A NOTE ON HEREDITARY RINGS OR NON-SINGULAR RINGS WITH CHAIN CONDITION

NGUYEN V. DUNG

1. Introduction.

A well-known result of Osofsky [10, 11] states that a ring R is semisimple artinian iff every cyclic R-module is injective. From this it follows that a right self-injective right hereditary ring is semisimple artinian. In this paper we shall extend these results of Osofsky in some ways.

Following Chatters-Hajarnavis [1] a ring R is called right CS if every closed right ideal of R is a direct summand of R_R. Clearly, the class of right CS rings includes all right self-injective rings. In Theorem 3.1 we prove that a right CS right hereditary ring is right noetherian. By the way, this result closely relates to a result of Goodearl [8] (see also Colby-Rutter [3]) characterizing rings over which non-singular right modules are projective (Corollary 3.2).

Further we study right hereditary rings R whose injective hulls $E(R_R)$ are finitely generated. In some cases we are able to show that the rings in question are right artinian (Theorem 3.4, Corollary 3.7). Finally we give a new characterization of semisimple artinian rings as right self-injective rings whose non-zero singular right modules contain non-zero injective submodules (Theorem 3.8). This result extends a result of Smith [15, Theorem 2.10].

2. Definitions and notations.

Throughout this paper we consider associative rings with identity and unitary modules. We write M_R to indicate that M is a right module over a ring R. To say that M_R has finite Goldie dimension means that M_R does not contain infinite direct sums of non-zero submodules. A submodule K of M is called essential in M if $K \cap L \neq 0$ for each non-zero submodule L of M. In this case M is called an essential extension of K. A submodule C of M is called a closed submodule of M if C is the only essential extension of C in M. A right ideal of R is closed (essential) if it is closed (essential) as submodule of R_R.

For a right R-module M, $E(M)$ denotes the injective hull of M, and the submodule

$$Z(M) = \{x \in M \mid xK = 0 \text{ for some essential right ideal } K \text{ of } R\}$$

is called the singular submodule of M. If $Z(M) = M$, M is called a singular module. If $Z(M) = 0$, M is said to be non-singular. A ring R is right non-singular if R_R is non-singular.

3. Results.

We start with proving the following theorem.

THEOREM 3.1. *A right* CS *right hereditary ring is right noetherian.*

PROOF. Let R be a right CS, right hereditary ring. First we show that R does not contain an infinite set of non-zero orthogonal idempotents. We do this by applying the technique of Osofsky in [11]. Assume on the contrary that there is an infinite set $\{e_i, i \in I\}$ of orthogonal idempotents $e_i \neq 0$ in R. Let P be a subset of I and put $A = \bigoplus_{i \in P} e_i R$. Since R is right CS, there exists an idempotent f in R such that A_R is essential in fR. Clearly we have $fe_i = e_i$ for all $i \in P$. Let $e_j \in \{e_i\}$ with $j \notin P$. Since R is right non-singular, R has a maximal right quotient ring Q which is von Neumann regular (see e.g. [9] or [17]). Hence there is an idempotent g of Q such that $Qe_j f = Qg$. Suppose that $fgQ \cap A \neq 0$. Then there is an x in A such that $0 \neq x = fgq$ with $q \in Q$. Hence $e_j fgq = e_j fq = 0$. Since $g \in Qe_j f$ we have $gq = 0$, implying $x = 0$, a contradiction. Thus $fgQ \cap A = 0$. Since fR is essential in fQ_R (see e.g. [9, Chapter 2]), it follows that A_R is essential in fQ_R, so we must have $fgQ = 0$, in particular $fg = 0$. Therefore $e_j f = e_j fg = 0$ for all $j \notin P$. Thus we can apply [11, Theorem] to arrive at a contradiction: Let $B = \sum_{i \in I} (e_i R + \ker \varphi)$ where φ: $R \to \prod_{i \in I} e_i R$, $\varphi(x) = \langle e_i x \rangle$, $x \in R$. Then B is a right ideal of R. Now by [11, Theorem] M_R/B_R is not injective for all $M_R \supset R_R$ (clearly $B \neq R$). But since R is right hereditary, $E(R_R)/B_R$ has to be injective. This contradiction shows that R does not contain an infinite set of non-zero orthogonal idempotents. Hence there are orthogonal idempotents $e_1, \ldots, e_n$ in R such that

$$R_R = e_1 R \oplus \ldots \oplus e_n R,$$

where each $e_i R$ is an indecomposable right R-module. From this we could use [1, Proposition 2.3], but we prefer giving a direct proof here for completeness. Let C be a non-zero closed submodule of $e_i R$. Then C_R is closed in R_R too. Hence $R_R = C \oplus C'$ for some submodule C' of R_R. From this we see that C_R is also a direct summand of $e_i R_R$. It follows that $C = e_i R$ are uniform right ideals of R.

Hence R_R has finite Goldie dimension. Since R is right hereditary, by [14, Corollary 2] we know that R is right noetherian.

The proof of Theorem 3.1 is complete.

Using Theorem 3.1 we can prove the equivalence (a) $\Leftrightarrow$ (b) in the following result given in [3, Theorem 3.2] and [8, Theorem 2.15].

COROLLARY 3.2. *For a ring R the following conditions are equipvalent:*
(a) R *is right hereditary and* $E(R_R)$ *is projective.*
(b) $Z(R_R) = 0$ *and every non-singular right R-module is projective.*
(c) R *is artinian, serial and hereditary on both right and left sides.*

PROOF. (b) $\Leftrightarrow$ (c) is proved in [8, Theorem 2.15]. (b) $\Rightarrow$ (a) is clear. (a) $\Rightarrow$ (b): Assume (a). Then every submodule of $E(R_R)$ is projective. Let A be any right ideal of R and B be a maximal essential extension of A_R in R_R. Then R/B_R is a non-singular cyclic right R-module. Using [6, Lemma 4] we see that R/B_R is embedded in $E(R_R)$. Therefore R/B_R is projective, so B_R is a direct summand of R_R, proving that R is right CS. Hence R is right noetherian by Theorem 3.1. Now let M be a non-singular right R-module, and let N denote the injective hull of M. Then N is also non-singular, and $N = \underset{i \in I}{\oplus} N_i$ where each N_i is indecomposable and injective. Let X_i be a non-zero cyclic submodule of N_i. Since X_i is non-singular we see by using [6, Lemma 4] that X_i is isomorphic to a submodule of $E(R_R)$, and since N_i is uniform, N_i can be embedded in $E(R_R)$, too. It follows that each N_i is projective, therefore N is also projective. Since R is right hereditary, M is projective, proving (b).

We note that if R is a (right non-singular) ring with $E(R_R)$ projective, then $E(R_R)$ is finitely generated. From this and Corollary 3.2 it arises a question: Does a right hereditary ring R with finitely generated $E(R_R)$ satisfy some kind of chain conditions on right ideals?

The purpose of the remainder of this paper is to answer this question in some special cases and to prove a related result. Recall that a module M is called completely injective if every homomorphic image of M is injective. By the same proof as that of [4, Proposition] without any change we obtain.

LEMMA 3.3. *Let M_R be a cyclic finitely presented completely injective module such that $\text{End}_R(M_R)$ is a von Neumann regular right self-injective ring. Then M_R has finite Goldie dimension.*

Now we can give a sufficient condition for a right hereditary ring to be right artinian.

THEOREM 3.4. *Let R be a right hereditary ring such that every closed right ideal*

of R is finitely generated and $E(R_R)$ is a direct sum of cyclic submodules. Then R is right artinian.

PROOF. Suppose that R is as in Theorem 3.4. Then $E(R_R) = E_1 \oplus \ldots \oplus E_n$, where each E_i is cyclic. Since R is right hereditary, in particular right non-singular, each E_i is non-singular. Hence for each i there is a closed right ideal A_i of R such that $E_i \approx R/A_i$. Moreover by hypothesis each A_i is finitely generated. From this and since R is right hereditary, each E_i is a cyclic finitely presented completely injective module. Then by [7, Corollary 19.29], each $\text{End}_R E_i$ is von Neumann regular and right self-injective. Then by Lemma 3.3, each E_i has finite Goldie dimension. It follows that R has finite right Goldie dimension. Hence R is right noetherian by [14, Corollary 2]. Since $E(R_R)$ is moreover finitely generated, it follows that R is right artinian by [18, Theorem A].

The proof of Theorem 3.4 is complete.

Next we consider right non-singular rings with cyclic injective hulls. The following lemma is similar to [12, Lemma 1.7].

LEMMA 3.5. *Let R be a ring such that $E(R_R)$ is cyclic. If R_R does not contain an infinite direct sum of isomorphic submodules, then R is right self-injective.*

PROOF. Let R be a ring such that $E(R_R) \approx R/A$ for some right ideal A of R. Then there is a short exact sequence

$$0 \to A \to R \xrightarrow{\;\varphi\;} E(R_R) \to 0.$$

Take $c \in R$ such that $\varphi(c) = 1$ where 1 is the identity of R. Suppose that $cx \in A$ for some $x \in R$. Then $0 = \varphi(cx) = \varphi(c)x = x$. It follows that $cx \in A$ iff $x = 0$ and hence $cR \cap A = 0$. Now assume that R_R does not contain an infinite direct sum of (non-zero) isomorphic submodules. Consider the infinite sum of submodules $c^i A$:

(*) $$A + cA + c^2 A + \ldots + c^n A + \ldots$$

Clearly $c^n A \approx A$ for all n. Suppose that there are $x_0, \ldots, x_n$ in A such that $x_0 + cx_1 + \ldots + c^n x_n = 0$. Then $-x_0 = c(x_1 + \ldots + c^{n-1} x_n)$. It follows that $x_0 = 0$ and $x_1 + cx_2 + \ldots + c^{n-1} x_n = 0$. From this we can easily verify that $x_0 = x_1 = \ldots = x_n = 0$, proving that the sum (*) is direct, a contradiction to the assumption in case $A \neq 0$. Hence $A = 0$, showing that R is right self-injective.

PROPOSITION 3.6. *Let R be a right non-singular ring such that $E(R_R)$ is a cyclic finitely presented completely injective right R-module. Then R is semisimple artinian.*

PROOF. Clearly $E(R_R)$ is non-singular, hence $S = \text{End}_R E(R_R)$ is von Neumann regular and right self-injective. From Lemma 3.3 we obtain that S is semisimple artinian. On the other hand, Lemma 3.5 shows that R is right self-injective. Hence R is semisimple artinian.

COROLLARY 3.7. *If R is a right hereditary ring such that $E(R_R)$ is cyclic and finitely presented, then R is semisimple artinian.*

QUESTION. Let R be a right hereditary ring such that $E(R_R)$ is cyclic. Is R necessarily semisimple artinian?

Following Goodearl [8], a ring R is defined to be a right SI ring if every singular right R-module is injective. If every singular cyclic right R-module is injective, then R is called right RIC by Smith [15, 16]. Recently Osofsky and Smith [13] have proved that every right RIC ring is right SI. Now a ring R is called right weakly SI (briefly, right WSI) if every singular right R-module $\neq 0$ contains a non-zero injective submodule. Clearly right SI rings are right WSI, but we do not know if the converse holds.

Let R be a right SI ring. Then for each essential right ideal B of R, R/B is semisimple (see [8, Theorem 3.11]). From this and [5, Theorem 5] we see that a right or left self-injective right SI ring is quasi-Frobenius. Then using [15, Theorem 2.10] we obtain that R is semisimple artinian. For right WSI rings we have the following result.

THEOREM 3.8. *A right self-injective, right WSI ring is semisimple artinian.*

PROOF. Let R be a right self-injective, right WSI ring, and let Z denote the singular submodule of R_R. Suppose that $Z \neq 0$. Then Z contains a non-zero injective right ideal of R which is clearly projective, a contradiction. Hence $Z = 0$, or with other words, R is right non-singular. Hence R is a von Neumann regular ring (see [7, Corollary 19.28]).

Let A be a countably generated right ideal of R which is not finitely generated. Since R_R is injective, there exists an indempotent e of R such that $E(A_R) = eR$. Let $C = A \oplus (1 - e)R$. Then C_R is countably generated and essential in R_R with $C \neq R$. Now, since R/C_R is a non-zero singular right R-module, by hypothesis we have $R/C_R = I \oplus N$ for some non-zero injective module I and a module N. Then there is a right ideal D of R such that $C \subset D \subset R$, $D/C \approx N$ and $R/D \approx I$. Since N_R is cyclic and C_R is countably generated, it follows that D_R is also countably generated. Clearly, $D \neq R$ and D_R is not finitely generated, for otherwise D_R would be a direct summand of R_R, a contradiction to the fact that D is essential in R_R. Now by using [7, Proposition 19.25] and by an easy induction proof we see that there is an infinite set of orthogonal idempotents $\{e_i\}_{i=1}^{\infty}$ in R such that $D = \bigoplus_{i=1}^{\infty} e_i R$. Then by [10, Lemma 5] we see that R/D_R is not injective, a contradiction to $R/D_R \approx I$ injective. This shows that every countably generated right ideal of R is finitely generated. Then, since R is von Neumann regular, R is semisimple artinian.

The proof of Theorem 3.8 is complete.

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INSTITUTE OF MATHEMATICS
P.O. BOX 631 BO HO
HANOI
VIETNAM