A note on high-order short-time expansions for ATM option prices under the CGMY model

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Abstract

The short-time asymptotic behavior of option prices for a variety of models with jumps has received much attention in recent years. In the present work, a novel third-order approximation for ATM option prices under the CGMY Lévy model is derived, and extended to a model with an additional independent Brownian component. Our results shed new light on the connection between both the volatility of the continuous component and the jump parameters and the behavior of ATM option prices near expiration.

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1 Introduction

Stemming in part from its importance for model calibration and testing, small-time asymptotics of option prices have received a lot of attention in recent years (see, e.g., [2], [3], [7], [8], [9], [12], and references therein). In the present paper, we study the small-time behavior for at-the-money (ATM) call (or equivalently, put) option prices

$$\Pi(t) := \mathbb{E}\left(S_t - S_0\right)^+ = S_0\mathbb{E}\left(e^{X_t} - 1\right)^+, \quad t \geq 0, \tag{1.1}$$

under the exponential Lévy model

$$S_t := S_0 e^{X_t}, \quad t \geq 0, \tag{1.2}$$

with $X_t := L_t + \sigma W_t$, where $(L_t)_{t\geq 0}$ is a CGMY Lévy process while $(W_t)_{t\geq 0}$ is an independent standard Brownian motion. Throughout, $x^+ := x\mathbf{1}_{\{x>0\}}$ and $x^- := x\mathbf{1}_{\{x<0\}}$ denote the positive and negative parts of a real x. The first order asymptotic behavior of (1.1) in short-time takes the form:

$$\lim_{t\to 0} t^{-\frac{1}{Y}}\frac{1}{S_0}\mathbb{E}(S_t - S_0)^+ = \mathbb{E}(Z^+), \tag{1.3}$$

where Z is a symmetric α-stable random variable with $\alpha = Y$ under $\mathbb{P}$. When $\sigma \neq 0$, $Z \sim \mathcal{N}(0, \sigma^2)$ $(Y = 2)$ and, thus, $\mathbb{E}(Z^+) = \sigma/\sqrt{2\pi}$ (see [9] and [12]). When $\sigma = 0$, the characteristic function of Z is explicitly given (see [3] and [12]) by

$$\mathbb{E}\left(e^{iuZ}\right) = \exp\left(-2C\Gamma(-Y)\left|\cos\left(\frac{\pi Y}{2}\right)\right| |u|^Y\right).$$

In that case, (see (25.6) in [11]),

$$\mathbb{E}(Z^+) = \frac{1}{\pi}\Gamma\left(1 - \frac{1}{Y}\right)\left(2C\Gamma(-Y)\left|\cos\left(\frac{\pi Y}{2}\right)\right|\right)^{\frac{1}{Y}}.$$

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Moreover (see [4]), in the pure-jump CGMY case ($\sigma = 0$), the second-order asymptotic behavior of the ATM call option price (1.1) in short-time is of the form

$$\frac{1}{S_0}\mathbb{E}(S_t - S_0)^+ = d_1 t^{\frac{1}{Y}} + d_2 t + o(t), \quad t \to 0, \tag{1.4}$$

while in the case of a non-zero independent Brownian component ($\sigma \neq 0$),

$$\frac{1}{S_0}\mathbb{E}(S_t - S_0)^+ = d_1 t^{\frac{1}{2}} + d_2 t^{\frac{3-Y}{2}} + o\left(t^{\frac{3-Y}{2}}\right), \quad t \to 0, \tag{1.5}$$

for different constants d_1 and d_2, which are determined explicitly in the sequel. For extensions of these results to a more general class of Lévy processes, we refer the reader to [5].

In this note, we derive the third-order asymptotic behavior of the ATM option prices in the CGMY model both with and without a Brownian component. As in [4] and [5], the main ingredient in our approach is a change of probability measure under which the process $(L_t)_{t\geq 0}$ becomes a stable Lévy process. There is an important motivation to consider the third-order expansion. As shown in the numerical examples provided in [5], though being a significant improvement over the first-order expansion, in some cases, the second-order expansion is not that accurate unless t is relatively small, especially under the presence of a Brownian component. Indeed, as it turns out, in the latter situation, the first two terms of the expansion do not even reflect the relative intensities of the negative or positive jumps (as dictated by the parameters G and M).

The remaining of the paper is organized as follows. Section 2 contains preliminary results on the CGMY model, some probability measure transformations, and asymptotic results for stable Lévy processes which will be needed throughout the paper. Section 3 establishes the third-order asymptotics of the ATM call option price under both the pure-jump CGMY model ($\sigma = 0$) and the CGMY model with an additional independent non-zero Brownian component ($\sigma \neq 0$). The proofs of our main results are deferred to the Appendix.

2 Setup and preliminary results

Throughout, $W = (W_t)_{t\geq 0}$ and $L = (L_t)_{t\geq 0}$ respectively stand for a standard Brownian motion and a CGMY Lévy process independent of each other (cf. [1]) defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. As usual, we denote the parameters of L by C, G, $M > 0$ and $Y \in (0, 2)$ so that the Lévy measure of L is given by

$$\nu(dx) = \left(\frac{Ce^{-Mx}}{x^{1+Y}}\mathbf{1}_{\{x>0\}} + \frac{Ce^{Gx}}{|x|^{1+Y}}\mathbf{1}_{\{x<0\}}\right)dx.$$

Hereafter, we assume $Y \in (1, 2)$, $M > 1$, zero interest rate, and that $\mathbb{P}$ is a martingale measure for the exponential Lévy model (1.2) with log-return process $X_t := L_t + \sigma W_t$, $t \geq 0$. In particular, the characteristic function of X_1 is given by

$$\mathbb{E}\left(e^{iuX_1}\right) = \exp\left(icu - \frac{\sigma^2 u^2}{2} + C\Gamma(-Y)\left((M - iu)^Y + (G + iu)^Y - M^Y - G^Y\right)\right), \tag{2.1}$$

with $c := -C\Gamma(-Y)\left((M-1)^Y + (G+1)^Y - M^Y - G^Y\right) - \sigma^2/2$. The following terminology will be needed in what follows:

$$M^* = M - 1, \quad G^* = G + 1, \quad c^* = c + \sigma^2, \quad \varphi(x) := M^* x\,\mathbf{1}_{\{x>0\}} - G^* x\,\mathbf{1}_{\{x<0\}}, \quad \nu^*(dx) = e^x \nu(dx). \tag{2.2}$$

We will make use of two density transformations of the Lévy process (see [11, Definition 33.4]). Hereafter, $\mathbb{P}^*$ and $\widetilde{\mathbb{P}}$ are probability measures on $(\Omega, \mathcal{F})$ such that for any $t \geq 0$:

$$\frac{d\mathbb{P}^*|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} = e^{X_t}, \qquad \frac{d\widetilde{\mathbb{P}}|_{\mathcal{F}_t}}{d\mathbb{P}^*|_{\mathcal{F}_t}} = e^{U_t}, \tag{2.3}$$

where

$$U_t := \lim_{\epsilon \to 0}\left(\sum_{s\leq t: |\Delta X_s| > \varepsilon} \varphi(\Delta X_s) - t\int_{|x|>\varepsilon}(e^{\varphi(x)} - 1)\nu^*(dx)\right).$$

Throughout, $\mathbb{E}^*$ and $\widetilde{\mathbb{E}}$ denote the expectations under $\mathbb{P}^*$ and $\widetilde{\mathbb{P}}$, respectively.

From the density transformation and the Lévy-Itô decomposition of a Lévy process ([11, Theorems 19.2 and Theorem 33.1]), $(X_t)_{t\geq 0}$ can be written as

$$X_t = L_t^* + \sigma W_t^*, \quad t \geq 0, \tag{2.4}$$

where, under $\mathbb{P}^*$, $(W_t^*)_{t\geq 0}$ is again a Wiener process while $(L_t^*)_{t\geq 0}$ is still a CGMY process, independent of W^*, but with parameters C, Y, $M = M^*$ and $G = G^*$. The Lévy triplet of $(X_t)_{t\geq 0}$ under $\mathbb{P}^*$ is given by $(b^*, (\sigma^*)^2, \nu^*)$ with $\sigma^* := \sigma$ and

$$b^* := c^* - \int_{|x|>1} x\nu^*(dx) - CY\Gamma(-Y)((M^*)^{Y-1} - (G^*)^{Y-1}). \tag{2.5}$$

Similarly, under the measure $\widetilde{\mathbb{P}}$, the process $(L_t^*)_{t\geq 0}$ is a stable Lévy process and $(W_t^*)_{t\geq 0}$ is still a Wiener process independent of L^*. Concretely, setting

$$\tilde{\nu}(dx) := C|x|^{-Y-1}dx, \qquad \tilde{b} = b^* + \int_{|x|\leq 1} x(\tilde{\nu} - \nu^*)(dx),$$

under $\widetilde{\mathbb{P}}$, $(X_t)_{t\geq 0}$ is a Lévy process with Lévy triplet $(\tilde{b}, \sigma^2, \tilde{\nu})$. In particular,

$$\tilde{\gamma} := \widetilde{\mathbb{E}} X_1 = -C\Gamma(-Y)\left((M-1)^Y + (G+1)^Y - M^Y - G^Y\right) + \frac{\sigma^2}{2}, \tag{2.6}$$

and the centered process

$$Z_t := L_t^* - t\tilde{\gamma}, \quad t \geq 0, \tag{2.7}$$

is symmetric and strictly Y-stable under $\widetilde{\mathbb{P}}$, and thus, is self-similar; i.e., $(t^{-1/Y} Z_{ut})_{u\geq 0} \stackrel{\mathfrak{D}}{=} (Z_u)_{u\geq 0}$, for any $t > 0$.

The process $(U_t)_{t\geq 0}$ can be expressed in terms of the jump-measure $N(dt, dx) := \#\{(s, \Delta X_s) \in dt \times dx\}$ of $(X_t)_{t\geq 0}$ and its compensator $\bar{N}(dt, dx) := N(dt, dx) - \tilde{\nu}(dx)dt$ (under $\widetilde{\mathbb{P}}$), namely,

$$U_t := \widetilde{U}_t + \eta t := M^* \bar{U}_t^{(p)} - G^* \bar{U}_t^{(n)} + \eta t, \quad t \geq 0, \tag{2.8}$$

where

$$\bar{U}_t^{(p)} := \int_0^t \int_{(0,\infty)} x\, \bar{N}(ds, dx), \qquad \bar{U}_t^{(n)} := \int_0^t \int_{(-\infty,0)} x\, \bar{N}(ds, dx), \tag{2.9}$$

$$\begin{aligned}\eta &:= C \int_{0^+}^{\infty} \left(e^{-M^*x} - 1 + M^*x\right) x^{-Y-1}\, dx + C \int_{-\infty}^{0^-} \left(e^{G^*x} - 1 - G^*x\right) |x|^{-Y-1}\, dx \\ &= C\Gamma(-Y)\left((M^*)^Y + (G^*)^Y\right).\end{aligned} \tag{2.10}$$

Finally, let us also note the following decomposition of the process X in terms of the previously defined processes:

$$X_t = Z_t + t\tilde{\gamma} + \sigma W_t^* = \bar{U}_t^{(p)} + \bar{U}_t^{(n)} + t\tilde{\gamma} + \sigma W_t^*. \tag{2.11}$$

To conclude this section, we recall some well-known results of stable Lévy processes needed in the sequel. First, note that, under $\widetilde{\mathbb{P}}$, $(\bar{U}_t^{(p)})_{t\geq 0}$ and $(-\bar{U}_t^{(n)})_{t\geq 0}$ are independent and identically distributed one-sided Y-stable processes with scale, skewness, and location parameters given by $C|\cos(\pi Y/2)|\Gamma(-Y)$, 1, and 0, respectively. The common transition density of $\bar{U}_t^{(p)}$ and $-\bar{U}_t^{(n)}$ is hereafter denoted by $p_U(t, x)$, $t \geq 0$. The following second-order approximation of $p_U(1, x)$ is well-known[1] (see e.g., (14.34) in [11]):

$$p_U(1, x) = Cx^{-Y-1} - \frac{C^2}{2\pi} \sin(2\pi Y)\Gamma(2Y+1)\Gamma^2(-Y)x^{-2Y-1} + o\left(x^{-2Y-1}\right), \quad x \to \infty. \tag{2.12}$$

In particular,

$$\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)} \geq x\right) = \widetilde{\mathbb{P}}\left(-\bar{U}_1^{(n)} \geq x\right) = \frac{C}{Y}x^{-Y} - \frac{C^2}{2\pi} \sin(2\pi Y)\Gamma(2Y)\Gamma^2(-Y)x^{-2Y} + o\left(x^{-2Y}\right), \quad x \to \infty. \tag{2.13}$$

The following result sharpens (2.12) and (2.13). Its proof is presented in the Appendix.

[1] In terms of the parameterization (α, β, τ, c) introduced in [11, Definition 14.16], (α, β, τ, c) of $\bar{U}_1^{(p)}$ and $-\bar{U}_1^{(n)}$ is $(Y, 1, 0, C|\cos(\pi Y/2)|\Gamma(-Y))$.

Lemma 2.1. *There exist constants* $0<\kappa_1,\kappa_2<\infty$ *such that, for* any $x>0$,

$$\text{(i) } \widetilde{\mathbb{P}}\Big(\bar{U}_1^{(p)}\geq x\Big)=\widetilde{\mathbb{P}}\Big(-\bar{U}_1^{(n)}\geq x\Big)\leq \kappa_1 x^{-Y}, \quad \text{(ii) } \left|\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)}\geq x\right)-\frac{C}{Y}x^{-Y}\right|=\left|\widetilde{\mathbb{P}}\left(-\bar{U}_1^{(n)}\geq x\right)-\frac{C}{Y}x^{-Y}\right|\leq \kappa_2 x^{-2Y}. \tag{2.14}$$

Similarly, the tail distribution and the probability density of Z_1, hereafter denoted by $p_Z(1,z)$, admit the following asymptotic behaviors[2] (see (14.34) in [11]),

$$\widetilde{\mathbb{P}}(Z_1\geq z)=\frac{C}{Y}z^{-Y}-\frac{C^2}{\pi Y}\sin(\pi Y)\cos^2\left(\frac{\pi Y}{2}\right)\Gamma(2Y+1)\Gamma^2(-Y)z^{-2Y}+o\left(z^{-2Y}\right), \quad z\to\infty, \tag{2.15}$$

$$p_Z(1,z)=Cz^{-Y-1}-\frac{2C^2}{\pi}\sin(\pi Y)\cos^2\left(\frac{\pi Y}{2}\right)\Gamma(2Y+1)\Gamma^2(-Y)z^{-2Y-1}+o\left(z^{-2Y-1}\right), \quad z\to\infty, \tag{2.16}$$

As in the proof of Lemma 2.1, there exists a constant $0<\kappa_3<\infty$ such that, for *any* $z>0$,

$$\widetilde{\mathbb{P}}(Z_1\geq z)\leq \kappa_3 z^{-Y}. \tag{2.17}$$

Finally, the following identity will also be of use:

$$\widetilde{\mathbb{E}}\left(e^{-\widetilde{U}_t}\right)=\widetilde{\mathbb{E}}\left(e^{-t^{1/Y}\widetilde{U}_1}\right)=\mathbb{E}^*\left(e^{-t^{1/Y}M^*\bar{U}_1^{(p)}}\right)\mathbb{E}^*\left(e^{t^{1/Y}G^*\bar{U}_1^{(n)}}\right)=e^{\eta t}. \tag{2.18}$$

3 The main results

In this section, we present the high-order asymptotic behavior for at-the-money call option prices (1.1). The proofs of all the results are deferred to the Appendix.

Let us first describe the asymptotics in the pure-jump CGMY model ($\sigma=0$), with the following notations:

$$d_1:=\widetilde{\mathbb{E}}\,(Z_1)^+=\frac{1}{\pi}\Gamma\left(1-\frac{1}{Y}\right)\left(2C\Gamma(-Y)\left|\cos\left(\frac{\pi Y}{2}\right)\right|\right)^{\frac{1}{Y}}, \tag{3.1}$$

$$d_2:=\frac{C\Gamma(-Y)}{2}\left((M-1)^Y-M^Y-(G+1)^Y+G^Y\right), \tag{3.2}$$

$$d_{31}:=\frac{\tilde{\gamma}^2}{2}\frac{\Gamma\left(\frac{1}{Y}+1\right)}{\pi}\left(-2C\Gamma(-Y)\cos\left(\frac{\pi Y}{2}\right)\right)^{-\frac{1}{Y}}, \tag{3.3}$$

$$d_{32}:=-\frac{1}{2}\widetilde{\mathbb{E}}\left(\left(Z_1^++\widetilde{U}_1\right)^2\mathbf{1}_{\{Z_1^++\widetilde{U}_1\leq 0\}}\right)-\int_0^\infty w\left(\widetilde{\mathbb{P}}\left(Z_1^++\widetilde{U}_1\geq w\right)-\frac{C(M^*+1)^Y}{Yw^Y}-\frac{C(G^*)^Y}{Yw^Y}\right)dw. \tag{3.4}$$

Theorem 3.1. *Under the exponential CGMY model (1.2) without Brownian component, as* $t\to 0$,

$$\frac{1}{S_0}\mathbb{E}(S_t-S_0)^+=d_1t^{\frac{1}{Y}}+d_2t+d_{31}t^{2-\frac{1}{Y}}+d_{32}t^{\frac{2}{Y}}+o\left(t^{2-\frac{1}{Y}}\right)+o\left(t^{\frac{2}{Y}}\right). \tag{3.5}$$

In particular, if $1<Y<3/2$*, the third-order term is* $d_{31}t^{2-\frac{1}{Y}}$*, while if* $3/2<Y<2$*, the third-order term is* $d_{32}t^{\frac{2}{Y}}$*.*

Next, we consider the asymptotic behavior of the ATM Black-Scholes implied volatility, which hereafter is denoted by $\hat{\sigma}$.

Proposition 3.2. *Let* $d_3=d_{31}\mathbf{1}_{\{Y\leq\frac{3}{2}\}}+d_{32}\mathbf{1}_{\{Y\geq\frac{3}{2}\}}$*. Then, under the exponential CGMY model (1.2) without Brownian component, as* $t\to 0$,

$$\frac{1}{\sqrt{2\pi}}\hat{\sigma}(t)=\begin{cases} d_1t^{\frac{1}{Y}-\frac{1}{2}}+d_2t^{\frac{1}{2}}+d_3t^{\frac{3}{2}-\frac{1}{Y}}+o\left(t^{\frac{3}{2}-\frac{1}{Y}}\right), & \text{if } 1<Y\leq\frac{3}{2},\\ d_1t^{\frac{1}{Y}-\frac{1}{2}}+d_2t^{\frac{1}{2}}+d_3t^{\frac{2}{Y}-\frac{1}{2}}+o\left(t^{\frac{2}{Y}-\frac{1}{2}}\right), & \text{if } \frac{3}{2}<Y<2.\end{cases} \tag{3.6}$$

[2] In terms of the parametrization in [11, Definition 14.16], (α,β,τ,c) of Z_1 therein is $(Y,0,0,2C|\cos(\pi Y/2)|\Gamma(-Y))$.

We now analyze the case of a CGMY model with non-zero Brownian component. In that instance, it was shown in [4] (see also [5] for extensions) that the second order correction term for the ATM European call option price is given via

$$\frac{1}{S_0}\mathbb{E}(S_t - S_0)^+ = d_1 t^{\frac{1}{2}} + d_2 t^{\frac{3-Y}{2}} + o\left(t^{\frac{3-Y}{2}}\right), \quad t \to 0, \tag{3.7}$$

with

$$d_1 := \mathbb{E}^*(\sigma W_1^*)^+ = \frac{\sigma}{\sqrt{2\pi}}, \qquad d_2 := \frac{C\sigma^{1-Y}}{Y(Y-1)}\widetilde{\mathbb{E}}\left(|W_1^*|^{1-Y}\right) = \frac{C2^{\frac{1-Y}{2}}\sigma^{1-Y}}{\sqrt{\pi}Y(Y-1)}\Gamma\left(1-\frac{Y}{2}\right). \tag{3.8}$$

As seen in the previous expressions, the first-order term only synthesizes the information about the continuous volatility parameter σ, while the second-order term incorporates also the information on the tail index parameter Y and the overall jump-intensity parameter C. However, these two-terms do not reflect the relative intensities of the negative or positive jumps (as given by the parameters G and M). This fact suggests the need of a high-order approximation as described in the following theorem and illustrated in Figure 1 below.

Theorem 3.3. *Let*

$$d_{31} := -C\Gamma(-Y)\left((G+1)^Y - G^Y\right), \quad d_{32} := -\frac{C^2\cos^2\left(\frac{\pi Y}{2}\right)\Gamma^2(-Y)2^{Y-\frac{1}{2}}\Gamma\left(Y-\frac{1}{2}\right)}{\pi\sigma^{2Y-1}}. \tag{3.9}$$

Then, under the exponential CGMY model (1.2) with non-zero Brownian component, as $t \to 0$,

$$\frac{1}{S_0}\mathbb{E}(S_t - S_0)^+ = d_1 t^{\frac{1}{2}} + d_2 t^{\frac{3-Y}{2}} + d_{31}t + d_{32}t^{\frac{5}{2}-Y} + o(t) + o\left(t^{\frac{5}{2}-Y}\right). \tag{3.10}$$

In particular, if $1 < Y < 3/2$, the third-order term is $d_{31}t$, while if $3/2 < Y < 2$, the third-order term is $d_{3,2}t^{\frac{5}{2}-Y}$.

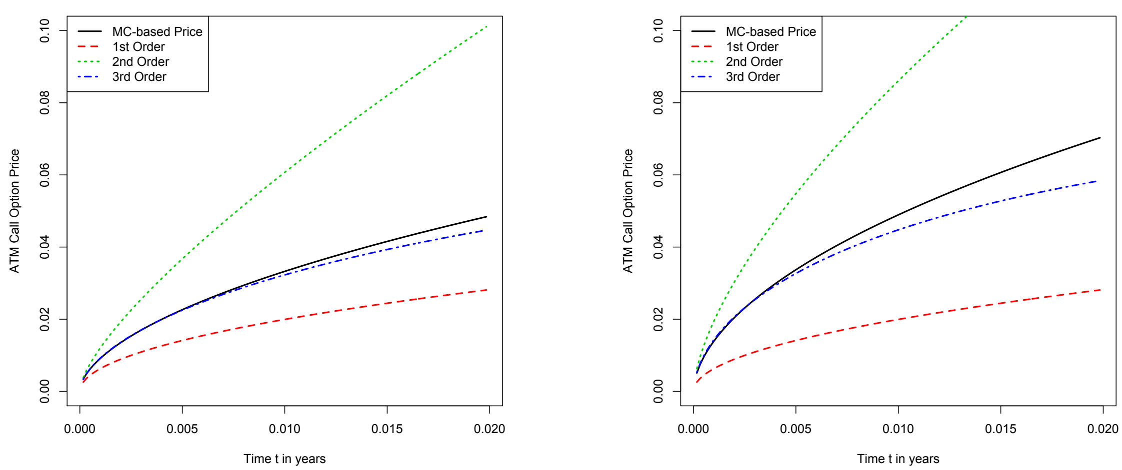

Figure 1: Comparisons of ATM call option prices with the first-, second-, and third-order approximations for $Y = 1.2$ (left panel) and $Y = 1.6$ (right panel). In both cases, $C = 0.5$, $G = 2$, $M = 3.6$, and $\sigma = 0.5$.

Our final proposition gives the small-time asymptotic behavior for the ATM Black-Scholes implied volatility, denoted again by $\hat{\sigma}$, under the generalized CGMY model. Unlike the pure-jump case, we can only derive the second order asymptotics using Theorem 3.3. In fact, the first order term of the ATM call option price under the generalized CGMY model is the same as the one under the Black-Scholes model. The third-order term of $\hat{\sigma}$ requires higher order asymptotics of the ATM call option price.

Proposition 3.4. *Let $d_3 = d_{31}\mathbf{1}_{\{Y\le\frac{3}{2}\}} + d_{32}\mathbf{1}_{\{Y\ge\frac{3}{2}\}}$. Then, under the exponential CGMY model (1.2) with non-zero Brownian component, as $t \to 0$,*

$$\frac{1}{\sqrt{2\pi}}\hat{\sigma}(t) = \begin{cases} \sigma + d_2 t^{1-\frac{Y}{2}} + d_3 t^{\frac{1}{2}} + o\left(t^{\frac{1}{2}}\right), & \text{if } 1 < Y \le \frac{3}{2}, \\ \sigma + d_2 t^{1-\frac{Y}{2}} + d_3 t^{2-Y} + o\left(t^{2-Y}\right), & \text{if } \frac{3}{2} < Y < 2. \end{cases} \tag{3.11}$$

A Proofs

For simplicity, throughout all the proofs, we fix $S_0 = 1$.

Proof of Lemma 2.1. From the leading term in the expansion (2.13), there exists $N > 0$ such that, for any $x > 0$,

$$\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)} \geq x\right) = \widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)} \geq x\right)\left(\mathbf{1}_{\{x\geq N\}} + \mathbf{1}_{\{x<N\}}\right) \leq \frac{2C}{Y}x^{-Y}\mathbf{1}_{\{x\geq N\}} + \frac{N^Y}{x^Y}\mathbf{1}_{\{x<N\}} \leq \left(2CY^{-1} + N^Y\right)x^{-Y},$$

and the first relationship in (2.14) follows by setting $\kappa_1 = 2CY^{-1} + N^Y$. Similarly, from (2.13), there exists $N > 0$ such that, for any $x > 0$,

$$\begin{aligned}
\left|\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)} \geq x\right) - \frac{C}{Y}x^{-Y}\right| &= \left|\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)} \geq x\right) - \frac{C}{Y}x^{-Y}\right|\left(\mathbf{1}_{\{x\geq N\}} + \mathbf{1}_{\{x<N\}}\right) \\
&\leq \frac{C^2}{\pi}\left|\sin(2\pi Y)\right|\Gamma(2Y)\Gamma^2(-Y)x^{-2Y}\mathbf{1}_{\{x\geq N\}} + \left(\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)} \geq x\right) + \frac{C}{Y}x^{-Y}\right)\mathbf{1}_{\{x<N\}} \\
&\leq \left(\frac{C^2}{\pi}\left|\sin(2\pi Y)\right|\Gamma(2Y)\Gamma^2(-Y) + N^{2Y} + CN^YY^{-1}\right)x^{-2Y}.
\end{aligned}$$

The second identity in (2.14) follows by setting $\kappa_2 = C^2\left|\sin(2\pi Y)\right|\Gamma(2Y)\Gamma^2(-Y)/\pi + N^{2Y} + CN^YY^{-1}$. □

Proof of Theorem 3.1 Set $\tilde{\gamma}_t := t^{1-1/Y}\tilde{\gamma}$ and $\vartheta := -C\Gamma(-Y)\left(M^Y + (G^*)^Y\right)$ and note that $d_2 = \vartheta + \eta + \tilde{\gamma}/2$ in view of (2.6). For future reference, it is also convenient to write ϑ as

$$\vartheta = \frac{C(M^Y + (G^*)^Y)}{Y}\int_0^\infty \frac{e^{-t^{\frac{1}{Y}}v} - 1}{t^{1-\frac{1}{Y}}}v^{-Y}dv, \tag{A.1}$$

which follows from the identity (see (14.18) in [11]):

$$-Y\Gamma(-Y) = \Gamma(1-Y) = \int_0^\infty \left(e^{-y} - 1\right)y^{-Y}dy = \int_0^\infty \frac{e^{-t^{\frac{1}{Y}}v} - 1}{t^{1-\frac{1}{Y}}}v^{-Y}dv. \tag{A.2}$$

Let us start by noting the following decomposition for the ATM option price (1.1) derived from (2.3), (2.8), (2.11), (2.18), and the fact that $(1-e^{-x})^+ = 1 - e^{-x^+}$:

$$\Pi(t) = \mathbb{E}\left(e^{X_t}\left(1 - e^{-X_t}\right)^+\right) = e^{-\eta t}\widetilde{\mathbb{E}}\left(e^{-\widetilde{U}_t}\left(1 - e^{-X_t^+}\right)\right) = 1 - e^{-\eta t}\widetilde{\mathbb{E}}\left(e^{-\widetilde{U}_t - X_t^+}\right).$$

Set

$$\Delta_1(t) := t^{-\frac{1}{Y}}\widetilde{\mathbb{E}}\left(1 - e^{-\left(\widetilde{U}_t - X_t^+\right)} - \left(\widetilde{U}_t - X_t^+\right)\right), \quad \Delta_2(t) := t^{-\frac{1}{Y}}\left(\widetilde{\mathbb{E}}\left(X_t^+\right) - \widetilde{\mathbb{E}}\left(Z_t^+\right)\right).$$

Then, recalling that $\widetilde{\mathbb{E}}(\widetilde{U}_t) = 0$ and $\widetilde{\mathbb{E}}(Z_t^+) = t^{1/Y}\widetilde{\mathbb{E}}Z_1^+$, we have the decomposition:

$$\begin{aligned}
A(t) &:= t^{\frac{1}{Y}-1}\left(t^{-\frac{1}{Y}}\Pi(t) - \widetilde{\mathbb{E}}(Z_1^+)\right) - d_2 \\
&= \left(t^{\frac{1}{Y}-1}\Delta_1(t) - \vartheta\right) + \left(t^{\frac{1}{Y}-1}\Delta_2(t) - \frac{\tilde{\gamma}}{2}\right) + \frac{e^{-\eta t} - 1 + \eta t}{t}\,\widetilde{\mathbb{E}}\left(e^{-\widetilde{U}_t - X_t^+}\right) - \eta t^{\frac{1}{Y}}\Delta_1(t) - \eta t^{\frac{1}{Y}}\,\widetilde{\mathbb{E}}\left(Z_t^+\right) \\
&:= A_1(t) + A_2(t) + A_3(t) - A_4(t) - A_5(t).
\end{aligned} \tag{A.3}$$

We shall prove that $A_1(t) = O(t^{\frac{2}{Y}-1})$ and $A_2 = O(t^{1-\frac{1}{Y}})$ and, hence, $t^{\frac{1}{Y}}\Delta_1(t) = O(t)$. These results, in turn, imply that $A_i(t) = O(t) = o(A_1(t)) = o(A_2(t))$, $i = 3, 4$, and $A_5(t) = O(t^{\frac{1}{Y}}) = o(A_1(t)) = o(A_2(t))$. So, it remains to analyze the asymptotic behaviors of $A_1(t)$ and $A_2(t)$. These two cases are analyzed in two steps:

Step 1. Using the identity $\widetilde{\mathbb{E}}\left(1 - e^{-V} - V\right) = \int_0^\infty (e^{-y} - 1)\,\widetilde{\mathbb{P}}\left(V \geq y\right)dy - \int_0^\infty (e^y - 1)\,\widetilde{\mathbb{P}}\left(V \leq -y\right)dy$ together with the change of variables $v = t^{-1/Y}y$, we can write

$$\begin{aligned}
A_1(t) &= \left(\int_0^\infty \frac{e^{-t^{\frac{1}{Y}}v} - 1}{t^{1-\frac{1}{Y}}}\widetilde{\mathbb{P}}\left(t^{-\frac{1}{Y}}X_t^+ + \widetilde{U}_1 \geq v\right)dv - \vartheta\right) - \int_0^\infty \frac{e^{t^{\frac{1}{Y}}v} - 1}{t^{1-\frac{1}{Y}}}\widetilde{\mathbb{P}}\left(t^{-\frac{1}{Y}}X_t^+ + \widetilde{U}_1 \leq -v\right)dv \\
&:= B_1(t) - B_2(t).
\end{aligned} \tag{A.4}$$

For $B_2(t)$, note that

$$\lim_{t\to 0}\frac{B_2(t)}{t^{\frac{2}{Y}-1}}=\lim_{t\to 0}\int_0^{\infty}\frac{e^{t^{\frac{1}{Y}}v}-1}{t^{\frac{1}{Y}}}\,\widetilde{\mathbb{P}}\left((Z_1+\tilde{\gamma}_t)^{+}+\widetilde{U}_1\leq -v\right)dv=\int_0^{\infty}v\widetilde{\mathbb{P}}\left(Z_1^{+}+\widetilde{U}_1\leq -v\right)dv \tag{A.5}$$

where the second equality follows from the dominated convergence theorem, which applies in view of the following direct consequence of (2.18):

$$\frac{e^{t^{\frac{1}{Y}}v}-1}{t^{\frac{1}{Y}}}\widetilde{\mathbb{P}}\left(t^{-\frac{1}{Y}}X_t^{+}+\widetilde{U}_1\leq -v\right)\leq\frac{e^{t^{\frac{1}{Y}}v}-1}{t^{\frac{1}{Y}}}\widetilde{\mathbb{P}}\left(\widetilde{U}_1\leq -v\right)\leq ve^{t^{\frac{1}{Y}}v}e^{-v}\widetilde{\mathbb{E}}\left(e^{-\widetilde{U}_1}\right)\leq e^{\eta}ve^{\left(t^{\frac{1}{Y}}-1\right)v}\leq e^{\eta}ve^{-v/2}.$$

We now analyze the asymptotic behavior of $B_1(t)$, which is shown to be $O(t^{\frac{2}{Y}-1})$. To this end, we decompose $t^{1-\frac{2}{Y}}B_1(t)$ as

$$\begin{aligned}
t^{1-\frac{2}{Y}}B_1(t)&=\int_0^{\infty}\frac{e^{-t^{\frac{1}{Y}}v}-1}{t^{\frac{1}{Y}}}\left(\widetilde{\mathbb{P}}\left(Z_1+\tilde{\gamma}_t>0,\,Z_1+\tilde{\gamma}_t+\widetilde{U}_1\geq v\right)-\frac{CM^Y}{Yv^Y}\right)dv\\
&\quad+\int_0^{\infty}\frac{e^{-t^{\frac{1}{Y}}v}-1}{t^{\frac{1}{Y}}}\left(\widetilde{\mathbb{P}}\left(Z_1+\tilde{\gamma}_t<0,\,Z_1+\tilde{\gamma}_t+\widetilde{U}_1\geq v\right)-\frac{C(G^*)^Y}{Yv^Y}\right)dv\\
&:=B_{11}(t)+B_{12}(t),
\end{aligned}\tag{A.6}$$

where we have used $t^{-1/Y}X_t=Z_1+\tilde{\gamma}_t$ and (A.1). As suggested by the previous equation, the limit of each of the terms therein can be obtained by passing $\lim_{t\to 0}$ into the various integrals. We now proceed to show that the latter operation is indeed valid. We begin with analyzing $B_{11}(t)$, for which we first apply the decomposition

$$\begin{aligned}
\widetilde{\mathbb{P}}\left(Z_1+\tilde{\gamma}_t>0,\,Z_1+\tilde{\gamma}_t+\widetilde{U}_1\geq v\right)&=\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)}+\bar{U}_1^{(n)}+\tilde{\gamma}_t>0,\,M\bar{U}_1^{(p)}-G\bar{U}_1^{(n)}+\tilde{\gamma}_t\geq v\right)\\
&=\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)}\geq\frac{v+G\bar{U}_1^{(n)}-\tilde{\gamma}_t}{M},\,-\bar{U}_1^{(n)}<\frac{v+M^*\tilde{\gamma}_t}{M+G}\right)\\
&\quad+\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)}+\tilde{\gamma}_t\geq -\bar{U}_1^{(n)}\geq\frac{v+M^*\tilde{\gamma}_t}{M+G}\right),
\end{aligned}$$

where we have used that $Z_1=\bar{U}_1^{(p)}+\bar{U}_1^{(n)}$ and $\widetilde{U}_1=M^*\bar{U}_1^{(p)}-G^*\bar{U}_1^{(n)}$. We then write:

$$B_{11}(t):=\int_0^{\infty}\frac{e^{-t^{\frac{1}{Y}}v}-1}{t^{\frac{1}{Y}}}\left(\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)}\geq\frac{v+G\bar{U}_1^{(n)}-\tilde{\gamma}_t}{M},\,-\bar{U}_1^{(n)}<\frac{v+M^*\tilde{\gamma}_t}{M+G}\right)-\frac{CM^Y}{Yv^Y}\right)du \tag{A.7}$$

$$+\int_0^{\infty}\frac{e^{-t^{\frac{1}{Y}}v}-1}{t^{\frac{1}{Y}}}\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)}+\tilde{\gamma}_t\geq -\bar{U}_1^{(n)}\geq\frac{v+M^*\tilde{\gamma}_t}{M+G}\right)du. \tag{A.8}$$

By (2.14-i), for any $v>0$ and t small enough (so that $G^*|\tilde{\gamma}_t|<1$ and $M^*|\tilde{\gamma}_t|<1$), the expression inside the integral in (A.8), which we denote $b_{11}^{(2)}(t;v)$, is such that

$$\left|b_{11}^{(2)}(t;v)\right|\leq v\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)}\geq\frac{v+M^*\tilde{\gamma}_t}{M+G}-\tilde{\gamma}_t\right)\widetilde{\mathbb{P}}\left(-\bar{U}_1^{(n)}\geq\frac{v+M^*\tilde{\gamma}_t}{M+G}\right)\leq v\mathbf{1}_{\{v\leq 1\}}+v\mathbf{1}_{\{v>1\}}\min\left\{1,\kappa_1^2(M+G)^{2Y}v^{-2Y}\right\},$$

where $\kappa_1\in(0,\infty)$ is given as in (2.14-i). Hence, by the dominated convergence theorem,

$$\lim_{t\to 0}\int_0^{\infty}b_{11}^{(2)}(t;v)\,dv=-\int_0^{\infty}v\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)}\geq -\bar{U}_1^{(n)}\geq\frac{v}{M+G}\right)dv. \tag{A.9}$$

We now bound the expression inside the integral in (A.7), which we denote $b_{11}^{(1)}(t;v)$. It suffices to consider $v>1$, since $|b_{11}^{(1)}(t;v)|\leq v(1+CY^{-1}M^Yv^{-Y})$, which is integrable on $\{v\leq 1\}$. We also let t be small enough so that

$|\tilde{\gamma}_t| < 1$, $G^*|\tilde{\gamma}_t| < 1$ and $M^*|\tilde{\gamma}_t| < 1$. Then, for any $v > 1$,

$$\begin{aligned}
\left|b_{11}^{(1)}(t;v)\right| &\le v\int_{\mathbb{R}} p_U(1,y)\left|\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)} \ge \frac{v-Gy-\tilde{\gamma}_t}{M}\right)\mathbf{1}_{\left\{y\le\frac{v+M^*\tilde{\gamma}_t}{M+G}\right\}} - \frac{CM^Y}{Yv^Y}\right| dy \\
&\le v\int_{-\infty}^{\frac{v+M^*\tilde{\gamma}_t}{M+G}} p_U(1,y)\left|\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)} \ge \frac{v-Gy-\tilde{\gamma}_t}{M}\right) - \frac{CM^Y}{Y(v-Gy-\tilde{\gamma}_t)^Y}\right| dy \\
&\quad + v\int_{-\infty}^{\frac{v+M^*\tilde{\gamma}_t}{M+G}} p_U(1,y)\frac{CM^Y}{Y}\left|(v-Gy-\tilde{\gamma}_t)^{-Y} - v^{-Y}\right| dy + \frac{CM^Y}{Yv^{Y-1}}\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)} \ge \frac{v+M^*\tilde{\gamma}_t}{M+G}\right) \\
&:= D_t^{(1)}(v) + D_t^{(2)}(v) + D_t^{(3)}(v).
\end{aligned} \tag{A.10}$$

Next, since

$$\frac{v-Gy-\tilde{\gamma}_t}{M} \ge \frac{v-G\frac{v+M^*\tilde{\gamma}_t}{M+G}-\tilde{\gamma}_t}{M} = \frac{v-G^*\tilde{\gamma}_t}{M+G} > 0, \quad \text{for any } \ y \le \frac{v+M^*\tilde{\gamma}_t}{M+G}, \tag{A.11}$$

the first integral in (A.10) can be bounded, using (2.14-ii), via:

$$D_t^{(1)}(v) \le \kappa_2 v\int_{-\infty}^{\frac{v+M^*\tilde{\gamma}_t}{M+G}} \frac{p_U(1,y)M^{2Y}}{(v-Gy-\tilde{\gamma}_t)^{2Y}}\,dy \le \kappa_2(M+G)^{2Y}v^{1-2Y}, \quad \text{for any } \ v > 1, \tag{A.12}$$

where $\kappa_2 \in (0,\infty)$ is given as in (2.14-ii). Moreover, using the convexity and monotonicity of the function $f(x) = x^{-Y}$ on $(0,\infty)$ and (A.11), the second integral in (A.10) can be upper estimated as

$$D_t^{(2)}(v) \le CM^Y v\int_{-\infty}^{\frac{v+M^*\tilde{\gamma}_t}{M^*+G^*}} p_U(1,y)v^{-Y-1}|Gy+\tilde{\gamma}_t|\,dy \le CM^Y v^{-Y}\left(G\widetilde{\mathbb{E}}\left|\bar{U}_1^{(p)}\right| + 1\right). \tag{A.13}$$

Finally, by (2.14-i), the last term in (A.10) can be upper bounded via

$$D_t^{(3)}(v) \le \kappa_1 CM^Y Y^{-1}v^{1-2Y}, \quad \text{for any } \ v > 1. \tag{A.14}$$

Combining (A.10) and (A.12)-(A.14), and by the dominated convergence theorem,

$$\lim_{t\to 0}\int_0^\infty b_{11}^{(1)}(t;v)\,dv = -\int_0^\infty v\left(\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)} \ge \frac{v+G\bar{U}_1^{(n)}}{M},\ -\bar{U}_1^{(n)} \le \frac{v}{M+G}\right) - \frac{CM^Y}{Yv^Y}\right) dv. \tag{A.15}$$

Putting together (A.9) and (A.15), we obtain

$$\begin{aligned}
\lim_{t\to 0} B_{11}(t) &= -\int_0^\infty v\,\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)} \ge -\bar{U}_1^{(n)} \ge \frac{v}{M+G}\right) dv - \int_0^\infty v\left(\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)} \ge \frac{v+G\bar{U}_1^{(n)}}{M},\ -\bar{U}_1^{(n)} \le \frac{v}{M+G}\right) - \frac{CM^Y}{Yv^Y}\right) dv \\
&= -\int_0^\infty v\left(\widetilde{\mathbb{P}}\left(Z_1 > 0,\ Z_1 + \widetilde{U}_1 \ge v\right) - \frac{CM^Y}{Yv^Y}\right) dv.
\end{aligned} \tag{A.16}$$

Applying the same arguments to the decomposition

$$\widetilde{\mathbb{P}}\left(Z_1+\tilde{\gamma}_t<0,\ Z_1+\tilde{\gamma}_t+\widetilde{U}_1\ge v\right) = \widetilde{\mathbb{P}}\left(-\bar{U}_1^{(n)} - \tilde{\gamma}_t \ge \bar{U}_1^{(p)} \ge \frac{v+G^*\tilde{\gamma}_t}{M+G}\right) + \widetilde{\mathbb{P}}\left(-\bar{U}_1^{(n)} \ge \frac{v-M^*\bar{U}_1^{(p)}}{G^*},\ \bar{U}_1^{(p)} < \frac{v-G^*\tilde{\gamma}_t}{M+G}\right),$$

it can be shown that

$$\begin{aligned}
\lim_{t\to 0} B_{12}(t) &= -\int_0^\infty v\widetilde{\mathbb{P}}\left(-\bar{U}_1^{(n)} \ge \bar{U}_1^{(p)} \ge \frac{v}{M+G}\right) dv - \int_0^\infty v\left(\widetilde{\mathbb{P}}\left(-\bar{U}_1^{(n)} \ge \frac{v-M^*\bar{U}_1^{(p)}}{G^*},\ \bar{U}_1^{(p)} \le \frac{v}{M+G}\right) - \frac{C(G^*)^Y}{Yv^Y}\right) dv \\
&= -\int_0^\infty v\left(\widetilde{\mathbb{P}}\left(Z_1 < 0,\ Z_1 + \widetilde{U}_1 \ge v\right) - \frac{C(G^*)^Y}{Yv^Y}\right) dv.
\end{aligned} \tag{A.17}$$

Combining (A.6), (A.16), and (A.17), we obtain

$$\lim_{t\to 0} t^{1-\frac{2}{Y}} B_1(t) = -\int_0^\infty v \left[\widetilde{\mathbb{P}}\left(Z_1^+ + \widetilde{U}_1 \geq v\right) - \frac{CM^Y}{Yv^Y} - \frac{C(G^*)^Y}{Yv^Y} \right] dv. \tag{A.18}$$

Combining (A.4), (A.5) and (A.18) together with the identity $\int_0^\infty v\widetilde{\mathbb{P}}(V \leq -v)dv = \widetilde{\mathbb{E}}\left((V^-)^2\right)/2$, we get

$$\lim_{t\to 0} t^{1-\frac{2}{Y}} A_1(t) = -\int_0^\infty v \left(\widetilde{\mathbb{P}}\left(Z_1^+ + \widetilde{U}_1 \geq v\right) - \frac{CM^Y}{Yv^Y} - \frac{C(G^*)^Y}{Yv^Y} \right) dv - \frac{1}{2}\widetilde{\mathbb{E}}\left(\left[\left(Z_1^+ + \widetilde{U}_1\right)^- \right]^2 \right) := d_{3,2}. \tag{A.19}$$

Step 2. Now, we analyze the behavior of $A_2 = t^{\frac{1}{Y}-1}\Delta_2(t) - \tilde{\gamma}/2$. By the self-similarity of $(Z_t)_{t\geq 0}$,

$$\Delta_2(t) = \widetilde{\mathbb{E}}\left((Z_1 + \tilde{\gamma}_t)^+ - Z_1^+\right) = \int_0^\infty \left(\widetilde{\mathbb{P}}\left(Z_1 \geq u - \tilde{\gamma}_t\right) - \widetilde{\mathbb{P}}\left(Z_1 \geq u\right)\right) du = \int_0^\infty \int_{u-\tilde{\gamma}_t}^u p_Z(w)\, dw\, du,$$

where for simplicity we have written $p_Z(u)$ for the density $p_Z(1,u)$ of Z_1. By the symmetry of Z_1, $\tilde{\gamma}/2 = \tilde{\gamma}\int_0^\infty p_Z(u)du$, and thus,

$$A_2(t) = \tilde{\gamma}\int_0^\infty \left(\frac{1}{\tilde{\gamma}_t}\int_{u-\tilde{\gamma}_t}^u p_Z(w)\, dw - p_Z(u) \right) du.$$

The identity $p_Z(w) = p_Z(u) + (w-u)\int_0^1 p_Z'(u + \beta(u-w))\, d\beta$, followed by the change of variables $v = \tilde{\gamma}_t^{-1}(w-u)$, gives

$$A_2(t) = \tilde{\gamma}\int_0^\infty \frac{1}{\tilde{\gamma}_t}\left[\int_{u-\tilde{\gamma}_t}^u (w-u)\left(\int_0^1 p_Z'(u+\beta(u-w))\, d\beta\right) dw\right] du = \tilde{\gamma}\tilde{\gamma}_t \int_0^\infty \left[\int_{-1}^0 v\left(\int_0^1 p_Z'(u+\beta\tilde{\gamma}_t v)\, d\beta\right) dv\right] du.$$

By Fubini's theorem and recalling that $\tilde{\gamma}_t = \tilde{\gamma}t^{1-1/Y}$,

$$A_2(t) = \tilde{\gamma}\tilde{\gamma}_t\int_{-1}^0 v\left(\int_0^1 \int_0^\infty p_Z'(u+\beta\tilde{\gamma}_t v)\, du\, d\beta\right) dv = -\tilde{\gamma}^2 t^{1-\frac{1}{Y}}\int_{-1}^0 v\left(\int_0^1 p_Z\left(\beta\tilde{\gamma}vt^{1-\frac{1}{Y}}\right) d\beta\right) dv.$$

It is now clear that

$$\lim_{t\to 0} t^{\frac{1}{Y}-1} A_2(t) = \frac{\tilde{\gamma}^2 p_Z(0)}{2} := d_{3,1}. \tag{A.20}$$

Next, using the power series representation of $p_Z(z)$ around $z = 0$ as given in (14.30) in [11], it follows that $\tilde{\gamma}^2 p_Z(0)/2$ reduces to the expression $d_{3,1}$ in (3.3). Finally, combining (A.19) and (A.20) with (A.3) (together with the remarks thereafter), we obtain (3.5). □

Proof of Proposition 3.2. The small-time asymptotic behavior of the ATM call-option price $C_{BS}(t,\sigma)$ at maturity t under the Black-Scholes model with volatility σ and zero interest rate (and fixing for simplicity $S_0 = 1$), is such that:

$$C_{BS}(t,\sigma) = \frac{\sigma}{\sqrt{2\pi}}\, t^{\frac{1}{2}} - \frac{\sigma^3}{24\sqrt{2\pi}}\, t^{\frac{3}{2}} + O(t^{\frac{5}{2}}), \quad t\to 0, \tag{A.21}$$

(see, e.g., [6, Corollary 3.4]). To derive the small-time asymptotics for the implied volatility, we need a result analogous to (A.21) when σ is replaced by $\hat{\sigma}(t)$. To obtain it, combining first the following representation

$$C_{BS}(t,\sigma) = F(\sigma\sqrt{t}) \quad \text{with} \quad F(\theta) := \int_0^\theta \Phi'\left(\frac{v}{2}\right) dv = \frac{1}{\sqrt{2\pi}}\int_0^\theta \exp\left(-v^2/8\right) dv,$$

originating in [10, Lemma 3.1], together with the Taylor expansion of F at $\theta = 0$ (see [10, Lemma 5.1]), we get

$$F(\theta) = \frac{1}{\sqrt{2\pi}}\,\theta - \frac{1}{24\sqrt{2\pi}}\,\theta^3 + O(\theta^5), \quad \theta\to 0.$$

Then, since $\hat{\sigma}(t)\to 0$ as $t\to 0$ (see, e.g., [12, Proposition 5]), we conclude that

$$C_{BS}(t,\hat{\sigma}(t)) = \frac{\hat{\sigma}(t)}{\sqrt{2\pi}}\, t^{\frac{1}{2}} - \frac{\hat{\sigma}(t)^3}{24\sqrt{2\pi}}\, t^{\frac{3}{2}} + O\left(\left(\hat{\sigma}(t)\, t^{\frac{1}{2}}\right)^5\right), \quad t\to 0. \tag{A.22}$$

Returning to the proof of Proposition 3.2, by comparing the first order terms in (3.5) and (A.22), it follows that $\widetilde{\mathbb{E}}(Z_1^+)\,t^{\frac{1}{Y}} \sim (2\pi)^{-1/2}\hat{\sigma}(t)\sqrt{t}$ as $t \to 0$, and thus,

$$\hat{\sigma}(t) \sim \sqrt{2\pi}\,\widetilde{\mathbb{E}}(Z_1^+)\,t^{\frac{1}{Y}-\frac{1}{2}} := \sigma_1 t^{\frac{1}{Y}-\frac{1}{2}}, \quad t \to 0. \tag{A.23}$$

Next, set $\tilde{\sigma}(t) := \hat{\sigma}(t) - \sigma_1 t^{\frac{1}{Y}-\frac{1}{2}}$. Comparing the first two terms in (3.5) with the first term in (A.22) (noting that the second term in (A.22) is $O(t^{1+1/Y})$) leads to

$$\tilde{\sigma}(t) \sim \sqrt{\frac{\pi}{2}} C\Gamma(-Y)\left((M-1)^Y - M^Y - (G+1)^Y + G^Y\right)\sqrt{t} := \sigma_2\sqrt{t}, \quad t \to 0. \tag{A.24}$$

Finally, to obtain the third-order expansion, set $\bar{\sigma}(t) := \hat{\sigma}(t) - \sigma_1 t^{\frac{1}{Y}-\frac{1}{2}} - \sigma_2\sqrt{t}$. By comparing the first three terms in (3.5) with the first term in (A.22), it follows that

$$\frac{\bar{\sigma}(t)}{\sqrt{2\pi}}\,t^{\frac{1}{2}} \sim \begin{cases} d_3\,t^{2-\frac{1}{Y}} & \text{if } \ 1 < Y \leq \frac{3}{2} \\ d_3\,t^{\frac{2}{Y}} & \text{if } \ \frac{3}{2} < Y < 2, \end{cases} \tag{A.25}$$

which leads to (3.6). □

Proof of Theorem 3.3. Let

$$\Delta_0(t) := \frac{1}{\sqrt{t}}\,\mathbb{E}(S_t - 1)^+ - d_1, \tag{A.26}$$

with constant d_1 given in (3.8). Let us start by noting the following easy representation

$$\frac{1}{\sqrt{t}}\,\mathbb{E}(S_t-1)^+ = \frac{1}{\sqrt{t}}\,\mathbb{E}\left(e^{X_t}\left(1-e^{-X_t}\right)^+\right) = \frac{1}{\sqrt{t}}\,\mathbb{E}^*\left(1-e^{-X_t^+}\right) = \int_0^\infty e^{-\sqrt{t}v}\,\mathbb{P}^*\left(t^{-1/2}X_t \geq v\right)dv,$$

where in the last equality we used the identity $\mathbb{E}^*(1-e^{-X_t^+}) = \int_0^\infty e^{-x}\mathbb{P}^*(X_t \geq x)\,dx$ together with the change of variables $v = t^{-1/2}x$. Next, recalling that $X_t = L_t^* + \sigma W_t^* = \tilde{\gamma}t + Z_t + \sigma W_t^*$ and using the self-similarity of W^* and the change of variables $y = v - t^{1/2}\tilde{\gamma}$, it follows that

$$\Delta_0(t) = \int_{-t^{\frac{1}{2}}\tilde{\gamma}}^\infty e^{-\sqrt{t}y - t\tilde{\gamma}}\mathbb{P}^*\left(\sigma W_1^* \geq y + t^{-\frac{1}{2}}Z_t\right)dy - \int_0^\infty \mathbb{P}^*(\sigma W_1^* \geq y)\,dy.$$

Furthermore, by changing the probability measure $\mathbb{P}^*$ to $\widetilde{\mathbb{P}}$, recalling that $U_t = \widetilde{U}_t + \eta t$, and using the self-similarity of both $(Z_t)_{t\geq 0}$ and $(\widetilde{U}_t)_{t\geq 0}$, we get

$$\begin{aligned}
\Delta_0(t) &= \int_{-\sqrt{t}\tilde{\gamma}}^\infty e^{-\sqrt{t}y-\tilde{\gamma}t}\widetilde{\mathbb{E}}\left(e^{-\widetilde{U}_t-\eta t}\mathbf{1}_{\left\{\sigma W_1^* \geq y - t^{-\frac{1}{2}}Z_t\right\}}\right)dy - \int_0^\infty \widetilde{\mathbb{E}}\left(e^{-\widetilde{U}_t-\eta t}\mathbf{1}_{\left\{\sigma W_1^* \geq u\right\}}\right)du \\
&= e^{-(\eta+\tilde{\gamma})t}\int_0^\infty e^{-\sqrt{t}y}\left[\widetilde{\mathbb{E}}\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1}\mathbf{1}_{\left\{\sigma W_1^* \geq y - t^{\frac{1}{Y}-\frac{1}{2}}Z_1\right\}}\right) - \widetilde{\mathbb{E}}\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1}\mathbf{1}_{\left\{\sigma W_1^* \geq y\right\}}\right)\right]dy \\
&\quad + e^{-(\eta+\tilde{\gamma})t}\int_{-\sqrt{t}\tilde{\gamma}}^0 e^{-\sqrt{t}y}\widetilde{\mathbb{E}}\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1}\mathbf{1}_{\left\{\sigma W_1^* \geq y - t^{\frac{1}{Y}-\frac{1}{2}}Z_1\right\}}\right)dy + \int_0^\infty\left(e^{-\tilde{\gamma}t-\sqrt{t}y}-1\right)\widetilde{\mathbb{P}}(\sigma W_1^* \geq y)\,dy \\
&:= A_1(t) + A_2(t) + A_3(t).
\end{aligned} \tag{A.27}$$

For $A_2(t)$, by changing variables to $u = t^{-1/2}y$ and the dominated convergence theorem,

$$\lim_{t\to 0} t^{-\frac{1}{2}}A_2(t) = \lim_{t\to 0} e^{-(\eta+\tilde{\gamma})t}\int_{-\tilde{\gamma}}^0 e^{-tu}\widetilde{\mathbb{E}}\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1}\mathbf{1}_{\left\{\sigma W_1^* \geq \sqrt{t}u - t^{\frac{1}{Y}-\frac{1}{2}}Z_1\right\}}\right)du = \int_{-\tilde{\gamma}}^0 \frac{1}{2}\,du = \frac{\tilde{\gamma}}{2}. \tag{A.28}$$

It is also clear that

$$\lim_{t\to 0} t^{-\frac{1}{2}}A_3(t) = \lim_{t\to 0}\int_0^\infty \frac{e^{-\tilde{\gamma}t-\sqrt{t}y}-1}{\sqrt{t}}\,\widetilde{\mathbb{P}}(\sigma W_1^* \geq y)\,dy = -\int_0^\infty y\,\widetilde{\mathbb{P}}(\sigma W_1^* \geq y)\,dy = -\frac{\sigma^2}{4}\widetilde{\mathbb{E}}\left(W_1^*\right)^2 = -\frac{\sigma^2}{4}. \tag{A.29}$$

It remains to analyze the asymptotic behavior of $A_1(t)$. To this end, let us first decompose it as follows:

$$
\begin{aligned}
A_1(t) &= e^{-(\eta+\tilde{\gamma})t}\widetilde{\mathbb{E}}\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1}\mathbf{1}_{\left\{W_1^*\geq 0,\,\sigma W_1^*+t^{\frac{1}{Y}-\frac{1}{2}}Z_1\geq 0\right\}}\int_{\sigma W_1^*}^{\sigma W_1^*+t^{\frac{1}{Y}-\frac{1}{2}}Z_1} e^{-\sqrt{t}y}\,dy\right)\\
&\quad - e^{-(\eta+\tilde{\gamma})t}\widetilde{\mathbb{E}}\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1}\mathbf{1}_{\left\{0\leq\sigma W_1^*\leq -t^{\frac{1}{Y}-\frac{1}{2}}Z_1\right\}}\int_0^{\sigma W_1^*} e^{-\sqrt{t}y}\,dy\right)\\
&\quad + e^{-(\eta+\tilde{\gamma})t}\widetilde{\mathbb{E}}\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1}\mathbf{1}_{\left\{0\leq -\sigma W_1^*\leq t^{\frac{1}{Y}-\frac{1}{2}}Z_1\right\}}\int_0^{\sigma W_1^*+t^{\frac{1}{Y}-\frac{1}{2}}Z_1} e^{-\sqrt{t}y}\,dy\right)\\
&:= B_1(t)-B_2(t)+B_3(t).
\end{aligned}
\tag{A.30}
$$

We analyze each of the above three terms in three steps:

Step 1. First, by the change of variable $u=t^{1/2-1/Y}y-\sigma t^{1/2-1/Y}W_1^*+\widetilde{U}_1$,

$$
\begin{aligned}
B_1(t) &= e^{-(\eta+\tilde{\gamma})t}t^{\frac{1}{Y}-\frac{1}{2}}\widetilde{\mathbb{E}}\left(e^{-\sqrt{t}\sigma W_1^*}\mathbf{1}_{\left\{W_1^*\geq 0,\,\sigma W_1^*+t^{\frac{1}{Y}-\frac{1}{2}}Z_1\geq 0\right\}}\int_{\widetilde{U}_1}^{\widetilde{U}_1+Z_1} e^{-t^{\frac{1}{Y}}u}\,du\right)\\
&= e^{-(\eta+\tilde{\gamma})t}t^{\frac{1}{Y}-\frac{1}{2}}\widetilde{\mathbb{E}}\left(e^{-\sqrt{t}\sigma W_1^*}\mathbf{1}_{\left\{W_1^*\geq 0,\,Z_1\geq 0\right\}}\int_{\widetilde{U}_1}^{\widetilde{U}_1+Z_1}\left(e^{-t^{\frac{1}{Y}}u}-1\right)du\right)\\
&\quad - e^{-(\eta+\tilde{\gamma})t}t^{\frac{1}{Y}-\frac{1}{2}}\widetilde{\mathbb{E}}\left(e^{-\sqrt{t}\sigma W_1^*}\mathbf{1}_{\left\{W_1^*\geq 0,\,-t^{\frac{1}{2}-\frac{1}{Y}}\sigma W_1^*\leq Z_1\leq 0\right\}}\int_{\widetilde{U}_1+Z_1}^{\widetilde{U}_1}\left(e^{-t^{\frac{1}{Y}}u}-1\right)du\right)\\
&\quad + e^{-(\eta+\tilde{\gamma})t}t^{\frac{1}{Y}-\frac{1}{2}}\widetilde{\mathbb{E}}\left(Z_1\mathbf{1}_{\left\{W_1^*\geq 0,\,Z_1\geq -t^{\frac{1}{2}-\frac{1}{Y}}\sigma W_1^*\right\}}e^{-\sqrt{t}\sigma W_1^*}\right).
\end{aligned}
$$

Next, by Fubini's theorem and the independence of W_1^* and $(Z_1,\widetilde{U}_1)$,

$$
\begin{aligned}
B_1(t) &= t^{\frac{1}{Y}-\frac{1}{2}}e^{-(\eta+\tilde{\gamma})t}\widetilde{\mathbb{E}}\left(e^{-\sqrt{t}\sigma W_1^*}\mathbf{1}_{\left\{W_1^*\geq 0\right\}}\right)\int_{\mathbb{R}}\left(e^{-t^{\frac{1}{Y}}u}-1\right)\widetilde{\mathbb{P}}\left(Z_1\geq 0,\,\widetilde{U}_1\leq u\leq \widetilde{U}_1+Z_1\right)du\\
&\quad - t^{\frac{1}{Y}-\frac{1}{2}}e^{-(\eta+\tilde{\gamma})t}\int_0^\infty\left[\int_{\mathbb{R}}\left(e^{-t^{\frac{1}{Y}}u}-1\right)\widetilde{\mathbb{P}}\left(-t^{\frac{1}{2}-\frac{1}{Y}}w\leq Z_1\leq 0,\,\widetilde{U}_1+Z_1\leq u\leq\widetilde{U}_1\right)du\right]e^{-\sqrt{t}w}\,\frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dw\\
&\quad + e^{-(\eta+\tilde{\gamma})t}t^{\frac{1}{Y}-\frac{1}{2}}\widetilde{\mathbb{E}}\left(Z_1\mathbf{1}_{\left\{W_1^*\geq 0,\,Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}\sigma W_1^*\right\}}e^{-\sqrt{t}\sigma W_1^*}\right)\\
&:= B_{11}(t)-B_{12}(t)+B_{13}(t),
\end{aligned}
\tag{A.31}
$$

where above we had used the symmetry of Z_1 and the following consequence thereof:

$$
\widetilde{\mathbb{E}}\left(Z_1\mathbf{1}_{\left\{Z_1\geq -t^{\frac{1}{2}-\frac{1}{Y}}w\right\}}\right)=\widetilde{\mathbb{E}}\left(Z_1\left(1-\mathbf{1}_{\left\{Z_1\leq -t^{\frac{1}{2}-\frac{1}{Y}}w\right\}}\right)\right)=\widetilde{\mathbb{E}}\left(-Z_1\mathbf{1}_{\left\{Z_1\leq -t^{\frac{1}{2}-\frac{1}{Y}}w\right\}}\right)=\widetilde{\mathbb{E}}\left(Z_1\mathbf{1}_{\left\{Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}w\right\}}\right).
$$

In order to obtain the asymptotic behavior of $B_{11}(t)$, consider

$$
\begin{aligned}
B_{11}^{(1)}(t) &:= \int_{-\infty}^0\left(e^{-t^{\frac{1}{Y}}u}-1\right)\widetilde{\mathbb{P}}\left(Z_1\geq 0,\,\widetilde{U}_1\leq u\leq\widetilde{U}_1+Z_1\right)du,\\
B_{11}^{(2)}(t) &:= \int_0^\infty\left(e^{-t^{\frac{1}{Y}}u}-1\right)\widetilde{\mathbb{P}}\left(Z_1\geq 0,\,\widetilde{U}_1\leq u\leq\widetilde{U}_1+Z_1\right)du.
\end{aligned}
$$

For $B_{11}^{(1)}(t)$, we use similar arguments as in (A.5). Concretely, for $u<0$, by (2.18),

$$
\frac{e^{-t^{\frac{1}{Y}}u}-1}{t^{\frac{1}{Y}}}\widetilde{\mathbb{P}}\left(Z_1\geq 0,\,\widetilde{U}_1\leq u\leq\widetilde{U}_1+Z_1\right)\leq(-u)\,e^{-t^{\frac{1}{Y}}u}\,\widetilde{\mathbb{P}}\left(\widetilde{U}_1\leq u\right)\leq(-u)\,e^{\left(1-t^{\frac{1}{Y}}\right)u}\,\widetilde{\mathbb{E}}\left(e^{-\widetilde{U}_1}\right)\leq e^{\eta}\,(-u)\,e^{\frac{u}{2}},
$$

and thus, by the dominated convergence theorem,

$$B_{11}^{(1)}(t) = t^{\frac{1}{Y}} \int_{-\infty}^{0} (-u)\,\widetilde{\mathbb{P}}\left(Z_1 \geq 0,\, u \leq \widetilde{U}_1 + Z_1\right) du + o(t^{\frac{1}{Y}}), \quad t \to 0. \tag{A.32}$$

For $B_{11}^{(2)}(t)$, we use arguments similar to those used to obtain (A.16). Concretely, let

$$\begin{aligned} B_{11}^{(21)}(t) &:= \int_0^{\infty} \left(e^{-t^{\frac{1}{Y}}u} - 1\right) \widetilde{\mathbb{P}}\left(Z_1 \geq 0,\, u \leq \widetilde{U}_1 + Z_1\right) du \\ &= t^{\frac{1}{Y}} \int_0^{\infty} \frac{e^{-t^{\frac{1}{Y}}u} - 1}{t^{\frac{1}{Y}}} \left(\widetilde{\mathbb{P}}\left(Z_1 \geq 0,\, u \leq \widetilde{U}_1 + Z_1\right) - \frac{CM^Y}{Yu^Y}\right) du - t^{1-\frac{1}{Y}} C\Gamma(-Y) M^Y, \end{aligned}$$

where in the second equality we had used the identity (A.2). The integral on the right-hand side of the previous equation is precisely the first integral defined in (A.6), and thus, in light of (A.16),

$$B_{11}^{(21)}(t) = -t^{1-\frac{1}{Y}} C\Gamma(-Y) M^Y + O(t^{\frac{1}{Y}}), \quad t \to 0.$$

Using similar arguments, it can be shown that

$$B_{11}^{(22)}(t) := \int_0^{\infty} \left(e^{-t^{\frac{1}{Y}}u} - 1\right) \widetilde{\mathbb{P}}\left(Z_1 \geq 0,\, u \leq \widetilde{U}_1\right) du = -t^{1-\frac{1}{Y}} C\Gamma(-Y) \left(M^*\right)^Y + O(t^{\frac{1}{Y}}), \quad t \to 0.$$

Therefore,

$$B_{11}^{(2)}(t) = -t^{1-\frac{1}{Y}} C\Gamma(-Y) \left(M^Y - (M^*)^Y\right) + O(t^{\frac{1}{Y}}), \quad t \to 0,$$

which, together with (A.32), implies that the term $B_{11}(t)$ introduced in (A.31) is such that

$$B_{11}(t) = -\frac{1}{2} t^{\frac{1}{2}} C\Gamma(-Y) \left(M^Y - (M^*)^Y\right) + O(t^{\frac{2}{Y}-\frac{1}{2}}), \quad t \to 0. \tag{A.33}$$

To deal with $B_{12}(t)$, we first make the change of variables $x = t^{\frac{1}{Y}} u$ in the integral appearing in this term so that

$$B_{12}(t) = t^{-\frac{1}{2}} e^{-(\eta+\tilde{\gamma})t} \int_0^{\infty} \left[\int_{\mathbb{R}} \left(e^{-x} - 1\right) \widetilde{\mathbb{P}}\left(-t^{\frac{1}{2}-\frac{1}{Y}} w \leq Z_1 \leq 0,\, \widetilde{U}_1 + Z_1 \leq t^{-\frac{1}{Y}} x \leq \widetilde{U}_1\right) dx\right] e^{-\sqrt{t}w} \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dw.$$

We shall prove that $B_{12}(t) = o(t^{1/2})$ as $t \to 0$. To this end, let

$$\begin{aligned} B_{12}^{(1)}(t) &= \int_0^{\infty} \left[\int_0^{\infty} \left(1 - e^{-x}\right) \widetilde{\mathbb{P}}\left(-t^{\frac{1}{2}-\frac{1}{Y}} w \leq Z_1 \leq 0,\, \widetilde{U}_1 + Z_1 \leq t^{-\frac{1}{Y}} x \leq \widetilde{U}_1\right) dx\right] e^{-\sqrt{t}w} \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dw, \\ B_{12}^{(2)}(t) &= \int_0^{\infty} \left[\int_{-\infty}^{0} \left(e^{-x} - 1\right) \widetilde{\mathbb{P}}\left(-t^{\frac{1}{2}-\frac{1}{Y}} w \leq Z_1 \leq 0,\, \widetilde{U}_1 + Z_1 \leq t^{-\frac{1}{Y}} x \leq \widetilde{U}_1\right) dx\right] e^{-\sqrt{t}w} \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dw. \end{aligned}$$

For $B_{12}^{(1)}(t)$, since by (2.14-i), for any $x > 0$ and $t > 0$,

$$\begin{aligned} \frac{1}{t}\widetilde{\mathbb{P}}\left(-t^{\frac{1}{2}-\frac{1}{Y}} w \leq Z_1 \leq 0,\, \widetilde{U}_1 + Z_1 \leq t^{-\frac{1}{Y}} x \leq \widetilde{U}_1\right) &\leq \frac{1}{t}\widetilde{\mathbb{P}}\left(\widetilde{U}_1 \geq t^{-\frac{1}{Y}} x\right) \leq \frac{1}{t}\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)} \geq \frac{t^{-\frac{1}{Y}} x}{2M^*}\right) + \frac{1}{t}\widetilde{\mathbb{P}}\left(-\bar{U}_1^{(n)} \geq \frac{t^{-\frac{1}{Y}} x}{2G^*}\right) \\ &\leq 2^Y \kappa_1 \left(\left(M^*\right)^Y + \left(G^*\right)^Y\right) x^{-Y}, \end{aligned} \tag{A.34}$$

by the dominated convergence theorem,

$$\lim_{t\to 0} \frac{1}{t} B_{12}^{(1)}(t) = \int_0^{\infty} \left[\int_0^{\infty} \left(1 - e^{-x}\right) \left(\lim_{t \to 0} \frac{1}{t}\widetilde{\mathbb{P}}\left(-t^{\frac{1}{2}-\frac{1}{Y}} w \leq Z_1 \leq 0,\, \widetilde{U}_1 + Z_1 \leq t^{-\frac{1}{Y}} x \leq \widetilde{U}_1\right)\right) dx\right] \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dw. \tag{A.35}$$

Moreover, for any $t>0$, $w>0$ and $x>0$, $P_t(x,w) := \widetilde{\mathbb{P}}\left(-t^{\frac{1}{2}-\frac{1}{Y}}w \le Z_1 \le 0, \widetilde{U}_1 + Z_1 \le t^{-\frac{1}{Y}}x \le \widetilde{U}_1\right)$ is such that

$$
\begin{aligned}
P_t(x,w) &= \widetilde{\mathbb{P}}\left(-t^{\frac{1}{2}-\frac{1}{Y}}w \le \bar{U}_1^{(p)} + \bar{U}_1^{(n)} \le 0,\, (M^*+1)\,\bar{U}_1^{(p)} + (1-G^*)\,\bar{U}_1^{(n)} \le t^{-\frac{1}{Y}}x \le M^*\bar{U}_1^{(p)} - G^*\bar{U}_1^{(n)}\right) \\
&\le \widetilde{\mathbb{P}}\left(\frac{t^{-\frac{1}{Y}}}{M^*+G^*} \le -\bar{U}_1^{(n)} \le \frac{t^{-\frac{1}{Y}}x + (M^*+1)\,t^{\frac{1}{2}-\frac{1}{Y}}w}{M^*+G^*},\ \frac{t^{-\frac{1}{Y}}x + G^*\bar{U}_1^{(n)}}{M^*} \le \bar{U}_1^{(p)}\right) \\
&\le \widetilde{\mathbb{P}}\left(\frac{t^{-\frac{1}{Y}}x}{M^*+G^*} \le -\bar{U}_1^{(n)}\right)\widetilde{\mathbb{P}}\left(\frac{t^{-\frac{1}{Y}}M^*x - G^*\,(M^*+1)\,t^{\frac{1}{2}-\frac{1}{Y}}w}{M^*\,(M^*+G^*)} \le \bar{U}_1^{(p)}\right).
\end{aligned}
$$

Hence, in view of (2.14-i),

$$
\lim_{t\to 0}\frac{1}{t}P_t(x,w) \le \kappa_1\,(M^*+G^*)^Y\,x^{-Y}\lim_{t\to 0}\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)} \ge \frac{t^{-\frac{1}{Y}}\left(M^*x - G^*\,(M^*+1)\sqrt{t}w\right)}{M^*\,(M^*+G^*)}\right) = 0. \tag{A.36}
$$

Combining (A.35) and (A.36) leads to $B_{12}^{(1)}(t) = o(t)$. For $B_{12}^{(2)}(t)$, note that, for any $t>0$, $w>0$ and $x<0$,

$$
\frac{1}{t}P_t(x,w) \le \frac{1}{t}\widetilde{\mathbb{P}}\left(\widetilde{U}_1 + Z_1 \le t^{-\frac{1}{Y}}x\right) \le \frac{1}{t}\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)} \le \frac{t^{-\frac{1}{Y}}x}{2\,(M^*+1)}\right) + \frac{1}{t}\widetilde{\mathbb{P}}\left(-\bar{U}_1^{(n)} \le \frac{t^{-\frac{1}{Y}}x}{2\,(G^*-1)}\right).
$$

Using that $M, G > 0$, it follows

$$
\frac{1}{t}P_t(x,w) \le \frac{2}{t}\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)} \le \frac{t^{-\frac{1}{Y}}x}{2\,(M^*+G^*)}\right) \le \frac{2}{t}\widetilde{\mathbb{E}}\left(e^{-\bar{U}_1^{(p)}}\right)\exp\left\{\frac{t^{-\frac{1}{Y}}x}{2\,(M^*+G^*)}\right\} \to 0, \quad t\to 0.
$$

Therefore, by the dominated convergence theorem,

$$
\lim_{t\to 0}\frac{1}{t}B_{12}^{(2)}(t) = \int_0^\infty \left[\int_{-\infty}^0 \left(e^{-x}-1\right)\left(\lim_{t\to 0}\frac{1}{t}P_t(x,w)\right)dx\right]\frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dw = 0,
$$

which in turn implies that

$$
B_{12}(t) = t^{-\frac{1}{2}}e^{-(\eta+\tilde{\gamma})t}\left(B_{12}^{(2)}(t) - B_{12}^{(1)}(t)\right) = o(t^{\frac{1}{2}}), \quad t\to 0. \tag{A.37}
$$

Finally, we deal with $B_{13}(t)$ and analyze the asymptotic behavior of the following expression:

$$
\widetilde{B}_{13}(t) := t^{\frac{Y}{2}-1}B_{13}(t) - \frac{C\sigma^{1-Y}}{2(Y-1)}\widetilde{\mathbb{E}}\left(|W_1^*|^{1-Y}\right). \tag{A.38}
$$

First, $\widetilde{B}_{13}(t)$ is further decomposed as:

$$
\begin{aligned}
\widetilde{B}_{13}(t) &= t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}}\left[e^{-(\eta+\tilde{\gamma})t}\,\widetilde{\mathbb{E}}\left(e^{-\sqrt{t}\sigma W_1^*}\mathbf{1}_{\{W_1^*\ge 0\}}\int_{t^{\frac{1}{2}-\frac{1}{Y}}\sigma W_1^*}^\infty z\,p_Z(1,z)\,dz\right) - \widetilde{\mathbb{E}}\left(\mathbf{1}_{\{W_1^*\ge 0\}}\int_{t^{\frac{1}{2}-\frac{1}{Y}}\sigma W_1^*}^\infty Cz^{-Y}\,dz\right)\right] \\
&= t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}}\left(e^{-(\eta+\tilde{\gamma})t}-1\right)\widetilde{\mathbb{E}}\left(e^{-\sqrt{t}\sigma W_1^*}\mathbf{1}_{\{W_1^*\ge 0\}}\int_{t^{\frac{1}{2}-\frac{1}{Y}}\sigma W_1^*}^\infty z\,p_Z(1,z)\,dz\right) \\
&\quad + t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}}\widetilde{\mathbb{E}}\left(\left(e^{-\sqrt{t}\sigma W_1^*}-1\right)\mathbf{1}_{\{W_1^*\ge 0\}}\int_{t^{\frac{1}{2}-\frac{1}{Y}}\sigma W_1^*}^\infty z\,p_Z(1,z)\,dz\right) \\
&\quad + t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}}\widetilde{\mathbb{E}}\left(\mathbf{1}_{\{W_1^*\ge 0\}}\int_{t^{\frac{1}{2}-\frac{1}{Y}}\sigma W_1^*}^\infty z\left(p_Z(1,z) - Cz^{-Y-1}\right)dz\right) \\
&:= B_{13}^{(1)}(t) + B_{13}^{(2)}(t) + B_{13}^{(3)}(t).
\end{aligned} \tag{A.39}
$$

As shown next,

$$
B_{13}^{(1)}(t) = O(t), \quad B_{13}^{(2)}(t) = O(\sqrt{t}), \quad t\to 0. \tag{A.40}
$$

Indeed, for $B_{13}^{(1)}(t)$, we first rewrite the expectation as

$$\widetilde{\mathbb{E}}\left(e^{-\sqrt{t}\sigma W_1^*}\mathbf{1}_{\{W_1^*\geq 0\}}\int_{t^{\frac{1}{2}-\frac{1}{Y}}\sigma W_1^*}^{\infty} z\, p_Z(1,z)\,dz\right)=\int_0^{\infty}\left(\int_{t^{\frac{1}{2}-\frac{1}{Y}}w}^{\infty} z\, p_Z(1,z)\,dz\right)e^{-\sqrt{t}w}\,\frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dw. \tag{A.41}$$

Next, by (2.16), there exists $H_1>0$ such that, for any $z\geq H_1$,

$$p_Z(1,z)\leq 2Cz^{-Y-1}. \tag{A.42}$$

Hence, for any $w>0$,

$$\begin{aligned} t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}}\int_{t^{\frac{1}{2}-\frac{1}{Y}}w}^{\infty} z\,p_Z(1,z)\,dz &\leq t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}}\left(\int_{t^{\frac{1}{2}-\frac{1}{Y}}w}^{\infty}2Cu^{-Y}du+\mathbf{1}_{\{t^{\frac{1}{2}-\frac{1}{Y}}w<H_1\}}H_1\widetilde{\mathbb{P}}\left(Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}w\right)\right)\\ &\leq \frac{2Cw^{1-Y}}{Y-1}+H_1^Y w^{1-Y}, \end{aligned}$$

where to derive the second term in the last inequality we used that $\widetilde{\mathbb{P}}(Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}w)\leq H_1^{Y-1}/(t^{\frac{1}{2}-\frac{1}{Y}}w)^{Y-1}$, when $t^{\frac{1}{2}-\frac{1}{Y}}w<H_1$. Together with (A.41) and since $Y\in(1,2)$, we obtain the first relationship in (A.40). The second relationship therein is obtained using similar arguments.

It remains to deal with $B_{13}^{(3)}(t)$, which can be rewritten as:

$$\begin{aligned} B_{13}^{(3)}(t)&=\frac{1}{2}t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}}\int_{\mathbb{R}}\left(\int_0^{t^{\frac{1}{Y}-\frac{1}{2}}|z|}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{w^2}{2\sigma^2}}\,dw\right)|z|\left(p_Z(1,z)-C|z|^{-Y-1}\right)dz\\ &=\frac{1}{2}t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}}\int_0^1\int_{\mathbb{R}}\frac{t^{\frac{1}{Y}-\frac{1}{2}}}{\sqrt{2\pi\sigma^2}}e^{-\frac{t^{\frac{2}{Y}-1}z^2u^2}{2\sigma^2}}|z|^2\left(p_Z(1,z)-C|z|^{-Y-1}\right)dz\,du, \end{aligned} \tag{A.43}$$

where we change variables $u=t^{\frac{1}{2}-\frac{1}{Y}}w/|z|$ and apply the Fubini Theorem in the second equality. For simplicity, we write $p_Z(z)$ instead of $p_Z(1,z)$ hereafter. Next, denoting the characteristic function of Z_1 by $\hat{p}_Z(x)$, we have

$$p_Z(z)=\mathcal{F}\left(\frac{1}{\sqrt{2\pi}}\,\hat{p}_Z\right)(z),\quad z^2p_Z(z)=\mathcal{F}\left(\frac{-1}{\sqrt{2\pi}}\,\hat{p}_Z''\right)(z), \tag{A.44}$$

where $\mathcal{F}(h)(z):=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-ivz}h(v)dv$ denotes the Fourier transformation of $h\in L_1(\mathbb{R})$. Also, regarding $|x|^{Y-2}$ as a tempered distribution, it is known that

$$|z|^{1-Y}=\mathcal{F}\left(K^{-1}|x|^{Y-2}\right)(z),$$

with $K:=-2\sin(\pi(Y-2)/2)\Gamma(Y-1)/\sqrt{2\pi}$. In particular, by definition,

$$\int_{\mathbb{R}}|z|^{1-Y}\phi(z)\,dz=\int_{\mathbb{R}}K^{-1}|x|^{Y-2}\mathcal{F}(\phi)(x)\,dx, \tag{A.45}$$

for any Schwartz function ϕ. Thus, combining (A.43)-(A.45),

$$\begin{aligned} B_{13}^{(3)}(t)&=\frac{1}{2}t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}}\int_0^1\int_{\mathbb{R}}\mathcal{F}\left(\frac{t^{\frac{1}{Y}-\frac{1}{2}}}{\sqrt{2\pi\sigma^2}}e^{-\frac{t^{\frac{2}{Y}-1}z^2u^2}{2\sigma^2}}\right)(x)\left(-\frac{1}{\sqrt{2\pi}}\,\hat{p}_Z''(x)-\frac{C}{K}|x|^{Y-2}\right)dx\,du\\ &=-\frac{1}{2\sqrt{2\pi}}t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}}\int_0^1\int_{\mathbb{R}}u^{-1}e^{-\frac{t^{1-\frac{2}{Y}}\sigma^2x^2}{2u^2}}\left(\frac{1}{\sqrt{2\pi}}\,\hat{p}_Z''(x)+\frac{C}{K}|x|^{Y-2}\right)dx\,du. \end{aligned}$$

Recalling that $\hat{p}_Z(x)=e^{-c|x|^Y}$ with $c:=2C|\cos(\pi Y/2)|\Gamma(-Y)$, we have:

$$\begin{aligned} \frac{1}{\sqrt{2\pi}}\,\hat{p}_Z''(x)+\frac{C}{K}|x|^{Y-2}&=-\frac{cY(Y-1)}{\sqrt{2\pi}}e^{-c|x|^Y}|x|^{Y-2}+\frac{1}{\sqrt{2\pi}}(cY|x|^{Y-1})^2e^{-c|x|^Y}+\frac{C}{K}|x|^{Y-2}\\ &=\frac{1}{\sqrt{2\pi}}(cY|x|^{Y-1})^2e^{-c|x|^Y}+\frac{cY(Y-1)}{\sqrt{2\pi}}|x|^{Y-2}\left(1-e^{-c|x|^Y}\right), \end{aligned}$$

where in the last equality we used $C/K = cY(Y-1)/\sqrt{2\pi}$. Hence,

$$\begin{aligned}B_{13}^{(3)}(t) &= \frac{-c^2Y^2}{2\pi} t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}} \int_0^1\int_0^\infty u^{-1} e^{-\frac{t^{1-\frac{2}{Y}}\sigma^2x^2}{2u^2}} x^{2Y-2} e^{-cx^Y}\,dx\,du \\ &\quad + \frac{-cY(Y-1)}{2\pi} t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}} \int_0^1\int_0^\infty u^{-1} e^{-\frac{t^{1-\frac{2}{Y}}\sigma^2x^2}{2u^2}} x^{Y-2}\left(1-e^{-cx^Y}\right)dx\,du \\ &:= B_{13}^{(3,1)}(t) + B_{13}^{(3,2)}(t). \end{aligned} \tag{A.46}$$

For $B_{13}^{(3,1)}(t)$, changing variables $v = t^{\frac{1}{2}-\frac{1}{Y}}\sigma x/u$,

$$\begin{aligned}B_{13}^{(3,1)}(t) &= -\frac{c^2Y^2}{2\pi} t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}} \int_0^1\int_0^\infty u^{-1} e^{-\frac{v^2}{2}} \left(\frac{t^{\frac{1}{Y}-\frac{1}{2}}vu}{\sigma}\right)^{2Y-2} e^{-c\left(\frac{t^{\frac{1}{Y}-\frac{1}{2}}uv}{\sigma}\right)^Y} \frac{t^{\frac{1}{Y}-\frac{1}{2}}u}{\sigma}\,dv\,du \\ &= -\frac{c^2Y^2}{2\pi\sigma^{2Y-1}} t^{1-\frac{Y}{2}} \int_0^1 \left(\int_0^\infty e^{-\frac{v^2}{2}} v^{2Y-2} e^{-c\sigma^{-Y}t^{1-\frac{Y}{2}}u^Yv^Y}\,dv\right) u^{2Y-2}\,du.\end{aligned}$$

Hence, by the dominated convergence theorem,

$$\lim_{t\to0} t^{\frac{Y}{2}-1} B_{13}^{(3,1)}(t) = -\frac{c^2Y^2}{2\sqrt{2\pi}(2Y-1)\sigma^{2Y-1}} \widetilde{\mathbb{E}}\left(|W_1^*|^{2Y-2}\right) = -\frac{2C^2Y^2\cos^2\left(\frac{\pi Y}{2}\right)\Gamma^2(-Y)}{\sqrt{2\pi}(2Y-1)\sigma^{2Y-1}} \widetilde{\mathbb{E}}\left(|W_1^*|^{2Y-2}\right). \tag{A.47}$$

Similarly, for $B_{13}^{(3,2)}(t)$,

$$\begin{aligned}B_{13}^{(3,2)}(t) &= -\frac{cY(Y-1)}{2\pi} t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}} \int_0^1\int_0^\infty u^{-1} e^{-\frac{v^2}{2}} \left(\frac{t^{\frac{1}{Y}-\frac{1}{2}}uv}{\sigma}\right)^{Y-2} \left[1-e^{-c\left(\frac{t^{\frac{1}{Y}-\frac{1}{2}}uv}{\sigma}\right)^Y}\right] \frac{t^{\frac{1}{Y}-\frac{1}{2}}u}{\sigma}\,dv\,du \\ &= -\frac{cY(Y-1)}{2\pi\sigma^{Y-1}} \int_0^1 \left[\int_0^\infty e^{-\frac{v^2}{2}} v^{Y-2}\left(1-e^{-c\sigma^{-Y}t^{1-\frac{Y}{2}}u^Yv^Y}\right)dv\right] u^{Y-2}\,du.\end{aligned}$$

Again, by the dominated convergence theorem,

$$\lim_{t\to0} t^{\frac{Y}{2}-1} B_{13}^{(3,2)}(t) = \frac{-c^2Y(Y-1)}{2\sqrt{2\pi}(2Y-1)\sigma^{2Y-1}} \widetilde{\mathbb{E}}\left(|W_1^*|^{2Y-2}\right) = -\frac{2C^2Y(Y-1)\cos^2\left(\frac{\pi Y}{2}\right)\Gamma^2(-Y)}{\sqrt{2\pi}(2Y-1)\sigma^{2Y-1}} \widetilde{\mathbb{E}}\left(|W_1^*|^{2Y-2}\right). \tag{A.48}$$

Combining (A.39), (A.40) and (A.46)-(A.48),

$$\lim_{t\to0} t^{\frac{Y}{2}-1} \widetilde{B}_{13}(t) = -\frac{2C^2Y\cos^2\left(\frac{\pi Y}{2}\right)\Gamma^2(-Y)}{\sqrt{2\pi}\sigma^{2Y-1}} \widetilde{\mathbb{E}}\left(|W_1^*|^{2Y-2}\right) := d'_{31}. \tag{A.49}$$

Combining (A.31), (A.33), (A.37), (A.38) and (A.49), the asymptotic behavior for $B_1(t)$, as $t \to 0$, is given by

$$\begin{aligned}B_1(t) &= -\frac{1}{2}C\Gamma(-Y)\left(M^Y - (M^*)^Y\right) t^{\frac{1}{2}} + t^{1-\frac{Y}{2}}\left(\widetilde{B}_{13}(t) + \frac{C\sigma^{1-Y}}{2(Y-1)}\widetilde{\mathbb{E}}\left(|W_1^*|^{1-Y}\right)\right) + o(t^{\frac{1}{2}}) \\ &= -\frac{1}{2}C\Gamma(-Y)\left(M^Y - (M^*)^Y\right) t^{\frac{1}{2}} + t^{1-\frac{Y}{2}}\left(t^{1-\frac{Y}{2}}d'_{31} + o(t^{1-\frac{Y}{2}}) + \frac{C\sigma^{1-Y}}{2(Y-1)}\widetilde{\mathbb{E}}\left(|W_1^*|^{1-Y}\right)\right) + o(t^{\frac{1}{2}}) \\ &= -\frac{1}{2}d'_3 t^{\frac{1}{2}} + \frac{C\sigma^{1-Y}}{2(Y-1)}\widetilde{\mathbb{E}}\left(|W_1^*|^{1-Y}\right)t^{1-\frac{Y}{2}} + d'_{31}t^{2-Y} + o(t^{\frac{1}{2}}) + o(t^{2-Y}),\end{aligned} \tag{A.50}$$

setting $d'_3 := C\Gamma(-Y)\left(M^Y - (M^*)^Y\right)$.

Step 2. Next, we analyze $B_2(t)$ by decomposing it as:

$$\begin{aligned}B_2(t) &= e^{-(\eta+\tilde{\gamma})t} \int_0^\infty \widetilde{\mathbb{E}}\left(\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1} - 1\right)\mathbf{1}_{\left\{Z_1 \le -t^{\frac{1}{2}-\frac{1}{Y}}w\right\}}\right) \frac{1-e^{-\sqrt{t}w}}{\sqrt{t}} \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dw \\ &\quad + e^{-(\eta+\tilde{\gamma})t} \int_0^\infty \widetilde{\mathbb{P}}\left(Z_1 \le -t^{\frac{1}{2}-\frac{1}{Y}}w\right) \frac{1-e^{-\sqrt{t}w}}{\sqrt{t}} \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dw \\ &:= B_{21}(t) + B_{22}(t).\end{aligned} \tag{A.51}$$

We begin with proving that $B_{21}(t) = o\left(t^{1/2}\right)$ as $t \to 0$. To this end, consider first

$$B_{21}^{(1)}(t) := \int_0^\infty b_{21}^{(1)}(t;w) \frac{1 - e^{-\sqrt{t}w}}{\sqrt{t}} \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dw,$$

where

$$b_{21}^{(1)}(t;w) := \widetilde{\mathbb{E}}\left(\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1} - 1\right)\mathbf{1}_{\left\{Z_1 \leq -t^{\frac{1}{2}-\frac{1}{Y}}w,\, \widetilde{U}_1 < 0\right\}}\right).$$

Note that, for any $0 < t < 1$ and $w > 0$, by (2.18),

$$\begin{aligned}
t^{-\frac{1}{2}} b_{21}^{(1)}(t;w) &= t^{-\frac{1}{2}}\widetilde{\mathbb{E}}\left(\mathbf{1}_{\left\{Z_1 \leq -t^{\frac{1}{2}-\frac{1}{Y}}w,\, \widetilde{U}_1 < 0\right\}} \int_{-\infty}^0 \mathbf{1}_{\{t^{\frac{1}{Y}}\widetilde{U}_1 \leq u \leq 0\}} e^{-u}\, du\right) \\
&\leq t^{-\frac{1}{2}} \int_{-\infty}^0 e^{-u}\widetilde{\mathbb{P}}\left(\widetilde{U}_1 \leq t^{-\frac{1}{Y}}u\right) du \leq \widetilde{\mathbb{E}}\left(e^{-\widetilde{U}_1}\right) t^{-\frac{1}{2}} \int_{-\infty}^0 e^{-u\left(1 - t^{-\frac{1}{Y}}\right)} du = e^{\eta}\frac{t^{\frac{1}{Y}-\frac{1}{2}}}{1 - t^{\frac{1}{Y}}}.
\end{aligned}$$

Since $Y \in (1,2)$, by the dominated convergence theorem,

$$\lim_{t\to 0} t^{-\frac{1}{2}} B_{21}^{(1)}(t) = \int_0^\infty \left(\lim_{t\to 0} t^{-\frac{1}{2}} b_{21}^{(1)}(t;w)\right) \frac{we^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dw \leq \frac{1}{\sqrt{\pi}} e^{\eta} \lim_{t\to 0} \frac{t^{\frac{1}{Y}-\frac{1}{2}}}{1 - t^{\frac{1}{Y}}} = 0. \tag{A.52}$$

Next, consider

$$B_{21}^{(2)}(t) := \int_0^\infty b_{21}^{(2)}(t;w) \frac{1 - e^{-\sqrt{t}w}}{\sqrt{t}} \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dw, \tag{A.53}$$

where $b_{21}^{(2)}(t;w)$ is defined and further decomposed as:

$$\begin{aligned}
b_{21}^{(2)}(t;w) &:= \widetilde{\mathbb{E}}\left(\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1} - 1\right)\mathbf{1}_{\left\{Z_1 \leq -t^{\frac{1}{2}-\frac{1}{Y}}w,\, \widetilde{U}_1 \geq 0\right\}}\right) \\
&= \widetilde{\mathbb{E}}\left(\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1} - 1 + t^{\frac{1}{Y}}\widetilde{U}_1\right)\mathbf{1}_{\left\{Z_1 \leq -t^{\frac{1}{2}-\frac{1}{Y}}w,\, \widetilde{U}_1 \geq 0\right\}}\right) - t^{\frac{1}{Y}}\widetilde{\mathbb{E}}\left(\widetilde{U}_1\mathbf{1}_{\left\{Z_1 \leq -t^{\frac{1}{2}-\frac{1}{Y}}w,\, \widetilde{U}_1 \geq 0\right\}}\right).
\end{aligned} \tag{A.54}$$

Note that (since $1 < Y < 2$) as $t \to 0$,

$$0 \leq t^{-\frac{1}{2}} \int_0^\infty t^{\frac{1}{Y}}\widetilde{\mathbb{E}}\left(\widetilde{U}_1\mathbf{1}_{\left\{Z_1 \leq -t^{\frac{1}{2}-\frac{1}{Y}}w,\, \widetilde{U}_1 \geq 0\right\}}\right) \frac{1 - e^{-\sqrt{t}w}}{\sqrt{t}} \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dw \leq t^{\frac{1}{Y}-\frac{1}{2}}\widetilde{\mathbb{E}}\left|\widetilde{U}_1\right| \int_0^\infty \frac{w\, e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dw \to 0. \tag{A.55}$$

Moreover, by (2.14-i) and the decomposition $\widetilde{U}_t = M^*\bar{U}_t^{(p)} - G^*\bar{U}_t^{(n)}$, for any $t > 0$ and $w > 0$,

$$\begin{aligned}
\widetilde{\mathbb{E}}\left(\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1} - 1 + t^{\frac{1}{Y}}\widetilde{U}_1\right)\mathbf{1}_{\left\{Z_1 \leq -t^{\frac{1}{2}-\frac{1}{Y}}w,\, \widetilde{U}_1 \geq 0\right\}}\right) &= \widetilde{\mathbb{E}}\left(\int_0^{t^{\frac{1}{Y}}\widetilde{U}_1} \left(1 - e^{-u}\right) du\, \mathbf{1}_{\left\{Z_1 \leq -t^{\frac{1}{2}-\frac{1}{Y}}w,\, \widetilde{U}_1 \geq 0\right\}}\right) \\
&= \int_0^\infty \left(1 - e^{-u}\right)\widetilde{\mathbb{P}}\left(Z_1 \leq -t^{\frac{1}{2}-\frac{1}{Y}}w,\, \widetilde{U}_1 \geq t^{-\frac{1}{Y}}u\right) du \\
&\leq \int_0^\infty \left(1 - e^{-u}\right)\widetilde{\mathbb{P}}\left(\widetilde{U}_1 \geq t^{-\frac{1}{Y}}u\right) du \\
&\leq 2^{Y+1}\kappa_1\left((M^*)^Y + (G^*)^Y\right) t \int_0^\infty \left(1 - e^{-u}\right) u^{-Y} du.
\end{aligned}$$

Hence, by the dominated convergence theorem,

$$\begin{aligned}
0 &\leq t^{-\frac{1}{2}} \int_0^\infty \widetilde{\mathbb{E}}\left(\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1} - 1 + t^{\frac{1}{Y}}\widetilde{U}_1\right)\mathbf{1}_{\left\{Z_1 \leq -t^{\frac{1}{2}-\frac{1}{Y}}w,\, \widetilde{U}_1 \geq 0\right\}}\right) \frac{1 - e^{-\sqrt{t}w}}{\sqrt{t}} \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dw \\
&\leq 2^{Y+1}\kappa_1\left(M^* + G^*\right)^Y \sqrt{t} \int_0^\infty \left(1 - e^{-u}\right) u^{-Y} du \int_0^\infty \frac{w\, e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dw \to 0, \quad t \to 0.
\end{aligned} \tag{A.56}$$

In light of (A.53)-(A.55) and (A.56), $B_{21}^{(2)}(t) = o(t^{1/2})$. Together with (A.52), and since $B_{21}(t) = e^{-(\eta+\tilde{\gamma})t}(B_{21}^{(1)}(t) + B_{21}^{(2)}(t)(t))$, we conclude that

$$\lim_{t\to 0} t^{-\frac{1}{2}} B_{21}(t) = 0. \tag{A.57}$$

Finally, we analyze $B_{22}(t)$ defined via (A.51). To this end, let

$$\begin{aligned}
\widetilde{B}_{22}(t) &:= t^{\frac{Y}{2}-1} B_{22}(t) - \frac{C\sigma^{1-Y}}{2Y} \widetilde{\mathbb{E}}\left(|W_1^*|^{1-Y}\right) \\
&= t^{\frac{Y}{2}-1} e^{-(\eta+\tilde{\gamma})t} \int_0^\infty \widetilde{\mathbb{E}}\left(\frac{1-e^{-\sqrt{t}\sigma W_1^*}}{\sqrt{t}} \mathbf{1}_{\{0\le \sigma W_1^* \le t^{\frac{1}{Y}-\frac{1}{2}} z\}}\right) p_Z(1,z)\, dz \\
&\quad - t^{\frac{Y}{2}-1} \int_0^\infty \widetilde{\mathbb{E}}\left(\sigma W_1^* \mathbf{1}_{\{0\le \sigma W_1^* \le t^{\frac{1}{Y}-\frac{1}{2}} z\}}\right) C z^{-Y-1}\, dz \\
&= t^{\frac{Y}{2}-1}\left(e^{-(\eta+\tilde{\gamma})t} - 1\right) \int_0^\infty \widetilde{\mathbb{E}}\left(\frac{1-e^{-\sqrt{t}\sigma W_1^*}}{\sqrt{t}} \mathbf{1}_{\{0\le \sigma W_1^* \le t^{\frac{1}{Y}-\frac{1}{2}} z\}}\right) p_Z(1,z)\, dz \\
&\quad + t^{\frac{Y}{2}-1} \int_0^\infty \widetilde{\mathbb{E}}\left(\left(\frac{1-e^{-\sqrt{t}\sigma W_1^*}}{\sqrt{t}} - \sigma W_1^*\right) \mathbf{1}_{\{0\le \sigma W_1^* \le t^{\frac{1}{Y}-\frac{1}{2}} z\}}\right) p_Z(1,z)\, dz \\
&\quad + t^{\frac{Y}{2}-1} \int_0^\infty \widetilde{\mathbb{E}}\left(\sigma W_1^* \mathbf{1}_{\{0\le \sigma W_1^* \le t^{\frac{1}{Y}-\frac{1}{2}} z\}}\right) \left(p_Z(1,z) - C z^{-Y-1}\right) dz \\
&:= B_{22}^{(1)}(t) + B_{22}^{(2)}(t) + B_{22}^{(3)}(t),
\end{aligned} \tag{A.58}$$

where we used the symmetry of Z_1 in the second equality. As shown next, the first two terms in (A.58) are such that

$$B_{22}^{(1)}(t) = O(t), \quad B_{22}^{(1)}(t) = O(\sqrt{t}), \quad t \to 0. \tag{A.59}$$

Indeed, for the first relation above, note that, by (2.17),

$$\begin{aligned}
&\int_0^\infty \widetilde{\mathbb{E}}\left(\frac{1-e^{-\sqrt{t}\sigma W_1^*}}{\sqrt{t}} \mathbf{1}_{\{0\le \sigma W_1^* \le t^{\frac{1}{Y}-\frac{1}{2}} z\}}\right) p_Z(1,z)\, dz \\
&\le \widetilde{\mathbb{E}}\left(\sigma W_1^* \mathbf{1}_{\{0\le \sigma W_1^* \le t^{\frac{1}{Y}-\frac{1}{2}} Z_1\}}\right) = \int_0^\infty w \widetilde{\mathbb{P}}\left(Z_1 \ge t^{\frac{1}{2}-\frac{1}{Y}} w\right) \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dw \le \kappa_3 t^{1-\frac{Y}{2}} \int_0^\infty w^{1-Y} \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dw.
\end{aligned}$$

The second relationship in (A.59) follows in a similar fashion.

It remains to deal with $B_{22}^{(3)}(t)$, which can be rewritten as:

$$\begin{aligned}
B_{22}^{(3)}(t) &= \frac{1}{2} t^{\frac{Y}{2}-1} \int_{\mathbb{R}} \left(\int_0^{t^{\frac{1}{Y}-\frac{1}{2}}|z|} \frac{w}{\sqrt{2\pi\sigma^2}} e^{-\frac{w^2}{2\sigma^2}}\, dw\right) \left(p_Z(1,z) - C|z|^{-Y-1}\right) dz \\
&= \frac{1}{2} t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}} \int_0^1 \int_{\mathbb{R}} \frac{t^{\frac{1}{Y}-\frac{1}{2}} u}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^{\frac{2}{Y}-1} u^2 z^2}{2\sigma^2}} z^2 \left(p_Z(1,z) - C|z|^{-Y-1}\right) dz\, du,
\end{aligned}$$

where we change variables $u = t^{\frac{1}{2}-\frac{1}{Y}} w/|z|$ and apply the Fubini Theorem in the second equality. Using the same argument given after (A.43), we get

$$\begin{aligned}
B_{22}^{(3)}(t) &= \frac{1}{2} t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}} \int_0^1 \int_{-\infty}^\infty \mathcal{F}\left(\frac{t^{\frac{1}{Y}-\frac{1}{2}} u}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^{\frac{2}{Y}-1} z^2 u^2}{2\sigma^2}}\right)(x) \left(-\frac{1}{\sqrt{2\pi}} \hat{p}_Z''(x) - \frac{C}{K}|x|^{Y-2}\right) dx\, du \\
&= -\frac{1}{2\sqrt{2\pi}} t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}} \int_0^1 \int_{-\infty}^\infty e^{-\frac{t^{1-\frac{2}{Y}} \sigma^2 x^2}{2u^2}} \left(\frac{1}{\sqrt{2\pi}} \hat{p}_Z''(x) + \frac{C}{K}|x|^{Y-2}\right) dx\, du \\
&:= B_{22}^{(3,1)}(t) + B_{22}^{(3,2)}(t),
\end{aligned} \tag{A.60}$$

with

$$B_{22}^{(3,1)}(t) := -\frac{c^2Y^2}{2\pi} t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}} \int_0^1 \int_0^\infty e^{-\frac{t^{1-\frac{2}{Y}}\sigma^2 x^2}{2u^2}} x^{2Y-2} e^{-cx^Y} dx\, du,$$

$$B_{22}^{(3,2)}(t) := -\frac{cY(Y-1)}{2\pi} t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}} \int_0^1 \int_0^\infty e^{-\frac{t^{1-\frac{2}{Y}}\sigma^2 x^2}{2u^2}} x^{Y-2} \left(1-e^{-cx^Y}\right) dxdu.$$

For $B_{22}^{(3,1)}$, changing variables to $v = t^{\frac{1}{2}-\frac{1}{Y}}\sigma x/u$,

$$B_{22}^{(3,1)}(t) = -\frac{c^2Y^2}{2\pi\sigma^{2Y-1}} t^{1-\frac{Y}{2}} \int_0^1 \left(\int_0^\infty e^{-\frac{v^2}{2}} v^{2Y-2} e^{-c\sigma^{-Y}t^{1-\frac{Y}{2}}u^Y v^Y} dv\right) u^{2Y-1} du.$$

Hence, by the dominated convergence theorem,

$$\lim_{t\to 0} t^{\frac{Y}{2}-1} B_{22}^{(3,1)}(t) = -\frac{c^2 Y}{4\sqrt{2\pi}\sigma^{2Y-1}} \widetilde{\mathbb{E}}\left(|W_1^*|^{2Y-2}\right) = -\frac{C^2 Y\cos^2\left(\frac{\pi Y}{2}\right)\Gamma^2(-Y)}{\sqrt{2\pi}\sigma^{2Y-1}} \widetilde{\mathbb{E}}\left(|W_1^*|^{2Y-2}\right). \tag{A.61}$$

Similarly, for $B_{22}^{(3,2)}(t)$,

$$B_{22}^{(3,2)}(t) = -\frac{cY(Y-1)}{2\pi\sigma^{Y-1}} \int_0^1 \left[\int_0^\infty e^{-\frac{v^2}{2}} v^{Y-2} \left(1-e^{-c\sigma^{-Y}t^{1-\frac{Y}{2}}u^Y v^Y}\right) dv\right] u^{Y-1} du.$$

Again, by the dominated convergence theorem,

$$\lim_{t\to 0} t^{\frac{Y}{2}-1} B_{22}^{(3,2)}(t) = -\frac{c^2(Y-1)}{4\sqrt{2\pi}\sigma^{2Y-1}} \widetilde{\mathbb{E}}\left(|W_1^*|^{2Y-2}\right) = -\frac{C^2(Y-1)\cos^2\left(\frac{\pi Y}{2}\right)\Gamma^2(-Y)}{\sqrt{2\pi}\sigma^{2Y-1}} \widetilde{\mathbb{E}}\left(|W_1^*|^{2Y-2}\right). \tag{A.62}$$

Combining (A.58)-(A.62),

$$\lim_{t\to 0} t^{\frac{Y}{2}-1} \widetilde{B}_{22}(t) = -\frac{C^2(2Y-1)\cos^2\left(\frac{\pi Y}{2}\right)\Gamma^2(-Y)}{\sqrt{2\pi}\sigma^{2Y-1}} \widetilde{\mathbb{E}}\left(|W_1^*|^{2Y-2}\right) := d'_{32}. \tag{A.63}$$

Hence, by combining (A.51), (A.57) and (A.63),

$$\begin{aligned} B_2(t) &= t^{1-\frac{Y}{2}} \left(\widetilde{B}_{22}(t) + \frac{C\sigma^{1-Y}}{2Y} \widetilde{\mathbb{E}}\left(|W_1^*|^{1-Y}\right)\right) + o(t^{\frac{1}{2}}) \\ &= t^{1-\frac{Y}{2}} \left(d'_{32} t^{1-\frac{Y}{2}} + o\left(t^{1-\frac{Y}{2}}\right) + \frac{C\sigma^{1-Y}}{2Y} \widetilde{\mathbb{E}}\left(|W_1^*|^{1-Y}\right)\right) + o(t^{\frac{1}{2}}) \\ &= \frac{C\sigma^{1-Y}}{2Y} \widetilde{\mathbb{E}}\left(|W_1^*|^{1-Y}\right) t^{1-\frac{Y}{2}} + d'_{32} t^{2-Y} + o(t^{\frac{1}{2}}) + o(t^{2-Y}), \quad t\to 0. \end{aligned} \tag{A.64}$$

Step 3. We finally study the behavior of $B_3(t)$ by further decomposing it as:

$$\begin{aligned} B_3(t) &= e^{-(\eta+\tilde{\gamma})t} \int_0^\infty \widetilde{\mathbb{E}}\left[\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1} - 1\right) \mathbf{1}_{\left\{Z_1 \geq t^{\frac{1}{2}-\frac{1}{Y}} w\right\}}\right] \frac{1-e^{\sqrt{t}w}}{\sqrt{t}} \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dw \\ &\quad + e^{-(\eta+\tilde{\gamma})t} \int_0^\infty \widetilde{\mathbb{P}}\left(Z_1 \geq t^{\frac{1}{2}-\frac{1}{Y}} w\right) \frac{1-e^{\sqrt{t}w}}{\sqrt{t}} \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dw \\ &\quad + e^{-(\eta+\tilde{\gamma})t} \int_0^\infty \widetilde{\mathbb{E}}\left[\mathbf{1}_{\left\{Z_1 \geq t^{\frac{1}{2}-\frac{1}{Y}} w\right\}} \left(\frac{e^{-t^{\frac{1}{Y}}\widetilde{U}_1} - e^{-t^{\frac{1}{Y}}(Z_1+\widetilde{U}_1)}}{\sqrt{t}} - t^{\frac{1}{Y}-\frac{1}{2}} Z_1\right)\right] e^{\sqrt{t}w} \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dw \\ &\quad + e^{-(\eta+\tilde{\gamma})t} \int_0^\infty t^{\frac{1}{Y}-\frac{1}{2}} \widetilde{\mathbb{E}}\left(Z_1 \mathbf{1}_{\left\{Z_1 \geq t^{\frac{1}{2}-\frac{1}{Y}} w\right\}}\right) e^{\sqrt{t}w} \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dw \\ &:= B_{31}(t) + B_{32}(t) + B_{33}(t) + B_{34}(t). \end{aligned} \tag{A.65}$$

First, note that the term $B_{32}(t)$ is similar to the term $B_{22}(t)$ in (A.51) and, thus, using arguments similar to those leading to (A.63) gives,

$$B_{32}(t) = -\frac{C\sigma^{1-Y}}{2Y}\widetilde{\mathbb{E}}\left(|W_1^*|^{1-Y}\right)t^{1-\frac{Y}{2}} - d'_{32}t^{2-Y} + o(t^{2-Y}), \quad t \to 0. \tag{A.66}$$

Next, the term $B_{34}(t)$ is similar to the term $B_{13}(t)$ introduced in (A.31) and, thus, using arguments similar to those leading to (A.49) gives,

$$B_{34}(t) = \frac{C\sigma^{1-Y}}{2(Y-1)}\widetilde{\mathbb{E}}\left(|W_1^*|^{1-Y}\right)t^{1-\frac{Y}{2}} + d'_{31}t^{2-Y} + o(t^{\frac{1}{2}}) + o(t^{2-Y}), \quad t \to 0. \tag{A.67}$$

It remains to analyze $B_{31}(t)$ and $B_{33}(t)$. For $B_{31}(t)$, note that the expectation appearing therein can be written as

$$\widetilde{\mathbb{E}}\left[\left(e^{-t^{\frac{1}{Y}}(M^*\bar{U}_1^{(p)}-G^*\bar{U}_1^{(n)})}-1\right)\mathbf{1}_{\left\{-\bar{U}_1^{(p)}-\bar{U}_1^{(n)}\leq -t^{\frac{1}{2}-\frac{1}{Y}}w\right\}}\right] = \widetilde{\mathbb{E}}\left[\left(e^{-t^{\frac{1}{Y}}(G^*\widehat{U}_1^{(p)}-M^*\widehat{U}_1^{(n)})}-1\right)\mathbf{1}_{\left\{\widehat{U}_1^{(p)}+\widehat{U}_1^{(n)}\leq -t^{\frac{1}{2}-\frac{1}{Y}}w\right\}}\right],$$

where $(\widehat{U}_1^{(p)}, \widehat{U}_1^{(n)}) := (-\bar{U}_1^{(n)}, -\bar{U}_1^{(p)}) \stackrel{\mathfrak{D}}{=} (\bar{U}_1^{(p)}, \bar{U}_1^{(n)})$. Thus, $B_{31}(t)$ is the same as the term $B_{21}(t)$ defined in (A.51) but with the role of the parameters M^* and G^* reversed. In other words, if we write $B_{21}(t; M^*, G^*) := B_{21}(t)$ to emphasize the dependence on the parameters M^* and G^*, we have that $B_{31}(t) = B_{21}(t; G^*, M^*)$. Therefore, in view of (A.57),

$$\lim_{t\to 0} t^{-\frac{1}{2}}B_{31}(t) = 0. \tag{A.68}$$

To finish, we further decompose $B_{33}(t)$ as:

$$\begin{aligned}
B_{33}(t) &= e^{-(\eta+\tilde{\gamma})t}\int_0^\infty \widetilde{\mathbb{E}}\left[\mathbf{1}_{\left\{Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}w\right\}}t^{\frac{1}{Y}-\frac{1}{2}}\int_{\widetilde{U}_1}^{Z_1+\widetilde{U}_1}\left(e^{-t^{\frac{1}{Y}}u}-1\right)du\right]e^{\sqrt{t}w}\frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dw \\
&= t^{-\frac{1}{2}}e^{-(\eta+\tilde{\gamma})t}\int_0^\infty\left[\int_{-\infty}^0 (e^{-x}-1)\,\widetilde{\mathbb{P}}\left(Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}w,\, \widetilde{U}_1\leq t^{-\frac{1}{Y}}x\leq Z_1+\widetilde{U}_1\right)dx\right]e^{\sqrt{t}w}\frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dw \\
&\quad + t^{-\frac{1}{2}}e^{-(\eta+\tilde{\gamma})t}\int_0^\infty\left[\int_0^\infty (e^{-x}-1)\,\widetilde{\mathbb{P}}\left(Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}w,\, \widetilde{U}_1\leq t^{-\frac{1}{Y}}x\leq Z_1+\widetilde{U}_1\right)dx\right]e^{\sqrt{t}w}\frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dw.
\end{aligned} \tag{A.69}$$

When $x<0$, by (2.18), for any $t>0$ and $w>0$,

$$P_t(w,x) := \widetilde{\mathbb{P}}\left(Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}w,\, \widetilde{U}_1\leq t^{-\frac{1}{Y}}x\leq Z_1+\widetilde{U}_1\right) \leq \widetilde{\mathbb{P}}\left(\widetilde{U}_1\leq t^{-\frac{1}{Y}}x\right) \leq \widetilde{\mathbb{E}}\left(e^{-\widetilde{U}_1}\right)e^{t^{-\frac{1}{Y}}x} = e^{\eta}e^{t^{-\frac{1}{Y}}x}.$$

Hence, for $0<t<1$ and since $1<Y<2$,

$$\begin{aligned}
0 &\leq t^{-1}e^{-(\eta+\tilde{\gamma})t}\int_0^\infty\left[\int_{-\infty}^0 (e^{-x}-1)\,\widetilde{\mathbb{P}}\left(Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}w, \widetilde{U}_1\leq t^{-\frac{1}{Y}}x\leq Z_1+\widetilde{U}_1\right)dx\right]e^{\sqrt{t}w}\frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dw \\
&\leq t^{-1}e^{-(\eta+\tilde{\gamma})t}e^{\eta}\int_{-\infty}^0 (e^{-x}-1)\,e^{t^{-\frac{1}{Y}}x}dx\int_0^\infty w\,e^{w}\frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dw \\
&= \frac{t^{\frac{2}{Y}-1}}{1-t^{\frac{1}{Y}}}e^{-(\eta+\tilde{\gamma})t}e^{\eta}\int_0^\infty w\,e^{w}\frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dw \to 0, \quad t\to 0.
\end{aligned} \tag{A.70}$$

For the second integral in (A.69), using arguments similar to those leading to (2.14-i), there exists a constant $\tilde{\kappa}\in(0,\infty)$, such that

$$t^{-1}\widetilde{\mathbb{P}}\left(t^{-\frac{1}{Y}}x\leq Z_1+\widetilde{U}_1\right)\leq\tilde{\kappa}x^{-Y}, \quad \text{for any } x>0,$$

and thus, by the dominated convergence theorem,

$$\begin{aligned}
&\lim_{t\to 0} t^{-1}e^{-(\eta+\tilde{\gamma})t}\int_0^\infty\left[\int_0^\infty (e^{-x}-1)\,\widetilde{\mathbb{P}}\left(Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}w,\, \widetilde{U}_1\leq t^{-\frac{1}{Y}}x\leq Z_1+\widetilde{U}_1\right)dx\right]e^{\sqrt{t}w}\frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dw \\
&= \int_0^\infty\left[\int_0^\infty (e^{-x}-1)\left(\lim_{t\to 0}\frac{1}{t}\widetilde{\mathbb{P}}\left(Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}w, \widetilde{U}_1\leq t^{-\frac{1}{Y}}x\leq Z_1+\widetilde{U}_1\right)\right)dx\right]\frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dw.
\end{aligned} \tag{A.71}$$

It remains to compute the limit in the above integrand. For any $t>0$, $x>0$ and $w>0$,

$$\begin{aligned}\frac{1}{t}P_t(w,x) &= \frac{1}{t}\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)}+\bar{U}_1^{(n)}\geq t^{\frac{1}{2}-\frac{1}{Y}}w,\ M^*\bar{U}_1^{(p)}-G^*\bar{U}_1^{(n)}\leq t^{-\frac{1}{Y}}x\leq M\bar{U}_1^{(p)}-G\bar{U}_1^{(n)}\right)\\ &= \frac{1}{t}\int_{\mathbb{R}}\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)}\geq t^{\frac{1}{2}-\frac{1}{Y}}w+u,\ \frac{t^{-\frac{1}{Y}}x-Gu}{M}\leq\bar{U}_1^{(p)}\leq\frac{t^{-\frac{1}{Y}}x-G^*u}{M^*}\right)p_U(1,u)\,du.\end{aligned}$$

Note that

$$\begin{aligned}&\frac{t^{-\frac{1}{Y}}x-Gu}{M}\leq\frac{t^{-\frac{1}{Y}}x-G^*u}{M^*}\ \Leftrightarrow\ u\leq\frac{t^{-\frac{1}{Y}}x}{M+G},\quad t^{\frac{1}{2}-\frac{1}{Y}}w+u\leq\frac{t^{-\frac{1}{Y}}x-G^*u}{M^*}\ \Leftrightarrow\ u\leq\frac{t^{-\frac{1}{Y}}x-M^*t^{\frac{1}{2}-\frac{1}{Y}}w}{M+G},\\ &t^{\frac{1}{2}-\frac{1}{Y}}w+u\leq\frac{t^{-\frac{1}{Y}}x-Gu}{M}\ \Leftrightarrow\ u\leq\frac{t^{-\frac{1}{Y}}x-Mt^{\frac{1}{2}-\frac{1}{Y}}w}{M+G}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{t}P_t(w,x) &= \frac{1}{t}\int_{-\infty}^{\frac{t^{-\frac{1}{Y}}x-Mt^{\frac{1}{2}-\frac{1}{Y}}w}{M+G}}\widetilde{\mathbb{P}}\left(\frac{t^{-\frac{1}{Y}}x-Gu}{M}\leq\bar{U}_1^{(p)}\leq\frac{t^{-\frac{1}{Y}}x-G^*u}{M^*}\right)p_U(1,u)\,du\\ &\quad+\frac{1}{t}\int_{\frac{t^{-\frac{1}{Y}}x-Mt^{\frac{1}{2}-\frac{1}{Y}}w}{M+G}}^{\frac{t^{-\frac{1}{Y}}x-M^*t^{\frac{1}{2}-\frac{1}{Y}}w}{M+G}}\widetilde{\mathbb{P}}\left(t^{\frac{1}{2}-\frac{1}{Y}}w+u\leq\bar{U}_1^{(p)}\leq\frac{t^{-\frac{1}{Y}}x-G^*u}{M^*}\right)p_U(1,u)\,du\\ &:= I_1(t;w,x)+I_2(t;w,x).\end{aligned}\tag{A.72}$$

For the first integral in (A.72), note that for any $t>0$, $x>0$ and $w>0$,

$$u\leq\frac{t^{-\frac{1}{Y}}x-Mt^{-\frac{1}{Y}}w}{M+G}<\frac{t^{-\frac{1}{Y}}x}{M+G}<\frac{t^{-\frac{1}{Y}}x}{G}\ \Rightarrow\ t^{-\frac{1}{Y}}x-Gu>0,\qquad\frac{x-Mw\sqrt{t}}{M+G}>0\ \Leftrightarrow\ t<\frac{x^2}{M^2w^2}.$$

Hence, by (2.13) and the dominated convergence theorem, for any $x>0$, $w>0$ and $u\leq t^{-\frac{1}{Y}}(x-Mw)/(M+G)$,

$$\begin{aligned}\lim_{t\to0}I_1(t;w,x) &= \int_{\mathbb{R}}p_U(1,u)\left[\lim_{t\to0}\frac{1}{t}\widetilde{\mathbb{P}}\left(\frac{t^{-\frac{1}{Y}}x-Gu}{M}\leq\bar{U}_1^+\leq\frac{t^{-\frac{1}{Y}}x-G^*u}{M^*}\right)\right]\mathbf{1}_{\left\{u\leq\frac{t^{-1/Y}x-Mt^{-1/Y}w}{M+G}\right\}}du\\ &= \int_{\mathbb{R}}p_U(1,u)\left[\lim_{t\to0}\frac{1}{t}\widetilde{\mathbb{P}}\left(\bar{U}_1^+\geq\frac{t^{-\frac{1}{Y}}x-Gu}{M}\right)\right]\mathbf{1}_{\left\{u\leq\frac{t^{-1/Y}x-Mt^{-1/Y}w}{M+G}\right\}}du\\ &\quad-\int_{\mathbb{R}}p_U(1,u)\left[\lim_{t\to0}\frac{1}{t}\widetilde{\mathbb{P}}\left(\bar{U}_1^+\geq\frac{t^{-\frac{1}{Y}}x-G^*u}{M^*}\right)\right]\mathbf{1}_{\left\{u\leq\frac{t^{-1/Y}x-Mt^{-1/Y}w}{M+G}\right\}}du\\ &= \frac{C}{Y}\left(M^Y-(M^*)^Y\right)x^{-Y}.\end{aligned}\tag{A.73}$$

For the second integral in (A.72), since for any $x>0$ and $w>0$, $t^{-\frac{1}{Y}}x-Mt^{-\frac{1}{Y}}w>0$ if and only if $t<w^2/(M^2w^2)$,

$$\begin{aligned}0 &\leq \frac{1}{t}\int_{\frac{t^{-\frac{1}{Y}}x-Mt^{\frac{1}{2}-\frac{1}{Y}}w}{M+G}}^{\frac{t^{-\frac{1}{Y}}x-M^*t^{\frac{1}{2}-\frac{1}{Y}}w}{M+G}}\widetilde{\mathbb{P}}\left(t^{\frac{1}{2}-\frac{1}{Y}}w+u\leq\bar{U}_1^{(p)}\leq\frac{t^{-\frac{1}{Y}}x-G^*u}{M^*}\right)p_U(1,u)\,du\\ &\leq\frac{1}{t}\widetilde{\mathbb{P}}\left(\bar{U}_1^{(p)}\geq t^{\frac{1}{2}-\frac{1}{Y}}w\right)\widetilde{\mathbb{P}}\left(-\bar{U}_1^{(n)}\geq\frac{t^{-\frac{1}{Y}}x-Mt^{\frac{1}{2}-\frac{1}{Y}}w}{M+G}\right)\to0,\quad t\to0.\end{aligned}\tag{A.74}$$

Combining (A.71)-(A.74), and using (A.2),

$$\lim_{t\to0}\frac{e^{-(\eta+\tilde{\gamma})t}}{t}\int_0^\infty\left[\int_0^\infty(e^{-x}-1)\,\widetilde{\mathbb{P}}\left(Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}w,\widetilde{U}_1\leq t^{-\frac{1}{Y}}x\leq Z_1+\widetilde{U}_1\right)dx\right]\frac{e^{\sqrt{t}w}\,e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dw=-\frac{C\Gamma(-Y)}{2}\left(M^Y-(M^*)^Y\right),$$

which, together with (A.70), leads to:

$$\lim_{t\to0}t^{-\frac{1}{2}}B_{33}(t)=-\frac{d_3'}{2},\tag{A.75}$$

where $d'_3 = C\Gamma(-Y)\left(M^Y - (M^*)^Y\right)$. Hence, by combining (A.65)-(A.68) and (A.75), we get

$$B_3(t) = -\frac{d'_3}{2}\, t^{\frac{1}{2}} + \frac{C\sigma^{1-Y}}{2Y(Y-1)}\widetilde{\mathbb{E}}\left(|W_1^*|^{1-Y}\right) t^{1-\frac{Y}{2}} + \left(d'_{31} - d'_{32}\right) t^{2-Y} + o(t^{\frac{1}{2}}) + o(t^{2-Y}), \quad t \to 0. \tag{A.76}$$

We are now in position of obtaining the higher-order asymptotic expansion. First, by combining (A.30), (A.50), (A.64) and (A.76),

$$A_1(t) = -d'_3\, t^{\frac{1}{2}} + \frac{C\sigma^{1-Y}}{Y(Y-1)}\widetilde{\mathbb{E}}\left(|W_1|^{1-Y}\right) t^{1-\frac{Y}{2}} + 2\left(d'_{31} - d'_{32}\right) t^{2-Y} + o(t^{\frac{1}{2}}) + o(t^{2-Y}), \quad t \to 0.$$

By combining the previous expression with (A.27)-(A.29),

$$\Delta_0(t) = \left(\frac{\tilde{\gamma}}{2} - \frac{\sigma^2}{4} - d'_3\right) t^{\frac{1}{2}} + \frac{C}{Y(Y-1)}\widetilde{\mathbb{E}}\left(|W_1|^{1-Y}\right) t^{1-\frac{Y}{2}} + 2\left(d'_{31} - d'_{32}\right) t^{2-Y} + o(t^{\frac{1}{2}}) + o(t^{2-Y}), \quad t \to 0,$$

which yields (3.11), by noting that the coefficient of the first term above reduces to the expression d_{31} in (3.9) and that $d_{32} = 2(d'_{31} - d'_{32})$. □

Proof of Proposition 3.4. When the diffusion component is present, [12, Proposition 5] implies that $\hat{\sigma}(t) \to \sigma$ as $t \to 0$. In particular, $\hat{\sigma}(t)t^{1/2} \to 0$ as $t \to 0$ and, thus, (A.22) above still holds. Let $\tilde{\sigma}(t) := \hat{\sigma}(t) - \sigma$, then $\tilde{\sigma}(t) \to 0$ as $t \to 0$, and (A.22) can be written as

$$C_{BS}(t, \hat{\sigma}(t)) = \frac{\sigma}{\sqrt{2\pi}}\, t^{\frac{1}{2}} + \frac{\tilde{\sigma}(t)}{\sqrt{2\pi}}\, t^{\frac{1}{2}} - \frac{\hat{\sigma}(t)^3}{24\sqrt{2\pi}}\, t^{\frac{3}{2}} + O\left(t^{\frac{5}{2}}\right), \quad t \to 0. \tag{A.77}$$

By comparing (3.7)-(3.8) and (A.77), and since the third term in (A.77) is $O(t^{3/2})$, we have

$$\frac{C 2^{\frac{1-Y}{2}} \sigma^{1-Y}}{Y(Y-1)\sqrt{\pi}}\Gamma\left(1 - \frac{Y}{2}\right) t^{\frac{3-Y}{2}} \sim \frac{\tilde{\sigma}(t)}{\sqrt{2\pi}}\, t^{\frac{1}{2}}, \quad t \to 0,$$

and, therefore,

$$\tilde{\sigma}(t) \sim \frac{C 2^{1-\frac{Y}{2}} \sigma^{1-Y}}{Y(Y-1)}\Gamma\left(1 - \frac{Y}{2}\right) t^{1-\frac{Y}{2}} := \sigma_1 t^{1-\frac{Y}{2}}, \quad t \to 0. \tag{A.78}$$

Next, set $\bar{\sigma}(t) := \hat{\sigma}(t) - \sigma - \sigma_1 t^{1-\frac{Y}{2}}$, then (A.77) can be rewritten as:

$$C_{BS}(t, \bar{\sigma}(t)) = \frac{\sigma}{\sqrt{2\pi}}\, t^{\frac{1}{2}} + \frac{C 2^{1-\frac{Y}{2}} \sigma^{1-Y}}{Y(Y-1)\sqrt{\pi}}\Gamma\left(1 - \frac{Y}{2}\right) t^{\frac{3-Y}{2}} + \frac{\bar{\sigma}(t)}{\sqrt{2\pi}}\, t^{\frac{1}{2}} - \frac{\hat{\sigma}(t)^3}{24\sqrt{2\pi}}\, t^{\frac{3}{2}} + O\left(t^{\frac{5}{2}}\right), \quad t \to 0. \tag{A.79}$$

We can finally deduce (3.11) by comparing the first three terms in (3.10) with (A.79). □

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