$ is contained in the image of $X^1_\Delta$. Factoring out by $X^1_\Delta$, we obtain an extension of a finite group by $\mathbb{R}$. Such an extension is necessarily split, so we get

$$0 \to X^1_\Delta \to X_\Delta(k) \to \mathbb{R} \oplus (\text{finite}) \to 0.$$

Since $X^1_\Delta$ is maximal compact, (finite) $= (0)$. Q.E.D.

Suppose now $N \neq N^0$, and let $A(k)_0 = \text{Image}(X^1_\Delta \to A(k))$. Note $A(k)/A(k)_0$ is finite, and we have a diagram (defining $Y$)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{G}^1_m & \longrightarrow & X^1_\Delta & \longrightarrow & A(k)_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & k^* & \longrightarrow & X_\Delta(k) & \longrightarrow & A(k) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & Y & \longrightarrow & A(k)/A(k)_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

In particular $Y \otimes \mathbb{Q} \cong \mathbb{Q}$ (canonically) so we get $X_\Delta(k) \to Q$.

In any of the above cases, let $\psi_\Delta\colon X_\Delta(k) \to F$ be the map just defined, and for $\mathfrak{A} \in Z_{\Delta,k}$ define

$$\langle \Delta, \mathfrak{A} \rangle_{\text{local}} = \psi_\Delta \sigma_\Delta(\mathfrak{A}). \tag{2.6}$$

When the given ground field is the completion of a global field at some place $v$, we write $\langle\ \rangle_v$ instead of $\langle\ \rangle_{\text{local}}$.

(2.7) **Theorem.** *Let $k$ be a number field. Let $a \in A(k)$, $a' \in A'(k)$. Let $\Delta$ (resp. $\mathfrak{A}$) be a divisor algebraically equivalent to zero defined over $k$ (resp. a zero cycle of degree* 0 *defined over $k$) on $A$ such that $[\Delta] = a'$ (resp. $\mathfrak{A}$ maps to $a \in A(k)$). Assume further that* Supp $\Delta$ *and* Supp $\mathfrak{A}$ *are disjoint. Then*

$$\langle a, a' \rangle = \sum_{\substack{v\,\text{place}\\ \text{of } k}} \langle \Delta, \mathfrak{A} \rangle_v,$$

*where $\langle a, a' \rangle$ is defined as in* (1.9).

*Proof.* Consider the global extension

$$0 \to T \to X \to N^0 \to 0$$

as in section 1 and push out along $a' \in \hat{T}$ to get

$$0 \to \mathbb{G}_m \to X_\Delta \to N^0 \to 0.$$

We can think of $\sigma_\Delta\colon Z_{\Delta,k} \to X_\Delta(k)$ just as in the local case. The problem therefore reduces to showing the map

$$X_\Delta(\mathbf{A}_k) \to \mathbb{R} \tag{2.8}$$

defined via the techniques of Sect. 1 coincides with the sum of the local maps

$$\psi_{\Delta,v}\colon X_\Delta(k_v) \to F = \begin{cases} \mathbb{Z} \\ \mathbb{Q} \\ \mathbb{R} \end{cases}$$

defined in the beginning of this paragraph.

Finally, this point is clear from the diagram

$$\begin{array}{ccccccc} & & \mathbb{G}^1_{m,k} & \longrightarrow & X_\Delta(\mathbf{A}_k)/\prod_v X^1_{\Delta,v} & \longrightarrow & X_\Delta(\mathbf{A}_k)/X^1_\Delta \longrightarrow 0 \\ & & \downarrow \Sigma v & & \downarrow \Sigma\psi_{\Delta,v} & & \downarrow (2.8) \\ 0 & \to & \mathrm{Ker(sum)} & \longrightarrow & \prod_{v\,\mathrm{arch.}} \mathbb{R} \times \coprod_{v\,\mathrm{non\text{-}arch}} \mathbb{Q} & \xrightarrow{\mathrm{sum}} & \mathbb{R} \longrightarrow 0 \end{array} \tag{2.9}$$

Néron has shown [3] that the height pairing can be written as a sum of local terms. With notation as in (2.7)

$$\langle a, a' \rangle = \sum_v \langle \Delta, \mathfrak{A} \rangle_{v,\mathrm{Neron}}. \tag{2.10}$$

(2.11) **Proposition.** $\langle \Delta, \mathfrak{A} \rangle_v = \langle \Delta, \mathfrak{A} \rangle_{v,\mathrm{Neron}}$.

*Proof.* We write $v\colon k_v^* \to \mathbb{R}$ for the logarithmic valuation, normalized in accordance with the global product formula. Let $D_a(A)_k$ denote the group of divisors on $A$ algebraically equivalent to zero and defined over $k$. The local Néron pairing is characterized by the following properties:

(1) $\langle\ \rangle_{v,\mathrm{Neron}}\colon \{(\Delta, \mathfrak{A}) \in D_a(A)_k \times Z_k(A) \mid |\Delta| \cap |\mathfrak{A}| = \emptyset\} \to \mathbb{R}$

(2) $\langle\ \rangle_{v,\mathrm{Neron}}$ is bilinear, assuming all terms in the desired equality are defined.

(3) If $\Delta = (f)$, then $\langle \Delta, \mathfrak{A} \rangle_{v,\mathrm{Neron}} = v(f(\mathfrak{A}))$, where for $\mathfrak{A} = \sum n_i(p_i)$, $f(\mathfrak{A}) = \prod_i f(p_i)^{n_i}$.

(4) $\langle \Delta, \mathfrak{A} \rangle_{v,\mathrm{Neron}} = \langle \Delta_a, \mathfrak{A}_a \rangle_{v,\mathrm{Neron}}$, where $a \in A(k_v)$ and the subscript indicates translation by $a$.

(5) For $\Delta \in D_a(A)_{k_v}$ and $x_0 \in A(k_v) - |\Delta|$, the map

$$x \mapsto \langle \Delta, (x) - (x_0) \rangle_{v,\text{Neron}}$$

is bounded on every $v$-bounded subset of $A(k_v) - |\Delta|$.
(Here $v$-bounded subset means subset of a coordiante neighborhood on which $v$ (coordinate functions) are bounded.)

We show that the pairing $(\Delta, \mathfrak{A}) \mapsto \langle \Delta, \mathfrak{A} \rangle_v$ satisfies condition (1)-(5), except that (4) will be proven only for $a \in N^0(\mathcal{O}) \subset A(k)$.

(2.12) **Lemma.** $\langle\ \rangle_v$ *satisfies* (3).

*Proof.* Let $\Delta = (f)$. Then $X_\Delta \cong \mathbb{G}_m \times N^0$ and

$$\sigma_\Delta \colon A - |\Delta| \to \mathbb{G}_m \times A$$
$$\sigma_\Delta(a) = (f(a), a).$$

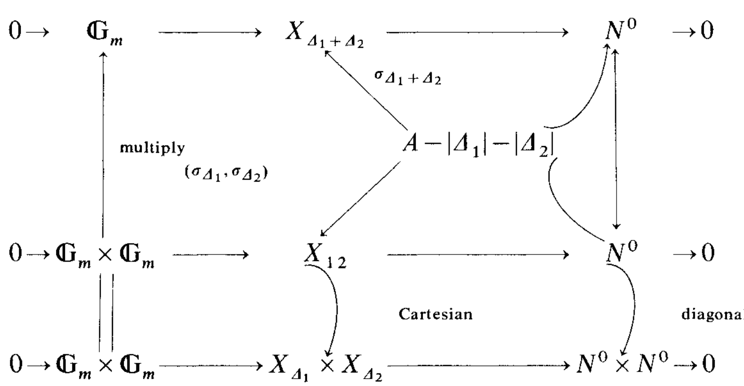
Since $\psi_\Delta = v$ on $\mathbb{G}_m(k_v) = k_v^*$, the lemma follows. Q.E.D.

(2.13) **Lemma.** $\langle\ \rangle_v$ *satisfies* (2), *i.e., it is bilinear.*

*Proof.* Bilinearity in $\mathfrak{A} \in Z_k(A)$ holds by definition. We must show

$$\langle \mathfrak{A}, \Delta_1 \rangle + \langle \mathfrak{A}, \Delta_2 \rangle = \langle \mathfrak{A}, \Delta_1 + \Delta_2 \rangle$$

whenever $\Delta_i \in D_a(A)$ and $|\mathfrak{A}| \cap (|\Delta_1| \cup |\Delta_2|) = \emptyset$. Note that $\sigma_{\Delta_1+\Delta_2}$ can be taken to be the "sum" in the sense of torseurs of $\sigma_{\Delta_1}$ and $\sigma_{\Delta_2}$, i.e., the rational section of $\mathcal{L}_{\Delta_1+\Delta_2}$ can be taken to be the tensor or rational sections of $\mathcal{L}_{\Delta_1}$ and $\mathcal{L}_{\Delta_2}$. The diagram




commutes, where $X_{12}$ is the pullback as indicated. Defining $X_{12}^1$ in the same way as $X_\Delta^1$ above, one finds

$$(X_{12}(k)/X_{12}^1) \otimes \mathbb{Q} \cong [(k^*/\mathbb{G}_m^1) \times (k^*/\mathbb{G}_m^1)] \otimes \mathbb{Q}$$

and the map $X_{12}(k)/X_{12}^1 \to X_{\Delta_1+\Delta_2}(k)/X_{\Delta_1+\Delta_2}^1$ corresponds to addition on $k^*/\mathbb{G}_m^1$. The assertion of the lemma now follows. Q.E.D.

(2.14) **Lemma.** *Let* $a \in N^0(\mathcal{O}) \subset A(k)$. *Then* $\langle \Delta, \mathfrak{A} \rangle = \langle \Delta_a, \mathfrak{A}_a \rangle$.

*Proof.* Let $\delta_a$: $N^0 \to N^0$ be translation by $a$. There is a map of $G_m$-torseurs $\tau_a$: $X_\Delta \to X_{\Delta_a}$ such that the diagram

$$\begin{array}{ccccc} X_\Delta & & \xrightarrow{\tau_a} & & X_{\Delta_a} \\ & \nwarrow^{\sigma_\Delta} & & {}^{\sigma_{\Delta_a}}\nearrow & \\ & A-|\Delta| & \xrightarrow{\delta_a} & A-|\Delta_a| & \\ & \swarrow & & \searrow & \\ A & & \xrightarrow[\delta_a]{} & & A \end{array}$$

commutes for suitable choice of $\sigma_\Delta, \sigma_{\Delta_a}$.

The key point is that we may choose $\tau_a$ such that $\tau_a(X_\Delta^1) \subset X_{\Delta_a}^1$. This is clear in the non-archimedean case because $X_\Delta^1 = X_\Delta(\mathcal{O})$ and it suffices to take $\tau_a$ defined over $\mathcal{O}$. In the archimedean case, choose $\tilde{a} \in X_{\Delta_a}^1$ lying over $a$ and consider the composition

$$\begin{array}{ccccc} X_\Delta(k) & \xrightarrow{\tau_a} & X_{\Delta_a}(k) & \xrightarrow{\delta_{-\tilde{a}}} & X_{\Delta_a}(k) \\ \downarrow & & \downarrow & & \downarrow \\ A(k) & \xrightarrow{\delta_a} & A(k) & \xrightarrow{\delta_{-a}} & A(k) \end{array}$$

Modifying $\tau_a$ by an element of $k^*$ we may assume $\delta_{-a} \circ \tau_a$ is the identity on $k^*$, whence an isomorphism of *groups* $X_\Delta(k) \xrightarrow{\cong} X_{\Delta_a}(k)$. Thus $\delta_{-\tilde{a}} \circ \tau_a(X_\Delta^1) = X_{\Delta_a}^1$. Since $\tilde{a} \in X_{\Delta_a}^1$ we get $\tau_a(X_\Delta^1) = X_{\Delta_a}^1$.

Since subtracting $\tilde{a}$ does not change the image of a point in $X_{\Delta_a}$ under $\psi_{\Delta_a}$, the above discussion actually shows that for any zero cycle $z$ on $X_\Delta$ defined over $k$ we have $\psi_\Delta(z) = \psi_{\Delta_a} \tau_a(z)$. Thus

$$\begin{aligned} \langle \Delta_a, \mathfrak{A}_a \rangle_v &= \psi_{\Delta_a}(\sigma_{\Delta_a} \delta_a)(\mathfrak{A}) = \psi_{\Delta_a}(\tau_a \sigma_\Delta)(\mathfrak{A}) \\ &= \psi_\Delta \sigma_\Delta(\mathfrak{A}) = \langle \Delta, \mathfrak{A} \rangle_v. \quad \text{Q.E.D.} \end{aligned}$$

(2.15) **Lemma.** *The pairing* $\langle\ \rangle_v$ *satisfies condition* (5).

*Proof.* The assignment $x \mapsto \langle \Delta, (x) - (x_0) \rangle_v$ is continuous, and $v$-bounded sets are compact. Q.E.D.

*Proof. of (2.11).* Let $\{\Delta, \mathfrak{A}\} = \langle \Delta, \mathfrak{A} \rangle_{v, \text{Neron}} - \langle \Delta, \mathfrak{A} \rangle_v$. We have $\{(f), \mathfrak{A}\} = 0$ so we may define

$$\{\ \}: A'(k) \times Z_k(A) \to \mathbb{R}.$$

Let $Z_k(A)^0 \subset Z_k(A)$ be those zero cycles $\sum n_i(p_i)$ such that $p_i \in N^0(\mathcal{O}) \subset A(k)$. There is a natural surjection $Z_k(A)^0 \twoheadrightarrow N^0(\mathcal{O})$ with kernel generated by elements $(a_1 + a_2) - (a_1) - (a_2) + (0)$, $a_i \in N^0(\mathcal{O})$. Translation invariance implies $\{\ \}$ factors through $\{\ \}: A'(k) \times N^0(\mathcal{O}) \to \mathbb{R}$. The image under $\{\Delta, \cdot\}$ of a subgroup of $N^0(\mathcal{O})$ contained in a $v$-bounded neighborhood of 0 is trivial by (5). It follows that $\{\ \} = 0$. Q.E.D.

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